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On entire solutions of some Fermat type differential-difference equations

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Abstract. On one hand, we study the existence of transcendental entire solutions with finite order of the Fermat type difference equations. On the other hand, we also investigate the existence and growth of solutions of nonlinear differential-difference equations. These results extend and improve some previous in [5, 14].

§1 Introduction

It is well-known that the growth of solutions of complex differential equations is an important topic in complex analysis theory, many results can be found in [11], in which Nevanlinna theory is an effective research tool. The growth of solutions of complex difference equations and differential-difference equations is a very interesting topic to which many related results have been obtained, see [2,3,10,15,17] and the references therein. This paper is devoted to considering the properties of solutions of complex difference equations and differential-difference equations. Nevanlinna theory will play an important role in this paper, we assume that the readers are familiar with standard notation and fundamental results of Nevanlinna theory, see [9, 11] for more details. Let f be a meromorphic function in the complex plane, we use $\rho(f)$ to denote the order of growth of f.

The following equation

$$f(z)^n + g(z)^n = 1 \tag{1}$$

can be regarded as the Fermat diophantine equations $x^n + y^n = 1$ over function fields, where n is a positive integer. Gross [6] obtained that (1) has no transcendental meromorphic solutions when $n \geq 4$. Montel [16] proved that (1) has no transcendental entire solutions when $n \geq 3$. If $n = 2$, Gross [6, 7] obtained that (1) has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is an entire function; no other entire function solutions exist. The other

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non-constant meromorphic solutions of (1) can be stated as $f(z)=\frac{2\beta(z)}{1+\beta(z)^2}$ and $g(z)=\frac{1-\beta(z)^2}{1+\beta(z)^2}$ for $n=2$, where $\beta(z)$ is a non-constant meromorphic function. Since then many researchers have obtained further results on Fermat type difference and differential-difference equations. For the case of $n\neq m$, Yang [21] investigated the generalization of the Fermat type functional equation (1) as the following equation

$$f(z)^n+g(z)^m=1, \tag{2}$$

and proved that if m and n satisfy $\frac{1}{m}+\frac{1}{n}<1$, then the equation (2) has no non-constant entire solutions. From which the cases of $m>2$ and $n>2$ are clear. Motivated by the above, many authors considered equations such that $g(z)$ has a special relationship, such as derivative $f'(z)$, shift $f(z+c)$ or q-difference $f(qz)$, with $f(z)$ in (2) when $m=n=2$, called Fermat type equations, for example see [14, 18, 19, 23] and the references therein.

Yang and Li [23, Theorem 1] considered the entire solutions of the equation

$$f(z)^2+f'(z)^2=1, \tag{3}$$

and obtained that the transcendental meromorphic solutions of (3) must have the form $f(z)=\frac{1}{2}(pe^{-iz}+pe^{iz})$, where p is a constant. Furthermore, Tang and Liao [19, Theorem 1] investigated the entire solutions of a generalization of (3) as the following equation

$$f(z)^2+P(z)^2f^{(k)}(z)^2=Q(z), \tag{4}$$

where $P(z)$ and $Q(z)$ are non-zero polynomials. They showed that if $f(z)$ is a transcendental meromorphic solution of (4), then $P(z)=A(constant)$, $Q(z)=B(constant)$, $k=2n+1$ for some nonnegative integer n and $f(z)=b\cos(az+c)$, where a,b,c are constants such that $Aa^k=\pm1$, $b^2=B$. Since the difference analogue of logarithmic derivative lemma [2, 10] is valid for finite order meromorphic functions, the finite order solutions of difference equations can always be considered. Liu [12, Proposition 5.1] considered the finite order entire solutions of the difference equation

$$f(z)^2+f(z+c)^2=1, \tag{5}$$

and proved that the transcendental entire solutions with finite order of (5) must satisfy $f(z)=\sin(Az+B)$, where B is a constant, $A=\frac{(4k+1)\pi}{2c}$, k is an integer, and c is a non-zero constant. If $f(z+c)$ is replaced by $f(z+c)-f(z)$ in (5), then the following equation

$$f(z)^2+[f(z+c)-f(z)]^2=1 \tag{6}$$

has no transcendental entire solutions with finite order, see [12, Proposition 5.3].

This paper is organized as follows. In Section 2, some results are shown for Fermat type difference equations, the proofs of these results are given in Section 3. Some results are shown for Fermat type differential-difference equations in Section 4, and the proofs of these results are given in Section 5 and Section 6 respectively.

§2 Fermat Type Difference Equations

Concerning the properties of solutions of Fermat type difference equations, many results have been obtained by different researchers. Liu considered the entire solutions of (7) and (8) below, which are generalizations of (5), and obtained the following two results.

Theorem 2.1. *[14, Theorem2.1] Let $P(z)$ and $Q(z)$ be two non-zero polynomials. If the difference equation*

$$f(z)^2 + P(z)^2 f(z+c)^2 = Q(z) \tag{7}$$

admits a transcendental entire solution of finite order, then $P(z) = \pm 1$ and $Q(z)$ reduces to a constant q. Thus $f(z) = \sqrt{q}\sin(Az+B)$, where B is a constant, $A = \frac{(4k+1)\pi}{2c}$, k is an integer.

Theorem 2.2. *[14, Theorem2.3] There is no transcendental entire solution with finite order of the equation*

$$f(z)^2 + P(z)^2(\triangle_c f(z))^2 = Q(z), \tag{8}$$

where $\triangle_c f(z) = f(z+c) - f(z)$, $P(z)$ and $Q(z)$ are two non-zero polynomials.

According to Theorems 2.1 and 2.2, it is natural to ask: what happens on the growth of solutions of differential equations when $P(z)$ and $Q(z)$ are replaced by transcendental entire functions in (7) and (8) respectively. Here, we consider the question, and obtain the following result, in which the coefficient $P(z)$ is replaced by $\alpha(z)e^{P(z)}$, improving the result of the Theorem 2.1.

Theorem 2.3. *Let $P(z)$ be non-constant polynomial, and $\alpha(z)$ and $Q(z)$ be non-zero polynomials. Then the difference equation*

$$f(z)^2 + \alpha(z)^2 (e^{P(z)})^2 f(z+c)^2 = Q(z) \tag{9}$$

do not have the transcendental entire solution with finite order.

We have the following result when the $Q(z)$ is replaced by $Q(z)e^{\lambda z}$ in (9) too, where λ is a non-zero constant. Here, we consider the case $Q(z) = \alpha(z) = 1$.

Theorem 2.4. *Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$, $a_n \neq 0$ and $|a_n| \neq |\lambda_1 - \lambda_2|$. Then the difference equation*

$$f(z)^2 + (e^{P(z)})^2 f(z+c)^2 = e^{\lambda z} \tag{10}$$

has no transcendental entire solution with finite order, where λ is a non-zero constant satisfying $\lambda = \lambda_1 + \lambda_2$, λ_1 and λ_2 are two constants.

Related Theorem 2.2, we have the following result.

Theorem 2.5. *Let $P(z)$ and $Q(z)$ be two non-constant polynomials, and $\alpha(z)$ be non-zero polynomial. Then the following equation*

$$f(z)^2 + \alpha(z)^2 (e^{P(z)})^2 (\triangle_c f(z))^2 = Q(z) \tag{11}$$

has no transcendental entire solution with finite order, where $\triangle_c f(z) = f(z+c) - f(z)$.

§3 Proofs of Theorems 2.3-2.5

We start the proof from the following two lemmas.

Lemma 3.1. *[24, Theorem 1.62] Let f_j $(j=1,2,\dots,n)$ be meromorphic functions, f_k $(k=1,2,\dots,n-1)$ be not constants, satisfying $\sum_{j=1}^{n} f_j = 1$ and $n \geq 3$. If $f_n(z) \not\equiv 0$ and*

$$\sum_{j=1}^{n} N\left(r, \frac{1}{f_j}\right) + (n-1)\sum_{j=1}^{n} \bar{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $\lambda(<1)$ is positive constant and $k = 1,2,\dots,n-1$, then $f_n(z) \equiv 1$.

Lemma 3.2. *[24, Theorem 1.51] Let $f_j (j=1,2,\dots,n, n\geq 2)$ be meromorphic functions, and g_j $(j=1,2,\dots,n)$ be entire functions. If f_j and g_j satisfy the following conditions,*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$,

(ii) $g_j(z) - g_k(z)$ *is not a constant*, $1 \leq j < k \leq n$,

(iii) $T(r, f_j) = o(T(r, e^{g_h - g_k}))$, $r \to \infty$, $r \bar{\in} E$, $1 \leq j \leq n$, $1 \leq h < k \leq n$, *where $E \subset (1,\infty)$ is of finite linear measure or finite logarithmic measure, then $f_j(z) \equiv 0$, $j = 1,2,\dots,n$.*

Proof of Theorem 2.3. Suppose on the contrary to the assertion that there exists a transcendental entire solution f of (9) with finite order. We aim for a contradiction. Then we can rewrite (9) with the form

$$[f(z) + i\alpha(z)e^{P(z)}f(z+c)][f(z) - i\alpha(z)e^{P(z)}f(z+c)] = Q(z). \tag{12}$$

Thus, $f(z)+i\alpha(z)e^{P(z)}f(z+c)$ and $f(z)-i\alpha(z)e^{P(z)}f(z+c)$ have finitely many zeros. Combining (12) with the Hadamard factorization theorem, we assume that

$$f(z) + i\alpha(z)e^{P(z)}f(z+c) = Q_1(z)e^{P_1(z)} \tag{13}$$

and

$$f(z) - i\alpha(z)e^{P(z)}f(z+c) = Q_2(z)e^{-P_1(z)}, \tag{14}$$

where $P_1(z)$ is a non-constant polynomial and $Q_1(z)Q_2(z) \equiv Q(z)$, $Q_1(z)$ and $Q_2(z)$ are non-zero polynomials. It follows from (13) and (14) that

$$f(z) = \frac{Q_1(z)e^{P_1(z)} + Q_2(z)e^{-P_1(z)}}{2} \tag{15}$$

and

$$f(z+c) = \frac{Q_1(z)e^{P_1(z)} - Q_2(z)e^{-P_1(z)}}{2i\alpha(z)e^{P(z)}}. \tag{16}$$

Combining (15) with (16), we get

$$\begin{aligned} f(z+c) &= \frac{Q_1(z+c)e^{P_1(z+c)} + Q_2(z+c)e^{-P_1(z+c)}}{2} \\ &= \frac{Q_1(z)e^{P_1(z)} - Q_2(z)e^{-P_1(z)}}{2i\alpha(z)e^{P(z)}}. \end{aligned}$$

Thus, we have

$$\frac{i\alpha(z)Q_1(z+c)e^{P(z)+P_1(z+c)+P_1(z)}}{-Q_2(z)} + \frac{i\alpha(z)Q_2(z+c)e^{P(z)-P_1(z+c)+P_1(z)}}{-Q_2(z)} \tag{17}$$

$$+\frac{Q_1(z)e^{2P_1(z)}}{Q_2(z)}=1.$$

If $P(z)-P_1(z+c)+P_1(z)$ is a constant, then $P(z)+P_1(z+c)+P_1(z)$ is not constant. Thus by Lemma 3.1 and (17), we have

$$i\alpha(z)Q_2(z+c)e^{P(z)-P_1(z+c)+P_1(z)}=-Q_2(z), \tag{18}$$

which means that $Q_2(z)$ is also a constant. Furthermore, we can rewrite (17) as the following form

$$i\alpha(z)Q_1(z+c)e^{P(z)+P_1(z+c)+P_1(z)}-Q_1(z)e^{2P_1(z)}=0. \tag{19}$$

Noticing that $P(z)+P_1(z+c)+P_1(z)$ is not a constant, thus we get $Q_1(z)=0$ by Lemma 3.2 and (19), which is a contradiction.

If $P(z)-P_1(z+c)+P_1(z)$ is not a constant, then by Lemma 3.1 and (17), we have

$$\frac{i\alpha(z)Q_1(z+c)e^{P(z)+P_1(z+c)+P_1(z)}}{-Q_2(z)}=1, \tag{20}$$

which means that $P(z)+P_1(z+c)+P_1(z)$ is a constant, thus $P(z)-P_1(z+c)-P_1(z)$ is not a constant. Combining (17) and (20), we get

$$\frac{i\alpha(z)Q_2(z+c)e^{P(z)-P_1(z+c)+P_1(z)}}{-Q_2(z)}+\frac{Q_1(z)e^{2P_1(z)}}{Q_2(z)}=0. \tag{21}$$

Thus we have $Q_1(z)=Q_2(z)=0$ by Lemma 3.2 and (21), which is also a contradiction. Therefore Theorem 2.3 is proved. □

Proof of Theorem 2.4. Suppose on the contrary to the assertion that there exists a transcendental entire solution f of (10) with finite order. We aim for a contradiction. By using the similar reason as in the proof of Theorem 2.3, we get

$$f(z)=\frac{e^{\lambda_1 z}+e^{\lambda_2 z}}{2} \tag{22}$$

and

$$f(z+c)=\frac{e^{\lambda_1 z}-e^{\lambda_2 z}}{2ie^{P(z)}}, \tag{23}$$

where λ_1 and λ_2 are constants and $\lambda_1+\lambda_2=\lambda$. Combining (22) with (23), we get

$$f(z+c)=\frac{e^{\lambda_1(z+c)}+e^{\lambda_2(z+c)}}{2}=\frac{e^{\lambda_1 z}-e^{\lambda_2 z}}{2ie^{P(z)}}.$$

Thus we have

$$ie^{\lambda_1(z+c)+P(z)}+ie^{\lambda_2(z+c)+P(z)}+e^{\lambda_2 z}-e^{\lambda_1 z}=0. \tag{24}$$

Next we claim that $\lambda_1\neq\lambda_2$. In fact, if $\lambda_1=\lambda_2$, then $f(z)=e^{\frac{\lambda z}{2}}$ by (22) and $\lambda_1+\lambda_2=\lambda$. It follows this and (10) that

$$e^{\lambda z}+(e^{P(z)})^2e^{\lambda(z+c)}=e^{\lambda z}. \tag{25}$$

This implies that $(e^{P(z)})^2e^{\lambda(z+c)}=0$, which is a contradiction.

From (24), we get

$$ie^{\lambda_1 c+P(z)}+ie^{(\lambda_2-\lambda_1)z+\lambda_2 c+P(z)}+e^{(\lambda_2-\lambda_1)z}=1 \tag{26}$$

and

$$-ie^{\lambda_2 c+P(z)}-ie^{(\lambda_1-\lambda_2)z+\lambda_1 c+P(z)}+e^{(\lambda_1-\lambda_2)z}=1. \tag{27}$$

Since $P(z)$ is a non-constant polynomial and $\lambda_1 \neq \lambda_2$, then from Lemma 3.1 and (26), we have

$$ie^{(\lambda_2-\lambda_1)z+\lambda_2 c+P(z)} = 1.$$

This shows that $P(z) = (\lambda_1 - \lambda_2)z - \lambda_2 c + (2k\pi - \frac{\pi}{2})i$. Similarly, from (27) we have

$$-ie^{(\lambda_1-\lambda_2)z+\lambda_1 c+P(z)} = 1,$$

which means that $P(z) = (\lambda_2-\lambda_1)z-\lambda_1 c+(2k\pi+\frac{\pi}{2})i$. This is a contraction with $|a_n|\neq|\lambda_1-\lambda_2|$. This shows that (10) has no the transcendental entire solutions with finite order. The proof of Theorem 2.4 is completed. □

Proof of Theorem 2.5. Suppose on the contrary to the assertion that there exists a transcendental entire solution f of (11) with finite order. We aim for a contradiction. By using the similar reason as in the proof of Theorem 2.3, we get

$$f(z) = \frac{Q_1(z)e^{P_1(z)} + Q_2(z)e^{-P_1(z)}}{2} \tag{28}$$

and

$$\triangle_c(f) = \frac{Q_1(z)e^{P_1(z)} - Q_2(z)e^{-P_1(z)}}{2i\alpha(z)e^{P(z)}}, \tag{29}$$

where $P_1(z)$ is a non-constant polynomial and $Q_1(z)Q_2(z) = Q(z)$, $Q_1(z)$, $Q_2(z)$ are non-zero polynomials and cannot be constants simultaneouslly. Thus, we have

$$\begin{aligned}&\frac{i\alpha(z)Q_1(z+c)e^{P_1(z+c)+P(z)+P_1(z)}}{-Q_2(z)} + \frac{i\alpha(z)Q_2(z+c)e^{P(z)+P_1(z)-P_1(z+c)}}{-Q_2(z)}\\ &+\frac{i\alpha(z)Q_1(z)e^{2P_1(z)+P(z)}}{Q_2(z)} + i\alpha(z)e^{P(z)} + \frac{Q_1(z)e^{2P_1(z)}}{Q_2(z)} = 1\end{aligned} \tag{30}$$

and

$$\begin{aligned}&\frac{i\alpha(z)Q_1(z+c)e^{P_1(z+c)+P(z)-P_1(z)}}{Q_1(z)} + \frac{i\alpha(z)Q_2(z+c)e^{P(z)-P_1(z)-P_1(z+c)}}{Q_1(z)}\\ &-\frac{i\alpha(z)Q_2(z)e^{P(z)-2P_1(z)}}{Q_1(z)} - i\alpha(z)e^{P(z)} + \frac{Q_2(z)e^{-2P_1(z)}}{Q_1(z)} = 1.\end{aligned} \tag{31}$$

Since $P(z)$ and $P_1(z)$ are non-constant polynomials, then $ie^{P(z)}$, $Q_1(z)e^{2P_1(z)}$, and $Q_2(z)e^{-2P_1(z)}$ are not constants.

If $e^{2P_1(z)+P(z)}$ is not a constant, then $e^{P_1(z+c)+P(z)+P_1(z)}$ is not a constant too. From (30) and Lemma 3.1 we have

$$i\alpha(z)Q_2(z+c)e^{P(z)+P_1(z)-P_1(z+c)} = -Q_2(z), \tag{32}$$

meaning that $P(z) + P_1(z) - P_1(z+c)$ is a constant. Thus $e^{P(z)-P_1(z)-P_1(z+c)}$ and $e^{P(z)-2P_1(z)}$ are not constants too. By (31) and Lemma 3.1 we have

$$i\alpha(z)Q_1(z+c)e^{P_1(z+c)+P(z)-P_1(z)} = Q_1(z). \tag{33}$$

Multiplying (32) and (33), we have $\alpha(z)^2 Q(z+c)e^{2P(z)} = Q(z)$, from which we get $P(z) = p$, $\alpha(z) = \alpha$ and $Q(z) = q$, where p, α and q are constants. This is a contradiction with $P(z)$ and $Q(z)$ are not-constant polynomials.

If $e^{2P_1(z)+P(z)}$ is a constant, denoting $2P_1(z) + P(z) = c_1$, that is, $P(z) = -2P_1(z) + c_1$, then $e^{P_1(z+c)+P(z)-P_1(z)}$ and $e^{P(z)-P_1(z)-P_1(z+c)}$ are not constants in (31). Thus from (31) and

Lemma 3.1, we have $-i\alpha(z)Q_2(z)e^{P(z)-2P_1(z)} = Q_1(z)$, that is

$$-i\alpha(z)Q_2(z)e^{-4P_1(z)+c_1} = Q_1(z).$$

This shows that $P_1(z)$ is a constant, which is a contradiction. The proof of Theorem 2.5 is completed. □

§4 Fermat type Differential-Difference Equations

It is interesting in the studying of solutions of differential-difference equations. Yang and Laine [22, Theorem 2.6] considered the existence of solutions of the differential-difference equation $f^n + M(z,f) = h$, where $M(z,f)$ is a linear differential-difference polynomial of f, not vanishing identically, and h is a meromorphic function of finite order. Later, Liu-Cao-Cao [13] considered the properties of entire solutions of the Fermat type differential-difference equations

$$f'(z)^n + f(z+c)^m = 1 \tag{34}$$

and

$$f'(z)^n + (\Delta_c f(z))^m = 1, \tag{35}$$

where m and n are positive integers. They proved the following results.

Theorem 4.1. *[13, Theorem 1.2] If $m \neq n$, then* (34) *has no transcendental entire solutions with finite order.*

Theorem 4.2. *[13, Theorem 1.4] The equation* (35) *has no transcendental entire solutions with finite order, provided that $m \neq n$, and $m > 1$, $n > 1$.*

From [21, Theorem 1], we know that there does not exist entire solutions of (34) when $m > 2, n > 2$. A natural question is what happen on the growth of solutions of differential equations for the case of $n = m = 2$. Liu-Cao-Cao studied the question, and proved the following results.

Theorem 4.3. *[13, Theorem 1.3] The transcendental entire solutions with finite order of differential-difference equation*

$$f'(z)^2 + f(z+c)^2 = 1 \tag{36}$$

must satisfy $f(z) = \sin(z \pm Bi)$, where B is a constant and $c = 2k\pi$ or $c = 2k\pi + \pi$, k is an integer.

Theorem 4.4. *[13, Theorem 1.5] The transcendental entire solutions with finite order of differential-difference equation*

$$f'(z)^2 + (\Delta_c f(z))^2 = 1 \tag{37}$$

must satisfy $f(z) = \frac{1}{2}\sin(2z + Bi)$, where B is a constant and $c = k\pi + \frac{\pi}{2}$, k is an integer.

As an improvement of Theorem 4.3, Liu-Yang proved the following result.

Theorem 4.5. *[14, Theorem 3.1] The transcendental entire solutions with finite order of differential-difference equation*

$$(f^{(k)}(z))^2 + f(z+c)^2 = 1 \tag{38}$$

must satisfy the following two cases:

(i) *if k is odd, then $f(z) = \mp sin(Aiz + Bi)$ and $c = \frac{k\pi i}{A}$, $A^k = \pm i$,*

(ii) *if k is even, then $f(z) = \pm cos(Aiz + Bi)$ and $c = \frac{k\pi i + \frac{\pi i}{2}}{A}$, $A^k = \pm 1$, where B is a constant.*

In 2016, Chen-Gao considered the following equation

$$(f'(z))^2 + P(z)^2 f(z+c)^2 = Q(z), \tag{39}$$

where $P(z)$ and $Q(z)$ are polynomials, and obtained the following result.

Theorem 4.6. *[4, Theorem 1.1] Let $P(z)$ and $Q(z)$ be two non-zero polynomials. If equation (39) has a transcendental entire solutions with finite order, then $P(z) = A(\neq 0)$ and $Q(z) = pq(\neq 0)$. Furthermore,*

$$f(z) = \frac{pe^{az+b} - qe^{-(az+b)}}{2a},$$

where $a = \pm iA, A = \frac{(-1)^k k\pi}{c}, b \in \mathbb{C}, p, q, c \in \mathbb{C}\backslash\{0\}$.

Motivation from Theorem 4.5, we will consider the high order derivative of (39), and get the following result.

Theorem 4.7. *There is no transcendental entire solutions with finite order of the equation*

$$f^{(k)}(z)^2 + P(z)^2 f(z+c)^2 = Q(z), \tag{40}$$

where $P(z)$ is non-constant polynomial, and $Q(z)$ is a non-zero polynomial.

In 2017, Chen-Gao-Du studied the existence of solutions of the following equation

$$f'(z)^2 + P(z)^2 f(z+c)^2 = Q(z)e^{\alpha(z)}, \tag{41}$$

where $P(z)$, $Q(z)$ and $\alpha(z)$ are polynomials, and obtained the following result.

Theorem 4.8. *[5] Let $P(z)$ and $Q(z)$ be two non-zero polynomials, $c \in \mathbb{C} \setminus \{0\}$ and $\alpha(z)$ be a polynomial. If the differential-difference equation (41) admits a transcendental entire solution of finite order, then $f(z)$ must satisfy one of the following cases:*

(i) *$P(z)$ and $Q(z)$ reduce to constants, and*

$$f(z) = \frac{q_1 e^{A_1 z + B_1}}{2A_1} + \frac{q_2 e^{A_2 z + B_2}}{2A_2},$$

where $A_1 = ie^{A_1 c}p$ and $A_2 = -ie^{A_2 c}p$, B_1 and B_2 are constants, and A_1, A_2, q_1, q_2, p, c are non-zero constants;

(ii) *$P(z)$ reduces to a constant, and $Q(z)$ is a polynomial with degree 1, and*

$$f(z) = \frac{(a_1 z + a_0 - \frac{a_1}{A_1})e^{A_1 z + B_1}}{2A_1} + \frac{q_2 e^{A_2 z + B_2}}{2A_2}, \quad \frac{1}{A_1} = c \quad and \quad \frac{1}{A_2} \neq c,$$

or

$$f(z) = \frac{q_1 e^{A_1 z + B_1}}{2A_1} + \frac{(b_1 z + b_0 - \frac{b_1}{A_2})e^{A_2 z + B_2}}{2A_2}, \quad \frac{1}{A_1} \neq c \quad and \quad \frac{1}{A_2} = c,$$

where $A_1 = ie^{A_1c}p$ *and* $A_2 = -ie^{A_2c}p$, B_1, B_2, a_0, b_0 *are constants, and* $A_1, A_2, q_1, q_2, a_1, b_1, p$, c *are non-zero constants;*

(iii) $f(z) = B(z)e^{Az}, \alpha(z) = 2Az + D$, *where* $B(z)$ *satisfies* $[B'(z) + AB(z)]^2 + P^2(z)B^2(z+c)e^{2Ac} = Q(z)e^D$, A, c *are non-zero constants,* D *is a constant.*

Here, we replace $P(z)$ by $e^{P(z)}$ in (41), and prove the following result.

Theorem 4.9. *Let* $P(z)$ *be non-zero polynomial,* $Q(z)$ *be non-constant polynomial,* $c \in \mathbb{C}\setminus\{0\}$, *and* $\alpha(z)$ *be a polynomial. If the differential-difference equation*

$$f'(z)^2 + (e^{P(z)})^2 f(z+c)^2 = Q(z)e^{\alpha(z)} \tag{42}$$

admits a transcendental entire solution of finite order, then $f(z)$ *must satisfy one of the following cases:*

(i) $P(z)$ *and* $Q(z)$ *reduce to constants, and*

$$f(z) = \frac{q_1 e^{A_1 z + B_1}}{2A_1} + \frac{q_2 e^{A_2 z + B_2}}{2A_2},$$

where $A_1 = ie^{A_1c+p}$ *and* $A_2 = -ie^{A_2c+p}$, B_1 *and* B_2 *are constants, and* A_1, A_2, q_1, q_2, p, c *are non-zero constants;*

(ii) $P(z)$ *reduce to a constant, and* $Q(z)$ *is a polynomial with degree 1, and*

$$f(z) = \frac{(a_1 z + a_0 - \frac{a_1}{A_1})e^{A_1 z + B_1}}{2A_1} + \frac{q_2 e^{A_2 z + B_2}}{2A_2}, \quad \frac{1}{A_1} = c \quad and \quad \frac{1}{A_2} \neq c,$$

or

$$f(z) = \frac{q_1 e^{A_1 z + B_1}}{2A_1} + \frac{(b_1 z + b_0 - \frac{b_1}{A_2})e^{A_2 z + B_2}}{2A_2}, \quad \frac{1}{A_1} \neq c \quad and \quad \frac{1}{A_2} = c,$$

where $A_1 = ie^{A_1c+p}$ *and* $A_2 = -ie^{A_2c+p}$, B_1, B_2, a_0, b_0 *are constants, and* $A_1, A_2, q_1, q_2, a_1, b_1, p$, c *are non-zero constants;*

(iii) $f(z) = B(z)e^{Az}$, $\alpha(z) = 2Az+D$, *where* $B(z)$ *satisfies* $[B'(z)+AB(z)]^2+(e^{P(z)})^2B^2(z+c)e^{2Ac} = Q(z)e^D$, A, c *are non-zero constants,* D *is a constant.*

Form the Malmquist-Yosida theorem [11, Theorem 10.2], it is easy to know that non-linear differential equation $f(z)^2 + f'(z)f(z) = 1$ (also can be written as $f'(z) = \frac{1-f(z)^2}{f(z)}$) has no non-constant entire solutions. Using the Yanagihara's result [20, Theorem 1] or combining Lemma 6.3 given in Section 6 with Valiron-Mohon'ko theorem [11, Theorem 2.2.5], we know the non-linear difference equation $f(z)^2 + f(z)f(z+c) = 1$ (also can be written as $f(z+c) = \frac{1-f(z)^2}{f(z)}$) has no finite order entire solutions. This shows that the existence of solutions of differential equations is very different with the existence of solutions of difference equations. Therefore, Liu-Yang considered the growth of entire solutions of the following two differential-difference equations

$$f(z)^2 + f'(z)f(z+c) = 1 \tag{43}$$

and

$$f'(z)^2 + f(z)f(z+c) = 1, \tag{44}$$

and obtained following result.

Theorem 4.10. *[14, Theorem3.4] The order of transcendental entire solutions of* (43) *and* (44) *must be at least one.*

As an improvement of Theorem 4.10, we consider the following equation

$$f(z)^2 + f^{(k)}(z)f(z+c) = 1 \tag{45}$$

and

$$(f^{(k)}(z))^2 + f(z)f(z+c) = 1, \tag{46}$$

and prove the following result.

Theorem 4.11. *The order of transcendental entire solutions of* (45) *and* (46) *must be at least one.*

The following examples will show that Theorem 4.11 is sharpness.

Example 1 The function $f(z) = \sin z$ is a solution of (45), when $k = 4n+1$, $c = \frac{\pi}{2}$, or $k = 4n+3$, $c = \frac{3\pi}{2}$, where n is a natural number. In addition, $f(z) = \cos z$ is also a solution of (45), when $k = 4n+1$, $c = \frac{\pi}{2}$, or $k = 4n+3$, $c = \frac{3\pi}{2}$.

Example 2 The function $f(z) = \sin z$ is a solution of (46), where $k = 2n+1$, with $c = 2m\pi$, m is an integer and n is a natural number.

Example 3 The functions $f(z) = 1 \pm e^z$ and $g(z) = -1 \pm e^z$ are the solutions of (46) when $e^c = -1$.

By using the similar idea as in the proof Theorem 4.11, we also consider the growth of entire solutions of the following differential-difference equations,

$$f(z)^2 + f(z)\triangle_c f(z) = 1, \tag{47}$$

$$f(z+c)^2 + f(z)\triangle_c f(z) = 1, \tag{48}$$

$$f'(z)^2 + f(z)\triangle_c f(z) = 1. \tag{49}$$

Theorem 4.12. *The order of transcendental entire solutions of* (47)*,* (48) *and* (49) *must be at least one.*

We also study the growth of solutions of the following equations,

$$f(z+c)^2 + f(z)f'(z) = 1, \tag{50}$$

$$\triangle_c^2 f(z)^2 + f(z)f'(z) = 1, \tag{51}$$

$$f(z+c)^2 + f(z)f(z+c) = 1. \tag{52}$$

Theorem 4.13. *The order of transcendental entire solutions of* (50),(51) *and* (52) *must be at least one.*

In [14], Liu-Yang also considered the generalization of (43) as the following equation

$$f(z)^n + f'(z)f(z+c) = 1. \tag{53}$$

It showed that there is no transcendental entire solutions of finite order when $n \geq 3$ or $n = 1$. In the following, we will consider the high order derivative of (53) when $n = 1$, that is

$$f(z) + f^{(k)}(z)f(z+c) = 1. \tag{54}$$

Theorem 4.14. *There is no transcendental entire solution with finite order of* (54)*.*

§5 Proofs of Theorems 4.7 and 4.9

We start the proof from the following lemma, which plays a key role in the proof of Theorem 4.9.

Lemma 5.1. *[5, Lemma 2.4] Let $Q(z)$ be a non-zero polynomial and satisfy*

$$Q(z+c)-Q(z)=aQ'(z)+b,$$

where a, c are non-zero constants, b is a constant, then one of the following statements holds:

(i) *If $b=0$ and $a\neq c$, then $Q(z)$ reduces to a non-zero constant;*

(ii) *If $b=0$ and $a=c$, then $Q(z)$ reduces to a non-zero constant or $Q(z)=a_1z+a_0$, where a_1 is a non-zero constant, a_0 is a constant;*

(iii) *If $b\neq 0$ and $a\neq c$, then $Q(z)=a_1z+a_0$ and $b=a_1(c-a)$, where a_1 is a non-zero constant, a_0 is a constant;*

(iv) *If $b\neq 0$ and $a=c$, then $Q(z)=a_2z^2+a_1z+a_0$, and $b=a_2c^2$, where a_2 is a non-zero constant, a_1, a_0 are constants.*

Proof of Theorem 4.7. Suppose on the contrary to the assertion that there exists a transcendental entire solution f of (40) with finite order. We aim for a contradiction. By using the similar reason as in the proof of Theorem 2.3, we get

$$f^{(k)}(z)=\frac{Q_1(z)e^{h(z)}+Q_2(z)e^{-h(z)}}{2} \tag{55}$$

and

$$f(z+c)=\frac{Q_1(z)e^{h(z)}-Q_2(z)e^{-h(z)}}{2iP(z)}, \tag{56}$$

where $h(z)$ is a non-constant polynomial, $Q_1(z)$ and $Q_2(z)$ are non-zero polynomials with $Q_1(z)Q_2(z)=Q(z)$. Combining (55) and (56), we get

$$\begin{aligned} f^{(k)}(z+c)&=\frac{Q_1(z+c)e^{h(z+c)}+Q_2(z+c)e^{-h(z+c)}}{2} \\ &=\frac{h_1(z)e^{h(z)}-h_2(z)e^{-h(z)}}{2iP(z)^{k+1}}, \end{aligned} \tag{57}$$

where

$$\begin{aligned} h_1(z)=&\sum_{h=0}^{k-1}C_{k-1}^{h}\sum_{t=0}^{k-h}C_{k-h}^{t}Q_1^{(k-t-h)}[(h')^t+M_t(h^{(t)},\cdots,h')]P^{(h)}P^{k-1} \\ &-\sum_{h=0}^{k-1}C_{k-1}^{h}\sum_{t=0}^{h}C_{h}^{t}Q_1^{(h-t)}[(h')^t+M_t(h^{(t)},\cdots,h')]P^{(k-h)}P^{k-1}+o(h_1(z)), \\ h_2(z)=&\sum_{h=0}^{k-1}C_{k-1}^{h}\sum_{t=0}^{k-h}(-1)^tC_{k-h}^{t}Q_2^{(k-t-h)}[(h')^t+N_t(h^{(t)},\cdots,h')]P^{(h)}P^{k-1} \\ &-\sum_{h=0}^{k-1}C_{k-1}^{h}\sum_{t=0}^{h}C_{h}^{t}Q_2^{(h-t)}[(h')^t+N_t(h^{(t)},\cdots,h')]P^{(k-h)}P^{k-1}+o(h_2(z)), \end{aligned}$$

M_t and N_t are differential polynomials of $(h^{(t)}, \cdots, h', h)$ respectively. Thus from (57), we get

$$\frac{h_1(z)e^{h(z)+h(z+c)}}{iP(z)^{k+1}Q_2(z+c)} - \frac{h_2(z)e^{h(z+c)-h(z)}}{iP(z)^{k+1}Q_2(z+c)} - \frac{Q_1(z+c)}{Q_2(z+c)}e^{2h(z+c)} = 1. \tag{58}$$

It is easy to see that both $\frac{h_1(z)e^{h(z)+h(z+c)}}{iP(z)^{k+1}Q_2(z+c)}$ and $\frac{Q_1(z+c)}{Q_2(z+c)}e^{2h(z+c)}$ are not constants. From Lemma 3.1, we get $-h_2(z)e^{h(z+c)-h(z)} = iP(z)^{k+1}Q_2(z+c)$, which implies that $h(z) = Az + B$, where A is a non-zero constant, B is a constant. Thus we get

$$-h_2(z)e^{Ac} = iP(z)^{k+1}Q_2(z+c). \tag{59}$$

Set $deg(P(z)) = p$, $deg(Q(z)) = q$, $deg(Q_1(z)) = q_1$, $deg(h(z)) = h$ and $deg(Q_2(z)) = q_2$. By comparing the both side degree of (59), it is not difficult to find the degree of left hand side is $kp + q_2 - 1$, and the degree of right hand side is $(k+1)p + q_2$, which is a contradiction. The proof is completed. □

Proof the Theorem 4.9. Assume that f is a finite order transcendental entire solution of (42). Then by using the similar reason as in the proof of Theorem 2.3, we get

$$f'(z) = \frac{Q_1(z)e^{\alpha_1(z)} + Q_2(z)e^{\alpha_2(z)}}{2} \tag{60}$$

and

$$f(z+c) = \frac{Q_1(z)e^{\alpha_1(z)} - Q_2(z)e^{\alpha_2(z)}}{2ie^{P(z)}}. \tag{61}$$

where $Q_1(z)$ and $Q_2(z)$ are two non-zero polynomials, and cannot be constants simultaneously, $Q(z) = Q_1(z)Q_2(z)$, $\alpha_1(z)$ and $\alpha_2(z)$ are two polynomials and cannot be constants simultaneously. In fact if both $\alpha_1(z)$ and $\alpha_2(z)$ are constants at the same time, then $f(z)$ is a polynomial. Shifting (60) and differentiating (61), we get

$$\frac{Q_1' + Q_1\alpha_1' - P'Q_1}{ie^{P(z)}Q_1(z+c)}e^{\alpha_1(z)-\alpha_1(z+c)} - \frac{Q_2' + Q_2\alpha_2' - P'Q_2}{ie^{P(z)}Q_1(z+c)}e^{\alpha_2(z)-\alpha_1(z+c)} - \frac{Q_2(z+c)}{Q_1(z+c)}e^{\alpha_2(z+c)-\alpha_1(z+c)} = 1. \tag{62}$$

Next we claim that $Q_1' + Q_1\alpha_1' - P'Q_1 \neq 0$, $Q_2' + Q_2\alpha_2' - P'Q_2 \neq 0$. If $Q_1' + Q_1\alpha_1' - P'Q_1 = 0$, then $Q_1(z) = k_1 e^{P(z)-\alpha_1(z)}$ and $P(z) = \ln Q_1(z) + \alpha_1(z) + k_2$, which means that $P(z) = \alpha_1(z) + c_1$ and $Q_1(z) = q_1(constant)$.

If $\alpha_1(z)$ is a constant denoting by s, then $P(z)$ must reduce to be a constant p and $\alpha_2(z)$ cannot be a constant, thus $Q_2' + Q_2\alpha_2' - P'Q_2 \neq 0$. Then (62) can be rewritten as

$$(Q_2' + Q_2\alpha_2')e^{\alpha_2(z)} + iQ_2(z+c)e^{\alpha_2(z+c)+p} + iq_1e^{p+s} = 0. \tag{63}$$

If $deg(\alpha_2(z)) \geq 2$, then $deg(\alpha_2(z+c) - \alpha_2(z)) \geq 1$. By Lemma 3.2, we have

$$Q_2' + Q_2\alpha_2' = iQ_2(z+c) = iq_1e^{p+s} = 0.$$

This is a contradiction. Thus $deg(\alpha_2(z)) \leq 1$. Noting that $\alpha_2(z)$ cannot be a constant, then $\alpha_2(z) = A_2z + B_2$, where A_2 is a non-zero constant. Rewriting (63) as

$$H(z)e^{A_2z} = -iq_1e^{p+s}, \tag{64}$$

where $H(z) = (Q_2' + Q_2\alpha_2')e^{B_2} + ie^{B_2+A_2c+p}Q_2(z+c)$. If $H(z) = 0$, then we get a contradiction by (64) and $iq_1e^{p+s} \neq 0$. If $H(z) \neq 0$, we can see that the left side of (64) is a transcendental entire function, and the right side of (64) is a non-zero constant, which is a contradiction, and then $Q_1' + Q_1\alpha_1' - P'Q_1 \neq 0$.

If $\alpha_1(z)$ is not a constant, then $Q_2(z)$ cannot be constant by $Q_1(z)$ and $Q_2(z)$ cannot be constants simultaneously. Thus $Q_2' + Q_2\alpha_2' - P'Q_2 \neq 0$. Then (62) can be rewritten as

$$\begin{aligned}&(Q_2'(z) + Q_2(z)\alpha_2'(z) - P'(z)Q_2(z))e^{\alpha_2(z)-\alpha_1(z+c)} \\ &+ iQ_2(z+c)e^{\alpha_2(z+c)-\alpha_1(z+c)+P(z)} + iq_1e^{P(z)} = 0\end{aligned} \tag{65}$$

and

$$\begin{aligned}&\frac{Q_2'(z)e^{\alpha_2(z)-\alpha_1(z+c)-P(z)}}{-iq_1} + \frac{Q_2(z)(\alpha_2'(z) - P'(z))e^{\alpha_2(z)-\alpha_1(z+c)-P(z)}}{-iq_1} \\ &- \frac{Q_2(z+c)}{q_1}e^{\alpha_2(z+c)-\alpha_1(z+c)} = 1.\end{aligned} \tag{66}$$

Next we claim that $\alpha_2(z) - \alpha_1(z)$ is a constant. In fact, if $\alpha_2(z) - \alpha_1(z)$ is not a constant, then $\alpha_2(z) - \alpha_1(z+c) - P(z)$ is not constant. Otherwise, from (62) we have

$$-\frac{Q_2' + Q_2\alpha_2' - P'Q_2}{iq_1}e^{\alpha_2(z)-\alpha_1(z+c)-P(z)} = 1 + \frac{Q_2(z+c)}{q_1}e^{\alpha_2(z+c)-\alpha_1(z+c)}. \tag{67}$$

It is not difficult to see that the order on both sides of the equation (67) are not equal. Furthermore, we get $\alpha_2(z+c) - \alpha_1(z+c) - (\alpha_2(z) - \alpha_1(z+c) - P(z))$ is not constant by (67). Thus by (67) and Lemma 3.2, we have

$$-\frac{Q_2' + Q_2\alpha_2' - P'Q_2}{iq_1} = -\frac{Q_2(z+c)}{q_1} = 0,$$

which is a contradiction. Thus $\alpha_2(z) - \alpha_1(z)$ is a constant, noting that $P(z) = \alpha_1(z) + c_1$, then $e^{\alpha_2(z)-\alpha_1(z+c)-P(z)}$ is not a constant. From (66) and Lemma 3.1, we have

$$-Q_2(z+c)e^{\alpha_2(z+c)-\alpha_1(z+c)} = q_1,$$

which means that Q_2 is also a constant. This is a contradiction, and then $Q_1' + Q_1\alpha_1' - P'Q_1 \neq 0$.

By using the similar way above we get $Q_2' + Q_2\alpha_2' - P'Q_2 \neq 0$.

By $Q_1' + Q_1\alpha_1' - P'Q_1 \neq 0$, $Q_2' + Q_2\alpha_2' - P'Q_2 \neq 0$, (62) and Lemma 3.1, we see that if any two of $e^{\alpha_1(z)-\alpha_1(z+c)}$, $e^{\alpha_2(z)-\alpha_1(z+c)}$, and $e^{\alpha_2(z+c)-\alpha_1(z+c)}$ are not constants, then the third term must be constant. If any two of them are constants, then the third term also must be constant. In what follows, we discuss four cases:

Case 1, $e^{\alpha_1(z)-\alpha_1(z+c)}$ and $e^{\alpha_2(z)-\alpha_1(z+c)}$ are not constants;

Case 2, $e^{\alpha_1(z)-\alpha_1(z+c)}$ and $e^{\alpha_2(z+c)-\alpha_1(z+c)}$ are not constants;

Case 3, $e^{\alpha_2(z)-\alpha_1(z+c)}$ and $e^{\alpha_2(z+c)-\alpha_1(z+c)}$ are not constants;

Case 4, $e^{\alpha_1(z)-\alpha_1(z+c)}$, $e^{\alpha_2(z)-\alpha_1(z+c)}$, and $e^{\alpha_2(z+c)-\alpha_1(z+c)}$ are all constants.

Case 1. If $e^{\alpha_1(z)-\alpha_1(z+c)}$ and $e^{\alpha_2(z)-\alpha_1(z+c)}$ are not constants, then by (62) and Lemma 3.1, we have

$$-\frac{Q_2(z+c)}{Q_1(z+c)}e^{\alpha_2(z+c)-\alpha_1(z+c)} = 1. \tag{68}$$

Combining and (62) and (68), we get

$$\frac{Q_1' + Q_1\alpha_1' - P'Q_1}{Q_2' + Q_2\alpha_2' - P'Q_2}e^{\alpha_1(z)-\alpha_2(z)} = 1, \tag{69}$$

which implies that $\alpha_2(z+c)-\alpha_1(z+c)$ and $\alpha_1(z)-\alpha_2(z)$ are constants. Denote $e^{\alpha_2(z+c)-\alpha_1(z+c)} =$

$e^{\alpha_2(z)-\alpha_1(z)} = k_3(\neq 0)$, by (68), we get $Q_1(z) = -k_3Q_2(z)$, substituting it into (69) yields $2(P'Q_2 - Q_2') = Q_2(\alpha_1' + \alpha_2')$. Then we get

$$Q_2(z) = e^{P(z)-\frac{1}{2}\alpha(z)},$$

which shows that $P(z) - \frac{1}{2}\alpha(z)$ is a constant and $Q_2(z) = q_2(constant)$. Thus by (68) we can get that $Q_1(z)$ is also a constant, which is a contradiction with $Q_1(z)$ and $Q_2(z)$ cannot be constants at the same time.

Case 2. If $e^{\alpha_1(z)-\alpha_1(z+c)}$ and $e^{\alpha_2(z+c)-\alpha_1(z+c)}$ are not constants, then by (62) and Lemma 3.1, we have

$$-\frac{Q_2' + Q_2\alpha_2' - P'Q_2}{ie^{P(z)}Q_1(z+c)}e^{\alpha_2(z)-\alpha_1(z+c)} = 1. \tag{70}$$

Combining (62) and (70), we get

$$\frac{Q_1' + Q_1\alpha_1' - P'Q_1}{ie^{P(z)}Q_2(z+c)}e^{\alpha_1(z)-\alpha_2(z+c)} = 1,$$

which means that $\alpha_2(z)-\alpha_1(z+c)$, $\alpha_2(z+c)-\alpha_1(z+2c)$, and $\alpha_1(z)-\alpha_2(z+c)$ are constants. By $\alpha_1(z) - \alpha_1(z+2c) = [\alpha_1(z) - \alpha_2(z+c)] + [\alpha_2(z+c) - \alpha_1(z+2c)]$, we see that $\alpha_1(z) - \alpha_1(z+2c)$ is a constant, then $\alpha_1(z)$ is a constant or a polynomial with degree 1, which means that $\alpha_1(z) - \alpha_1(z+c)$ is also a constant, and this is a contradiction.

Case 3. If $e^{\alpha_2(z)-\alpha_1(z+c)}$ and $e^{\alpha_2(z+c)-\alpha_1(z+c)}$ are not constants, then $e^{\alpha_1(z)-\alpha_1(z+c)}$ is a constant, which means that $\alpha_1(z)-\alpha_1(z+c)$ is a constant. From (62) and Lemma 3.1, we have

$$\frac{Q_1' + Q_1\alpha_1' - P'Q_1}{iQ_1(z+c)}e^{\alpha_1(z)-\alpha_1(z+c)-P(z)} = 1, \tag{71}$$

which means that $P(z) = p$ is a non-zero constant. Therefore $\alpha_1(z)$ cannot be a constant, otherwise we get $Q_1' = ie^pQ_1(z+c)$, which is impossible. Therefore, $\alpha_1(z)$ can only be a polynomial with degree 1. Denote $\alpha_1(z) = A_1z + B_1$, where A_1 is a non-zero constant, and B_1 is a constant. Rewriting (71) as

$$Q_1(z+c) - Q_1(z) = \frac{1}{A_1}Q_1'(z) \quad and \quad A_1 = ie^{A_1c+p}. \tag{72}$$

By Lemma 5.1, we have

(1) if $\frac{1}{A_1} \neq c$, then $Q_1(z) = q_1(constant)$;

(2) if $\frac{1}{A_1} = c$, then $Q_1(z) = q_1(constant)$ or $Q_1(z) = a_1z + a_0$,

where a_1 is a non-zero constant and a_0 is a constant.

By (62) and (71), we have

$$-\frac{Q_2' + Q_2\alpha_2'}{iQ_2(z+c)}e^{\alpha_2(z)-\alpha_2(z+c)-p} = 1, \tag{73}$$

which means that $\alpha_2(z) - \alpha_2(z+c)$ is a constant, thus $\alpha_2(z)$ cannot be a constant, otherwise we get $-Q_2' = ie^pQ_2(z+c)$, which is impossible. Thus $\alpha_2(z)$ can only be a polynomial with degree 1. Denote $\alpha_2(z) = A_2z + B_2$, where A_2 is a non-zero constant, and B_2 is a constant. Rewriting (73) as

$$Q_2(z+c) - Q_2(z) = \frac{1}{A_2}Q_2'(z), \quad and \quad A_2 = -ie^{A_2c+p}.$$

By Lemma 5.1, we have

(1) if $\frac{1}{A_2} \neq c$, then $Q_2(z) = q_2(constant)$;

(2) if $\frac{1}{A_2} = c$, then $Q_2(z) = q_2(constant)$ or $Q_2(z) = b_1 z + b_0$,
where b_1 is a non-zero constant and b_0 is a constant.

Noting that $A_1 = ie^{A_1 c+p}$ and $A_2 = -ie^{A_2 c+p}$, we see that $A_1 \neq A_2$, that is $\frac{1}{A_1} \neq \frac{1}{A_2}$. In what follows, we discuss three subcases:

Subcase 3.1, $\frac{1}{A_1} \neq c$ and $\frac{1}{A_2} \neq c$;

Subcase 3.2, $\frac{1}{A_1} = c$ and $\frac{1}{A_2} \neq c$;

Subcase 3.3, $\frac{1}{A_1} \neq c$ and $\frac{1}{A_2} = c$.

Subcase 3.1. If $\frac{1}{A_1} \neq c$ and $\frac{1}{A_2} \neq c$, then $Q_1(z) = q_1$ and $Q_2(z) = q_2$. By (42), (60), and (61), we get

$$f'(z) = \frac{q_1 e^{A_1 z+B_1} + q_2 e^{A_2 z+B_2}}{2} \tag{74}$$

and

$$f(z) = \frac{q_1 e^{A_1 z+B_1}}{2A_1} + \frac{q_2 e^{A_2 z+B_2}}{2A_2}. \tag{75}$$

Subcase 3.2. If $\frac{1}{A_1} = c$ and $\frac{1}{A_2} \neq c$, then $Q_1(z) = q_1$ and $Q_2(z) = q_2$, or $Q_1(z) = a_1 z + a_0$ and $Q_2(z) = q_2$. If $Q_1(z) = q_1$ and $Q_2(z) = q_2$, then we get (74) and (75). If $Q_1(z) = a_1 z + a_0$ and $Q_2(z) = q_2$, then by (42), (60) and (61), we get

$$f'(z) = \frac{(a_1 z + a_0)e^{A_1 z+B_1} + q_2 e^{A_2 z+B_2}}{2}$$

and

$$f(z) = \frac{(a_1 z + a_0 - \frac{a_1}{A_1})e^{A_1 z+B_1}}{2A_1} + \frac{q_2 e^{A_2 z+B_2}}{2A_2}.$$

Subcase 3.3. If $\frac{1}{A_1} \neq c$ and $\frac{1}{A_2} = c$, then $Q_1(z) = q_1$ and $Q_2(z) = q_2$, or $Q_1(z) = q_1$ and $Q_2(z) = b_1 z + b_0$. If $Q_1(z) = q_1$ and $Q_2(z) = q_2$, then we get (74) and (75). If $Q_1(z) = q_1$ and $Q_2(z) = b_1 z + b_0$, then by (42), (60) and (61), we get

$$f'(z) = \frac{q_1 e^{A_1 z+B_1} + (b_1 z + b_0)e^{A_2 z+B_2}}{2}$$

and

$$f(z) = \frac{q_1 e^{A_1 z+B_1}}{2A_1} + \frac{(b_1 z + b_0 - \frac{b_1}{A_2})e^{A_2 z+B_2}}{2A_2}.$$

Case 4. If $e^{\alpha_1(z)-\alpha_1(z+c)}$, $e^{\alpha_2(z)-\alpha_1(z+c)}$ and $e^{\alpha_2(z+c)-\alpha_1(z+c)}$ are all constants, then $\alpha_1(z) - \alpha_1(z+c)$, $\alpha_2(z) - \alpha_1(z+c)$ and $\alpha_2(z+c) - \alpha_1(z+c)$ are all constants. Note that $\alpha_1(z)$ and $\alpha_2(z)$ are not constants simultaneously, then $\alpha_1(z) = Az + B_1$, $\alpha_2(z) = Az + B_2$ and $\alpha(z) = 2Az + D$, where A is non-zero constant and B_1, B_2, $D(= B_1 + B_2)$ are constants. Therefore, we get that $f(z) = B(z)e^{Az}$ by (42), (60) and (61), where $B(z)$ satisfies $[B'(z) + AB(z)]^2 + (e^{P(z)})^2 B^2(z + c)e^{2Ac} = Q(z)e^D$. This proof is completed. □

§6 Proofs of Theorems 4.11-4.14

In order to prove Theorems 4.11-4.13, we need Lemma 6.1. The definition of $\varepsilon - set\ E$ can be found in [8, p. 75-76].

Lemma 6.1. *[1, Lemma 3.5] Let f be a transcendental meromorphic function with $\rho(f) < 1$.*

Let $h > 0$. Then there exists an $\varepsilon - set$ E such that

$$f(z+c) - f(z) = cf'(z)(1+o(1))$$

as $z \to \infty$ in $\mathbb{C} \setminus E$, uniformly in c for $|c| \leq h$.

Lemma 6.2. *[2, Corollary 2.5] Let f be a transcendental meromorphic function of finite order $\rho(f)$. For each $\varepsilon > 0$.*

$$m(r, \frac{f(z+c)}{f(z)}) = O(r^{\rho(f)-1+\varepsilon}) = S(r,f),$$

where $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure.

Lemma 6.3. *[2, Theorem 2.1] Let f be a transcendental meromorphic function of finite order $\rho(f)$. For each $\varepsilon > 0$.*

$$T(r, f(z+c)) = T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + O(\log r).$$

Furthermore, if $\rho(f) < 1$, then

$$T(r, f(z+c)) = T(r,f) + S(r,f),$$

where $S(r,f)$ is defined as in the Lemma 6.2.

Proof of Theorem 4.11. Suppose on the contrary to the assertion that there exists a transcendental entire solution f of (45) with $\rho(f) < 1$. We aim for a contradiction. From Lemma 6.1, there exist an $\varepsilon - set$ E_1, such that for $z \to \infty$ and $z \bar{\in} E_1$, we get

$$f(z)^2 + f^{(k)}(z)f(z) + cf^{(k)}(z)f'(z)(1+o(1)) = 1.$$

Then

$$1 + \frac{f^{(k)}(z)}{f(z)} + \frac{f^{(k)}(z)f'(z)}{f(z)^2} = \frac{1}{f(z)^2}. \tag{76}$$

From the Wiman-Valiron theory which can be found in [11, p. 51], we see that there exists a subset $E_2 \subset (1,\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \bar{\in} E_2$ and $|f(z)| = M(r,f)$, we have

$$\frac{f^{(k)}(z)}{f(z)} = (\frac{v(r)}{z})^k(1+o(1)), \tag{77}$$

where $v(r)$ is the central index of $f(z)$. Set $E_3 = \{|z| : z \in E_1\}$, then E_3 is of finite logarithmic measure. By (76) and (77), for all z satisfying $|z| = r \bar{\in} [0,1] \bigcup E_2 \bigcup E_3$ and $|f(z)| = M(r,f)$, we have

$$(\frac{v(r)}{z})^k(1+o(1)) + c(\frac{v(r)}{z})^{k+1} - \frac{1}{f(z)^2} = -1. \tag{78}$$

Since $f(z)$ is a transcendental entire function, and

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ v(r)}{\log r} < 1,$$

then we have

$$|\frac{v(r)}{z}| \to 0, |\frac{1}{f(z)^2}| = |\frac{1}{M(r,f)^2}| \to 0, \quad as \quad z \to \infty. \tag{79}$$

Combining (78) and (79), we get a contradiction. Hence, the order of transcendental solution of (45) must be at least one.

By using similar method as in the equation (45), we can also get the same conclusion for equation (46). □

Proof of the Theorem 4.12. Suppose on the contrary to the assertion that there exists a transcendental entire solution f of (47) with $\rho(f) < 1$. We aim for a contradiction. From Lemma 6.1, there exist an $\varepsilon - set\ E_4$, for $z \to \infty$ and $z \overline{\in} E_4$, we get

$$f(z)^2 + cf(z)f'(z)(1 + o(1)) = 1.$$

Then

$$1 + \frac{f'(z)}{f(z)}c(1 + o(1)) = \frac{1}{f(z)^2}. \tag{80}$$

From the Wiman-Valiron theory, there exists a subset $E_5 \subset (1, \infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \overline{\in} E_5$ and $|f(z)| = M(r, f)$, we have

$$\frac{f'(z)}{f(z)} = \frac{v(r)}{z}(1 + o(1)). \tag{81}$$

Set $E_6 = \{|z| : z \in E_4\}$, then E_6 is of finite logarithmic measure. Thus, by (80) and (81), for all z satisfying $|z| = r\ \overline{\in}[0, 1] \bigcup E_5 \bigcup E_6$ and $|f(z)| = M(r, f)$, we have

$$c\frac{v(r)}{z}(1 + o(1)) - \frac{1}{f(z)^2} = -1. \tag{82}$$

Since $f(z)$ is a transcendental entire function with $\rho(f) < 1$, then we get (79) holds. Combining (79) and (82), we get a condition. Hence, (47) has no transcendental entire solution of order less than one.

Similarly, we get (48) and (49) also has no transcendental entire solution of order less than one. □

Proof of the Theorem 4.13. Suppose on the contrary to the assertion that there exists a transcendental entire solution f of (50) with $\rho(f) < 1$. We aim for a contradiction. From Lemma 6.1, there exist an $\varepsilon - set\ E_7$, for $z \to \infty$ and $z \overline{\in} E_7$, we get

$$f(z)^2 + f(z)f'(z)(2c(1 + o(1)) + 1) + c^2 f'(z)^2(1 + o(1))^2 = 1.$$

Then

$$1 + c^2(\frac{f'(z)}{f(z)})^2(1 + o(1))^2 + \frac{f'(z)}{f(z)}(2c(1 + o(1)) + 1) = \frac{1}{f(z)^2}. \tag{83}$$

From the Wiman-Valiron theory, there exists a subset $E_8 \subset (1, \infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \overline{\in} E_8$ and $|f(z)| = M(r, f)$, we have (81) holds. Set $E_9 = \{|z| : z \in E_7\}$, then E_9 has finite logarithmic measure. Thus, by (81) and (83), for all z satisfying $|z| = r\ \overline{\in}[0, 1] \bigcup E_8 \bigcup E_9$ and $|f(z)| = M(r, f)$, we have

$$(\frac{v(r)}{z})^2 c^2(1 + o(1))^2 + \frac{v(r)}{z}(2c(1 + o(1)) + 1) - \frac{1}{f(z)^2} = -1. \tag{84}$$

Since f is a transcendental entire function with $\rho(f) < 1$ then we get (79) holds. Combining (79) and (84), we get a contradiction. Hence, (41) has no transcendental entire solution of order less than one.

Similarly, we get (51) and (52) also has no transcendental entire solution of order less than one. □

Proof of the Theorem 4.14. Suppose that f is a transcendental entire solution of (54). By (54) we get

$$f(z)f^{(k)}(z)\frac{f(z+c)}{f(z)} = 1 - f(z).$$

By the Clunies lemma [11, Lemma 2.4.2], we have

$$m(r, f^{(k)}(z)\frac{f(z+c)}{f(z)}) = S(r, f),$$

where $S(r, f)$ is defined as in the Lemma 6.2. It follows from this and Lemma 6.2 that

$$m(r, f^{(k)}(z)) \leq m(r, f^{(k)}\frac{f(z+c)}{f(z)}) + m(r, \frac{f(z)}{f(z+c)}) = S(r, f),$$

which is a contradiction. This proof is completed. □

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Conflict of interest The authors declare no conflict of interest.

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