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Triple reverse order law for the Drazin inverse*

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Abstract. In this paper, we investigate the reverse order law for Drazin inverse of three bounded linear operators under some commutation relations. Moreover, the Drazin invertibility of sum is also obtained for two bounded linear operators and its expression is presented.

§1 Introduction

Generalized inverses such as Moore-Penrose inverse, Drazin inverse and group inverse have attracted many researchers because of its application in singular differential equations, Markov chains, statistics, numerical analysis and so on.

It is well known that the reverse order law holds for the ordinary inverse, but is not necessarily true for generalized inverses. So, many authors had studied the reverse order law of generalized inverses. In [1,7,8,10,11], the authors obtained the conditions for the reverse order law to hold concerning Moore-Penrose inverse in the finite dimensional space, Hilbert space and ring. In [5,16,18], the reverse order law of group inverse in the Hilbert space, semigroup and ring was investigated. The mixed-type reverse order law in the Hilbert space and ring was studied in [6,14,20,26]. The reverse order law of Drazin inverse was considered in [19,24,25] in the finite dimensional space, Hilbert space, Banach algebra and ring. In addition, [3,15,21,22,27] discussed the reverse order law of {1}-, {1,2}-, {1,3}-, {1,4}-, {1,2,3}- and {1,2,4}-inverse.

In [7,11,15,22,24,25], the authors considered the reverse order law for different generalized inverses of multiple operator and matrix products. To our knowledge, the reverse order law for Drazin inverses of three operator products has not been studied yet in literature.

This paper contains three parts. The first part is the Drazin invertibility of three bounded linear operator products, the second is devoted to the reverse order law for the Drazin inverse of three bounded linear operator products, and the last part deals with the Drazin invertibility of

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sum for two bounded linear operators. We derive the Drazin invertibility of product PQR and some equivalent conditions for the reverse order law $(PQR)^D = R^D Q^D P^D$ to hold under some commutation relations (1) $[P, P^2Q] = 0$, $[P, R] = 0$, $[R, RQ] = 0$, $[P, QPQ] = 0$, (2) $[P, PQ] = 0$, $[P, R] = 0$, $[R, RQ] = 0$, respectively. In particular, we obtain that $(PQR)^D = R^D Q^D P^D$ and P^D, Q^D, R^D all commute if P, Q, R all commute. As special cases, some equivalent conditions for the reverse order law $(PQ)^D = Q^D P^D$ to hold are also presented. Finally, we consider the equivalent condition that $P + Q$ is Drazin invertible when $[P, P^2Q] = 0$, $[P, QPQ] = 0$, $[Q, PQ]P^\pi = 0$, and the expression of $(P + Q)^D$ is also given.

From now on in this paper we will let $\mathcal{X}$ and $\mathcal{Y}$ denote Banach spaces, and let the set $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denote the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ and $\mathcal{B}(\mathcal{X}, \mathcal{X})$ be written as $\mathcal{B}(\mathcal{X})$. Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is Drazin invertible, if there exists an operator $T^D \in \mathcal{B}(\mathcal{X})$ such that

$$TT^D = T^D T, \quad T^D = T(T^D)^2 \quad \text{and} \quad T^{k+1}T^D = T^k \text{ for some integer } k \geq 0.$$

Here, T^D is the unique Drazin inverse of T, and the smallest integer k, denoted by $\operatorname{ind}(T)$, is called the index of T. If T is Drazin invertible, then T has the operator matrix form $T = \begin{bmatrix} T_1 & 0 \\ 0 & N \end{bmatrix}$ with respect to the invariant space decomposition $\mathcal{X} = \mathcal{N}(T^\pi) \oplus \mathcal{R}(T^\pi)$, where T_1 is invertible, N is nilpotent and $T^\pi = I - TT^D$.

We need the following results about the Drazin inverse, which will be useful tools for proving the reverse order law. Moreover, write $[P, Q] = PQ - QP$.

Lemma 1.1. [12, Theorem 5.5] *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be Drazin invertible with $[P, Q] = 0$, then the operators P, Q, P^D, Q^D all commute and*

$$(PQ)^D = Q^D P^D = P^D Q^D.$$

Lemma 1.2. *Let $A \in \mathcal{B}(\mathcal{X}), C \in \mathcal{B}(\mathcal{X}, \mathcal{Y}), D \in \mathcal{B}(\mathcal{Y})$, and*

$$M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}.$$

(1) [13, Lemma 2.4] *If two of the operators A, D and M are Drazin invertible, then so is the third.*

(2) [9, Theorem 5.1] *If A, D are Drazin invertible, then*

$$M^D = \begin{bmatrix} A^D & 0 \\ X & D^D \end{bmatrix},$$

where $X = \sum_{i=0}^{\operatorname{ind}(A)-1} (D^D)^{i+2} C A^i A^\pi + D^\pi \sum_{i=0}^{\operatorname{ind}(D)-1} D^i C (A^D)^{i+2} - D^D C A^D.$

§2 Drazin invertibility of product

In this section, we will consider the Drazin invertibility of three linear bounded operator products under some commutation relations, which is necessary to proof the reverse order law.

Theorem 2.1. *Let $P, Q, R \in \mathcal{B}(\mathcal{X})$, and P, Q, R, P^DQ, R^DP^DQ be Drazin invertible. If $[P, P^2Q] = 0$, $[P, R] = 0$, $[R, RQ] = 0$ and $[P, QPQ] = 0$, then the product PQR is Drazin invertible.*

Proof. Since P is Drazin invertible, P can be written as the operator matrix form

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & N_1 \end{bmatrix} \tag{1}$$

with respect to the space decomposition $\mathcal{X} = \mathcal{N}(P^\pi) \oplus \mathcal{R}(P^\pi)$, where P_1 is invertible, $N_1^s = 0$, $s = \mathrm{ind}(P)$ and $P^D = \begin{bmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. Let Q and R be decomposed as $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$ and $R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$ with respect to the above space decomposition. From $[P, P^2Q] = 0$, we get

$$P_1Q_1 = Q_1P_1, \quad P_1Q_2 = Q_2N_1 \tag{2}$$

and

$$N_1^2Q_4N_1 = N_1^3Q_4. \tag{3}$$

The second equality in (2) implies that

$$Q_2 = P_1^{-1}Q_2N_1 = P_1^{-2}Q_2N_1^2 = \cdots = P_1^{-s}Q_2N_1^s = 0,$$

and hence

$$Q = \begin{bmatrix} Q_1 & 0 \\ Q_3 & Q_4 \end{bmatrix}. \tag{4}$$

Substituting (1) and (4) into the assumption $[P, QPQ] = 0$, we have

$$N_1Q_3P_1Q_1 + N_1Q_4N_1Q_3 = Q_3P_1Q_1P_1 + Q_4N_1Q_3P_1 \tag{5}$$

and

$$N_1Q_4N_1Q_4 = Q_4N_1Q_4N_1. \tag{6}$$

From (3) and (6), it follows that $(N_1Q_4)^{2k+1} = N_1^{2k+1}Q_4^{2k+1}$ $(k \in \mathbb{N})$, which means that N_1Q_4 is nilpotent. Since P_1 is invertible, by (5), we derive that

$$Q_3P_1Q_1 + Q_4N_1Q_3 = N_1(Q_3P_1Q_1 + Q_4N_1Q_3)P_1^{-1} = N_1^s(Q_3P_1Q_1 + Q_4N_1Q_3)(P_1^{-1})^s,$$

and hence

$$Q_3P_1Q_1 + Q_4N_1Q_3 = 0.$$

The assumption that P^DQ is Drazin invertible means that Q_1 is Drazin invertible. Then the above equality multiplied by $(Q_1^D)^2$, together with $Q_1^D = Q_1(Q_1^D)^2$, suggests that

$$Q_3P_1Q_1^D = -Q_4N_1Q_3(Q_1^D)^2. \tag{7}$$

Applying (2) and Lemma 1.1, we see that $P_1Q_1^D = Q_1^DP_1$, $P_1^{-1}Q_1^D = Q_1^DP_1^{-1}$. This and (7) yield

$$Q_3Q_1^D = -Q_4N_1Q_3(Q_1^D)^2P_1^{-1} = -Q_4N_1(Q_3Q_1^D)P_1^{-1}Q_1^D = (-Q_4N_1)^kQ_3Q_1^D(P_1^{-1}Q_1^D)^k, \ k \in \mathbb{N}.$$

Note that N_1Q_4 is nilpotent and $(Q_4N_1)^k = Q_4(N_1Q_4)^{k-1}N_1$, then Q_4N_1 is nilpotent. Consequently,

$$Q_3Q_1^D = 0. \tag{8}$$

On the other hand, by $[P, R] = 0$, we can conclude

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & R_4 \end{bmatrix} \tag{9}$$

with

$$P_1R_1 = R_1P_1, \quad N_1R_4 = R_4N_1. \tag{10}$$

Obviously, R_1, R_4 are Drazin invertible, and

$$R^D = \begin{bmatrix} R_1^D & 0 \\ 0 & R_4^D \end{bmatrix}. \tag{11}$$

In order to complete the proof, we further decompose R_1 and R_4, respectively, as

$$R_1 = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{14} \end{bmatrix}, \quad R_4 = \begin{bmatrix} R_{41} & 0 \\ 0 & R_{44} \end{bmatrix} \tag{12}$$

with respect to the space decompositions $\mathcal{N}(P^\pi) = \mathcal{N}(R_1^\pi) \oplus \mathcal{R}(R_1^\pi)$ and $\mathcal{R}(P^\pi) = \mathcal{N}(R_4^\pi) \oplus \mathcal{R}(R_4^\pi)$, where R_{11}, R_{41} are invertible, and R_{14}, R_{44} are nilpotent. Then

$$R = \begin{bmatrix} R_{11} & 0 & 0 & 0 \\ 0 & R_{14} & 0 & 0 \\ 0 & 0 & R_{41} & 0 \\ 0 & 0 & 0 & R_{44} \end{bmatrix}. \tag{13}$$

From (10), we deduce that P_1 and N_1 in (1) have the following operator matrix form

$$P_1 = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{14} \end{bmatrix}, \quad N_1 = \begin{bmatrix} N_{11} & 0 \\ 0 & N_{14} \end{bmatrix} \tag{14}$$

with

$$P_{11}R_{11} = R_{11}P_{11}, \quad P_{14}R_{14} = R_{14}P_{14} \tag{15}$$

and

$$N_{11}R_{41} = R_{41}N_{11}, \quad N_{14}R_{44} = R_{44}N_{14}. \tag{16}$$

Since P_1 is invertible and N_1 is nilpotent, P_{11}, P_{14} are invertible and N_{11}, N_{14} are nilpotent. Thus,

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & N_1 \end{bmatrix} = \begin{bmatrix} P_{11} & 0 & 0 & 0 \\ 0 & P_{14} & 0 & 0 \\ 0 & 0 & N_{11} & 0 \\ 0 & 0 & 0 & N_{14} \end{bmatrix}. \tag{17}$$

Next we let Q_1, Q_4 and Q_3 have the following operator matrix forms

$$Q_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{13} & Q_{14} \end{bmatrix}, \quad Q_4 = \begin{bmatrix} Q_{41} & Q_{42} \\ Q_{43} & Q_{44} \end{bmatrix}, \quad Q_3 = \begin{bmatrix} Q_{31} & Q_{32} \\ Q_{33} & Q_{34} \end{bmatrix}. \tag{18}$$

Substituting (13) and (18) into the assumption $[R, RQ] = 0$ yields

$$\begin{aligned} &R_{11}^2Q_{11} = R_{11}Q_{11}R_{11}, \ R_{41}^2Q_{41} = R_{41}Q_{41}R_{41}, \\ &R_{11}^2Q_{12} = R_{11}Q_{12}R_{14}, \ R_{41}^2Q_{32} = R_{41}Q_{32}R_{14}, \ R_{41}^2Q_{42} = R_{41}Q_{42}R_{44}, \\ &R_{14}^2Q_{13} = R_{14}Q_{13}R_{11}, \ R_{44}^2Q_{33} = R_{44}Q_{33}R_{11}, \ R_{44}^2Q_{43} = R_{44}Q_{43}R_{41}, \\ &R_{14}^2Q_{14} = R_{14}Q_{14}R_{14}, \ R_{44}^2Q_{44} = R_{44}Q_{44}R_{44}. \end{aligned} \tag{19}$$

Because R_{11}, R_{41} are invertible, we obtain

$$R_{11}Q_{11} = Q_{11}R_{11}, \quad R_{41}Q_{41} = Q_{41}R_{41}, \tag{20}$$

and

$$Q_{12}=R_{11}^{-k}Q_{12}R_{14}^{k},\qquad Q_{32}=R_{41}^{-k}Q_{32}R_{14}^{k},\qquad Q_{42}=R_{41}^{-k}Q_{32}R_{44}^{k},$$
$$R_{14}Q_{13}=R_{14}^{k+1}Q_{13}R_{11}^{-k},\ R_{44}Q_{33}=R_{44}^{k+1}Q_{33}R_{11}^{-k},\ R_{44}Q_{43}=R_{44}^{k+1}Q_{43}R_{41}^{-k},$$

$k\in\mathbb{N}$, which, by the nilpotence of R_{14} and R_{44}, imply

$$Q_{12}=0,\ Q_{32}=0,\ Q_{42}=0,\ R_{14}Q_{13}=0,\ R_{44}Q_{33}=0,\ R_{44}Q_{43}=0. \tag{21}$$

Hence,

$$Q=\begin{bmatrix}Q_{11}&0&0&0\\Q_{13}&Q_{14}&0&0\\Q_{31}&0&Q_{41}&0\\Q_{33}&Q_{34}&Q_{43}&Q_{44}\end{bmatrix}, \tag{22}$$

and

$$PQR=\begin{bmatrix}P_1Q_1R_1&0\\N_1Q_3R_1&N_1Q_4R_4\end{bmatrix}=\begin{bmatrix}P_{11}Q_{11}R_{11}&0&0&0\\P_{14}Q_{13}R_{11}&P_{14}Q_{14}R_{14}&0&0\\N_{11}Q_{31}R_{11}&0&N_{11}Q_{41}R_{41}&0\\N_{14}Q_{33}R_{11}&N_{14}Q_{34}R_{14}&N_{14}Q_{43}R_{41}&N_{14}Q_{44}R_{44}\end{bmatrix}. \tag{23}$$

In order to prove that PQR is Drazin invertible, it is sufficient to show that $P_1Q_1R_1$ and $N_1Q_4R_4$ are both Drazin invertible. Note that Q_{11} is Drazin invertible, since R^DP^DQ is Drazin invertible. Then $Q_{11}R_{11}$ is Drazin invertible by (20) and Lemma 1.1. Moreover, $Q_{14}R_{14}$ is nilpotent by the first equality in (19). Hence, $Q_1R_1=\begin{bmatrix}Q_{11}R_{11}&0\\Q_{13}R_{11}&Q_{14}R_{14}\end{bmatrix}$ is Drazin invertible and

$$(Q_1R_1)^D=\begin{bmatrix}R_{11}^{-1}Q_{11}^D&0\\X_{Q_1R_1}&0\end{bmatrix}, \tag{24}$$

where

$$X_{Q_1R_1}=(Q_{14}R_{14})^{\pi}\sum_{i=0}^{ind(Q_{14}R_{14})-1}(Q_{14}R_{14})^iQ_{13}R_{11}((Q_{11}R_{11})^D)^{i+2}=Q_{13}(Q_{11}^D)^2R_{11}^{-1} \tag{25}$$

by $R_{14}Q_{13}=0$ in (21). In addition, $[P_1,Q_1R_1]=0$ follows from the first equalities in (2) and (10), then $P_1Q_1R_1$ is Drazin invertible with

$$(P_1Q_1R_1)^D=(Q_1R_1)^DP_1^{-1}=P_1^{-1}(Q_1R_1)^D. \tag{26}$$

In the following, we prove that $N_1Q_4R_4=\begin{bmatrix}N_{11}Q_{41}R_{41}&0\\N_{14}Q_{43}R_{41}&N_{14}Q_{44}R_{44}\end{bmatrix}$ is Drazin invertible. In fact, by (16) and the second equality in (20), we have

$$(N_{11}Q_{41}R_{41})^k=(N_{11}Q_{41})^kR_{41}^k\ (k\in\mathbb{N}).$$

By (19) and (16), we have

$$(N_{14}Q_{44}R_{44})^k=(N_{14}Q_{44})R_{44}^k(N_{14}Q_{44})^{k-1}\ (k\in\mathbb{N}).$$

Since N_1Q_4 is nilpotent, we obtain that $N_{11}Q_{41}$ is nilpotent, which together with the nilpontence of R_{44} indicates that $N_{11}Q_{41}R_{41}$ and $N_{14}Q_{44}R_{44}$ are nilpotent. Thus, $N_1Q_4R_4$ is Drazin invertible with $(N_1Q_4R_4)^D=0$. Therefore, by Lemma 1.2, PQR is Drazin invertible and

$$(PQR)^D=\begin{bmatrix}(Q_1R_1)^DP_1^{-1}&0\\X_{PQR}&0\end{bmatrix}, \tag{27}$$

where

$$X_{PQR} = \sum_{i=0}^{ind(N_1Q_4R_4)-1} (N_1Q_4R_4)^i(N_1Q_3R_1)[(P_1Q_1R_1)^D]^{i+2}. \tag{28}$$

□

Replacing the conditions $[P, P^2Q] = 0$, $[P, QPQ] = 0$ by $[P, PQ] = 0$ in Theorem 2.1, we conclude that $N_1Q_3 = 0$ and N_1Q_4 is nilpotent. Similar to the proof of Theorem 2.1, we have the following result.

Theorem 2.2. *Let $P, Q, R \in \mathcal{B}(\mathcal{X})$, and P, Q, R, P^DQ, R^DP^DQ be Drazin invertible. If $[P, PQ] = 0$, $[P, R] = 0$ and $[R, RQ] = 0$, then the product PQR is Drazin invertible.*

The next theorem is a symmetrical formulation of Theorem 2.1.

Theorem 2.3. *Let $P, Q, R \in \mathcal{B}(\mathcal{X})$, and P, Q, R, PQ^D, PQ^DR^D be Drazin invertible. If $[Q, PQ^2] = 0$, $[Q, R] = 0$, $[R, PR] = 0$ and $[P, QPQ] = 0$, then the product PQR is Drazin invertible.*

§3 Reverse order law

In this section, we will investigate the reverse order law for the Drazin inverse of three linear bounded operator products. As a special case, the reverse order law of the Drazin inverse is obtained for two linear bounded operator products.

Theorem 3.1. *Under the conditions of Theorem 2.1, the following reverse order law statements are equivalent:*

(i) $(PQR)^D = R^DQ^DP^D$,

(ii) $(PQR)^DP = R^DQ^DP^DP$,

(ii) $(PP^DQR)^D = R^DQ^DPP^D$.

Proof. From (4) and the Drazin invertibility of Q and Q_1, we know that Q_4 is Drazin invertible, and

$$Q^D = \begin{bmatrix} Q_1^D & 0 \\ X_Q & Q_4^D \end{bmatrix}. \tag{29}$$

Then

$$R^DQ^DP^D = \begin{bmatrix} R_1^DQ_1^DP_1^{-1} & 0 \\ R_4^DX_QP_1^{-1} & 0 \end{bmatrix}, \quad R^DQ^DP^DP = \begin{bmatrix} R_1^DQ_1^D & 0 \\ R_4^DX_Q & 0 \end{bmatrix}$$

and

$$(PQR)^DP = \begin{bmatrix} (P_1Q_1R_1)^DP_1 & 0 \\ X_{PQR}P_1 & 0 \end{bmatrix} = \begin{bmatrix} (Q_1R_1)^D & 0 \\ X_{PQR}P_1 & 0 \end{bmatrix}$$

by (11), (1), (27) and (26), where

$$X_Q = \sum_{i=0}^{ind(Q_1)-1} (Q_4^D)^{i+2}Q_3Q_1^iQ_1^\pi + Q_4^\pi \sum_{i=0}^{ind(Q_4)-1} Q_4^iQ_3(Q_1^D)^{i+2} - Q_4^DQ_3Q_1^D.$$

Obviously, it can be seen that

$$(PQR)^D = R^D Q^D P^D \Longleftrightarrow \begin{cases} (Q_1R_1)^D = R_1^D Q_1^D, \\ R_4^D X_Q = X_{PQR} P_1. \end{cases}$$

In the following, we show that $(Q_1R_1)^D = R_1^D Q_1^D$ implies $X_{PQR} = 0$. If $(Q_1R_1)^D = R_1^D Q_1^D$, then $Q_{13}Q_{11}^D = 0$ by (24), (25) and $R_1^D Q_1^D = \begin{bmatrix} R_{11}^{-1}Q_{11}^D & 0 \\ 0 & 0 \end{bmatrix}$. From (8) and

$$Q_3 Q_1^D = \begin{bmatrix} Q_{31}Q_{11}^D & 0 \\ Q_{33}Q_{11}^D + Q_{34}X_{Q_1} & Q_{34}Q_{14}^D \end{bmatrix}, \tag{30}$$

where

$$\begin{aligned} X_{Q_1} &= \sum_{i=0}^{ind(Q_{11})-1} (Q_{14}^D)^{i+2} Q_{13} Q_{11}^i Q_{11}^\pi + Q_{14}^\pi \sum_{i=0}^{ind(Q_{14})-1} Q_{14}^i Q_{13} (Q_{11}^D)^{i+2} - Q_{14}^D Q_{13} R_{11} Q_{11}^D \\ &= \sum_{i=0}^{ind(Q_{11})-1} (Q_{14}^D)^{i+2} Q_{13} Q_{11}^i Q_{11}^\pi - Q_{14}^D Q_{13} R_{11} Q_{11}^D, \end{aligned}$$

we have $Q_{31}Q_{11}^D = 0, Q_{34}Q_{14}^D = 0, Q_{33}Q_{11}^D + Q_{34}X_{Q_1} = 0$, and we further obtain $Q_{33}Q_{11}^D = 0$ according to $Q_{34}X_{Q_1} = 0$. Thus, by (26),

$$Q_3R_1(P_1Q_1R_1)^D = Q_3R_1R_1^DQ_1^DP_1^{-1} = \begin{bmatrix} Q_{31}Q_{11}^D & 0 \\ Q_{33}Q_{11}^D & 0 \end{bmatrix} P_1^{-1} = 0, \tag{31}$$

which implies $X_{PQR} = 0$ in (28). Therefore,

$$\begin{aligned} (PQR)^D = R^D Q^D P^D \quad &\Longleftrightarrow \quad \begin{cases} (Q_1R_1)^D = R_1^D Q_1^D \\ R_4^D X_Q = 0 \end{cases} \\ &\Longleftrightarrow \quad (PQR)^D P = R^D Q^D P^D P \\ &\Longleftrightarrow \quad (PP^DQR)^D = R^D Q^D PP^D. \end{aligned}$$

□

Similar to the proof of Theorem 3.1, we obtain the reverse order law associated with Theorem 2.2.

Theorem 3.2. *Under the conditions of Theorem 2.2, the following reverse order law statements are equivalent:*
(i) $(PQR)^D = R^D Q^D P^D$,
(ii) $(PQR)^D P = R^D Q^D P^D P$,
(ii) $(PP^DQR)^D = R^D Q^D PP^D$.

If $[P, Q] = 0$, $[P, R] = 0$ and $[Q, R] = 0$, then $Q_3 = 0$ in (4), and so the next result is obtained directly from Theorem 2.1 and Theorem 3.1.

Corollary 3.1. *Let P, Q, $R \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. If $[P, Q] = 0$, $[P, R] = 0$ and $[Q, R] = 0$, then PQR is Drazin invertible, and*
(i) $(PQR)^D = R^D Q^D P^D$,
(ii) P^D, Q^D, R^D *are commutative.*

In Theorems 2.1, 2.2, 3.1 and 3.2, if $R = I$, then we can obtain the results on the Drazin invertibility of PQ and its reverse order law.

Corollary 3.2. *Let $P,\ Q \in \mathcal{B}(\mathcal{H})$, and $P,\ Q,\ P^DQ$ be Drazin invertible. If P, Q satisfy (1) $[P, P^2Q] = 0$, $[P, QPQ] = 0$, or (2) $[P, PQ] = 0$, then PQ is Drazin invertible, and the following reverse order law statements are equivalent:*

(i) $(PQ)^D = Q^DP^D$,

(ii) $(PQ)^DP = Q^DP^DP$,

(iii) $(PP^DQ)^D = Q^DPP^D$.

The next theorem is a symmetrical formulation of Theorem 3.1.

Theorem 3.3. *Under the conditions of Theorem 2.3, the following reverse order law statements are equivalent:*

(i) $(PQR)^D = R^DQ^DP^D$,

(ii) $Q(PQR)^D = QR^DQ^DP^D$,

(ii) $(PQQ^DR)^D = R^DQQ^DP^D$.

§4 Drazin invertibility of $P + Q$

Combing with the proof of Theorem 2.1, we can describe the Drazin invertibility of sum of two bounded linear operators. Related results can be founded in [2,4,23].

Theorem 4.1. *Let $P, Q \in \mathcal{B}(\mathcal{X})$, and P, Q, P^DQ be Drazin invertible. If $[P, P^2Q] = 0$, $[P, QPQ] = 0$ and $[Q, PQ]P^\pi = 0$, then $P + Q$ is Drazin invertible if and only if $I + P^DQ$ is Drazin invertible, and*

$$(P+Q)^D = \alpha + \beta + ((I - (P+Q)P^\pi\beta)\sum_{k=0}^{\infty}(P+Q)^kP^\pi Q\alpha^{k+2} - \beta Q\alpha, \tag{32}$$

where

$$\alpha = P^D(I + P^DQ)QQ^D + \sum_{i=0}^{\infty}(P^D)^{i+1}(-Q)^iQ^\pi,$$

$$\beta = \sum_{i=0}^{\infty}(Q^D)^{i+1}(-P)^iQQ^DP^\pi + \sum_{i=0}^{\infty}(Q^DP^\pi)^{i+2}PQ^\pi(P+Q)^iP^\pi.$$

Proof. From the proof of Theorem 2.1, (1) and (4) are the matrix form of P and Q, respectively. Then, substituting (1) and (4) into the assumption $[Q, PQ]P^\pi = 0$, we get

$$N_1Q_4^2 = Q_4N_1Q_4. \tag{33}$$

In the proof of Theorems 2.1, we have shown Q_1 and Q_4 are Drazin invertible, then Q_1 and Q_4 can be written as

$$Q_1 = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{14} \end{bmatrix}, \quad Q_4 = \begin{bmatrix} Q_{41} & 0 \\ 0 & Q_{44} \end{bmatrix} \tag{34}$$

with respect to the space decompositions $\mathcal{N}(P^\pi) = \mathcal{N}(Q_1^\pi) \oplus \mathcal{R}(Q_1^\pi)$ and $\mathcal{R}(P^\pi) = \mathcal{N}(Q_4^\pi) \oplus \mathcal{R}(Q_4^\pi)$, where Q_{11}, Q_{41} are invertible and Q_{14}, Q_{44} are nilpotent. Let Q_3 have the corresponding matrix form $Q_3 = \begin{bmatrix} Q_{31} & Q_{32} \\ Q_{33} & Q_{34} \end{bmatrix}$. From (8) and $Q_3Q_1^D = \begin{bmatrix} Q_{31}Q_{11}^{-1} & 0 \\ Q_{33}Q_{11}^{-1} & 0 \end{bmatrix}$, it is obvious that $Q_{31} = 0$, $Q_{33} = 0$, and hence

$$Q_3 = \begin{bmatrix} 0 & Q_{32} \\ 0 & Q_{34} \end{bmatrix}. \tag{35}$$

Thus,

$$Q = \begin{bmatrix} Q_{11} & 0 & 0 & 0 \\ 0 & Q_{14} & 0 & 0 \\ 0 & Q_{32} & Q_{41} & 0 \\ 0 & Q_{34} & 0 & Q_{44} \end{bmatrix}. \tag{36}$$

Under the previous space decompositions of $\mathcal{N}(P^\pi)$ and $\mathcal{R}(P^\pi)$, by $P_1Q_1 = Q_1P_1$ and (33), we have

$$P_1 = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{14} \end{bmatrix}, \tag{37}$$

and

$$N_1 = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{14} \end{bmatrix} \tag{38}$$

with

$$P_{11}Q_{11} = Q_{11}P_{11}, \quad P_{14}Q_{14} = Q_{14}P_{14} \tag{39}$$

and

$$N_{14}Q_{44}^2 = Q_{44}N_{14}Q_{44},\ N_{11}Q_{41} = Q_{41}N_{11},\ N_{12}Q_{44} = 0. \tag{40}$$

Since P_1 is invertible and N_1 is nilpotent, $P_{11},\ P_{14}$ are invertible and $N_{11},\ N_{14}$ are nilpotent. Thus,

$$P = \begin{bmatrix} P_{11} & 0 & 0 & 0 \\ 0 & P_{14} & 0 & 0 \\ 0 & 0 & N_{11} & N_{12} \\ 0 & 0 & 0 & N_{14} \end{bmatrix}, \tag{41}$$

and then

$$P+Q=\begin{bmatrix} P_1 + Q_1 & 0 \\ Q_3 & N_1 + Q_4 \end{bmatrix}=\begin{bmatrix} P_{11} + Q_{11} & 0 & 0 & 0 \\ 0 & P_{14} + Q_{14} & 0 & 0 \\ 0 & Q_{32} & N_{11} + Q_{41} & N_{12} \\ 0 & Q_{34} & 0 & N_{14} + Q_{44} \end{bmatrix}. \tag{42}$$

The second equality in (39) means that $P_{14}^{-1}Q_{14}$ is nilpotent, because P_{14} is invertible and Q_{14} is nilpotent. Then $I + P_{14}^{-1}Q_{14}$ is invertible and $(I + P_{14}^{-1}Q_{14})^{-1} = \sum_{i=0}^{\infty} P_{14}^{-i}(-Q_{14})^i$, and we further see that $P_{14} + Q_{14}$ is invertible and

$$\begin{aligned} 0 \oplus (P_{14} + Q_{14})^{-1} \oplus 0 \oplus 0 &= 0 \oplus (I + P_{14}^{-1}Q_{14})^{-1}P_{14}^{-1} \oplus 0 \oplus 0 \\ &= 0 \oplus \sum_{i=0}^{\infty} P_{14}^{-i}(-Q_{14})^i P_{14}^{-1} \oplus 0 \oplus 0 \\ &= \sum_{i=0}^{\infty}(P^D)^{i+1}(-Q)^i Q^\pi. \end{aligned} \tag{43}$$

Similarly, from the second equality in (40), we can conclude that $N_{11}+Q_{41}$ is invertible and

$$\begin{aligned} 0\oplus 0\oplus (N_{11}+Q_{41})^{-1}\oplus 0 &= 0\oplus 0\oplus (I+Q_{41}^{-1}N_{11})^{-1}Q_{41}^{-1}\oplus 0 \\ &= 0\oplus 0\oplus \sum_{i=0}^{\infty} Q_{41}^{-i}(-N_{11})^{i}Q_{41}^{-1}\oplus 0 \\ &= \sum_{i=0}^{\infty}(Q^{D})^{i+1}(-P)^{i}QQ^{D}P^{\pi}. \end{aligned} \tag{44}$$

Substituting (34) and (38) into (3) and (6), we get

$$N_{14}^{3}Q_{44}=N_{14}^{2}Q_{44}N_{14},\quad N_{14}Q_{44}N_{14}Q_{44}=Q_{44}N_{14}Q_{44}N_{14},$$

which, together with (40), imply that $N_{14}+Q_{44}$ is nilpotent. Then, $N_1+Q_4=\begin{bmatrix} N_{11}+Q_{41} & N_{12} \\ 0 & N_{14}+Q_{44}\end{bmatrix}$ is Drazin invertible, and

$$(N_1+Q_4)^{D}=\begin{bmatrix} (N_{11}+Q_{41})^{-1} & \sum_{i=0}^{\infty}((N_{11}+Q_{41})^{-1})^{i+2}N_{12}(N_{14}+Q_{44})^{i} \\ 0 & 0 \end{bmatrix}. \tag{45}$$

Thus, from (42) and (39), it can be seen that

$$\begin{aligned} P+Q \text{ is Drazin invertible} &\iff P_1+Q_1 \text{ is Drazin invertible} \\ &\iff P_{11}+Q_{11} \text{ is Drazin invertible} \\ &\iff I+P_{11}^{-1}Q_{11} \text{ is Drazin invertible} \\ &\iff I+P^{D}Q \text{ is Drazin invertible.} \end{aligned}$$

Next, we are going to give the expression of $(P+Q)^D$. By (42) and Lemma 1.2, we get

$$(P+Q)^{D} = \begin{bmatrix} (P_1+Q_1)^{D} & 0 \\ X_3 & (N_1+Q_4)^{D} \end{bmatrix}, \tag{46}$$

where

$$\begin{aligned} X_3 &= \sum_{i=0}^{\infty}((N_1+Q_4)^{D})^{i+2}Q_3(P_1+Q_1)^{i}(P_1+Q_1)^{\pi} \\ &\quad + (N_1+Q_4)^{\pi}\sum_{i=0}^{\infty}(N_1+Q_4)^{i}Q_3((P_1+Q_1)^{D})^{i+2}-(N_1+Q_4)^{D}Q_3(P_1+Q_1)^{D}. \end{aligned} \tag{47}$$

Notice that

$$(P_1+Q_1)^{D}=\begin{bmatrix} (P_{11}+Q_{11})^{D} & 0 \\ 0 & (P_{14}+Q_{14})^{-1} \end{bmatrix} \tag{48}$$

and

$$(P_{11}+Q_{11})^{D}\oplus 0\oplus 0\oplus 0 = P_{11}^{-1}(I+P_{11}^{-1}Q_{11})^{D}\oplus 0\oplus 0\oplus 0 = P^{D}(I+P^{D}Q)^{D}QQ^{D}, \tag{49}$$

then it follows from (43) and (49) that

$$\begin{bmatrix} (P_1+Q_1)^{D} & 0 \\ 0 & 0 \end{bmatrix} = P^{D}(I+P^{D}Q)^{D}QQ^{D}+\sum_{i=0}^{\infty}(P^{D})^{i+1}(-Q)^{i}Q^{\pi}\triangleq \alpha. \tag{50}$$

On the other hand, (6) implies that $N_1Q_4N_1Q_4^D = Q_4^DN_1Q_4N_1$. Substituting (34), (38) and

(40) into this equality yields $N_{11}N_{12}=0$. Then, in (45),

$$\sum_{i=0}^{\infty}((N_{11}+Q_{41})^{-1})^{i+2}N_{12}(N_{14}+Q_{44})^{i}=\sum_{i=0}^{\infty}(Q_{41}^{-1})^{i+2}N_{12}(N_{14}+Q_{44})^{i},$$

and hence

$$(N_1+Q_4)^D=\begin{bmatrix}(N_{11}+Q_{41})^{-1} & \sum_{i=0}^{\infty}(Q_{41}^{-1})^{i+2}N_{12}(N_{14}+Q_{44})^{i}\\ 0 & 0\end{bmatrix}. \tag{51}$$

This and (44) give

$$\begin{bmatrix}0 & 0\\ 0 & (N_1+Q_4)^D\end{bmatrix}=\sum_{i=0}^{\infty}(Q^D)^{i+1}(-P)^iQQ^DP^{\pi}+\sum_{i=0}^{\infty}(Q^DP^{\pi})^{i+2}PQ^{\pi}(P+Q)^iP^{\pi}\triangleq\beta. \tag{52}$$

Due to (51), (35) and (48), we have

$$(N_1+Q_4)^DQ_3(P_1+Q_1)^{\pi}=0.$$

Note that $(P_1+Q_1)(P_1+Q_1)^{\pi}=(P_1+Q_1)^{\pi}(P_1+Q_1)$. Then X_3 in (47) can be simplified as

$$X_3=(N_1+Q_4)^{\pi}\sum_{i=0}^{\infty}(N_1+Q_4)^iQ_3((P_1+Q_1)^D)^{i+2}-(N_1+Q_4)^DQ_3(P_1+Q_1)^D. \tag{53}$$

Again, $(P+Q)P^{\pi}=\left[\begin{smallmatrix}0 & 0\\ 0 & N_1+Q_4\end{smallmatrix}\right]$ and $P^{\pi}QPP^D=\left[\begin{smallmatrix}0 & 0\\ Q_3 & 0\end{smallmatrix}\right]$. So, from (53), we get

$$\begin{bmatrix}0 & 0\\ X_3 & 0\end{bmatrix}=(I-(P+Q)P^{\pi}\beta)\sum_{i=0}^{\infty}(P+Q)^iP^{\pi}QPP^D\alpha^{i+2}-\beta P^{\pi}QPP^D\alpha. \tag{54}$$

Therefore, by(46), (50), (52) and (54), it follows that

$$(P+Q)^D=\alpha+\beta+(I-(P+Q)P^{\pi}\beta)\sum_{i=0}^{\infty}(P+Q)^iP^{\pi}QPP^D\alpha^{i+2}-\beta P^{\pi}QPP^D\alpha.$$

Taking into account $\beta P^{\pi}=\beta$ and $PP^D\alpha=\alpha$, we have

$$(P+Q)^D = \alpha+\beta+(I-(P+Q)P^{\pi}\beta)\sum_{k=0}^{\infty}(P+Q)^kP^{\pi}Q\alpha^{k+2}-\beta Q\alpha.$$

□

In the following, some special cases of Theorem 4.1 are given.

Corollary 4.1. *Let $P,\ Q\in\mathcal{B}(\mathcal{H})$, and P,Q,P^DQ be Drazin invertible. If $[P,PQ]=0$, $[Q,PQ]=0$ and $[P,QPQ]=0$, then $P+Q$ is Drazin invertible if and only if $I+P^DQ$ is Drazin invertible. In this case,*

$$\begin{aligned}(P+Q)^D &= \alpha+\beta+((I-(P+Q)P^{\pi}\beta)\sum_{k=0}^{\infty}(P+Q)^kP^{\pi}Q(P^D)^{k+2}\\ &- \alpha_1^2QP^{\pi}QP^D-(Q^DP^{\pi})^4P^2Q^{\pi}QP^{\pi}QP^D,\end{aligned}$$

where $\alpha,\ \beta$ are defined in Theorem 4.1, and $\alpha_1=\sum_{i=0}^{\infty}(Q^D)^{i+1}(-P)^iQQ^DP^{\pi}$.

Proof. By the assumptions and the proof of Theorem 4.1, it follows that $N_1Q_3=0$ and $Q_3Q_1=0$. Then, from (35), (38) and (34), we have

$$N_{11}Q_{32}+N_{12}Q_{34}=0,\ N_{14}Q_{34}=0 \tag{55}$$

and

$$Q_{32}Q_{14} = 0,\ Q_{34}Q_{14} = 0. \tag{56}$$

By (45),

$$(N_1+Q_4)^DQ_3 = \begin{bmatrix} (N_{11}+Q_{41})^{-1} & \sum_{i=0}^{\infty}((N_{11}+Q_{41})^{-1})^{i+2}N_{12}(N_{14}+Q_{44})^i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & Q_{32} \\ 0 & Q_{34} \end{bmatrix}.$$

From (40) and (55), it can be seen that

$$N_{12}(N_{14}+Q_{44})Q_{34}{=}0, N_{12}(N_{14}+Q_{44})^2Q_{34}{=}N_{12}N_{14}Q_{44}Q_{34}, N_{12}(N_{14}+Q_{44})^iQ_{34}{=}0, i \geq 3.$$

Thus,

$$\begin{aligned} &(N_1+Q_4)^DQ_3 \\ = &\begin{bmatrix} 0 & (N_{11}+Q_{41})^{-1}Q_{32}+(N_{11}+Q_{41})^{-2}N_{12}Q_{34}+(N_{11}+Q_{41})^{-4}N_{12}N_{14}Q_{44}Q_{34} \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} (N_{11}+Q_{41})^{-1}Q_{32}+(N_{11}+Q_{41})^{-2}N_{12}Q_{34} &= (N_{11}+Q_{41})^{-2}(N_{11}Q_{32}+Q_{41}Q_{32}+N_{12}Q_{34}) \\ &= (N_{11}+Q_{41})^{-2}Q_{41}Q_{32} \end{aligned}$$

and $N_{11}N_{12} = 0$, we then obtain

$$(N_1+Q_4)^DQ_3 = \begin{bmatrix} 0 & (N_{11}+Q_{41})^{-2}Q_{41}Q_{32}+Q_{41}^{-4}N_{12}N_{14}Q_{44}Q_{34} \\ 0 & 0 \end{bmatrix}.$$

In addition, (56) implies $Q_{32}(P_{14}+Q_{14})^{-1} = Q_{32}P_{14}^{-1}$ and $Q_{34}(P_{14}+Q_{14})^{-1} = Q_{34}P_{14}^{-1}$. Hence,

$$(N_1{+}Q_4)^DQ_3(P_1{+}Q_1)^D = \begin{bmatrix} 0 & (N_{11}{+}Q_{41})^{-2}Q_{41}Q_{32}P_{14}^{-1}+Q_{41}^{-4}N_{12}N_{14}Q_{44}Q_{34}P_{14}^{-1} \\ 0 & 0 \end{bmatrix}.$$

In view of (44), we get

$$\begin{bmatrix} 0 & 0 \\ (N_1+Q_4)^DQ_3(P_1+Q_1)^D & 0 \end{bmatrix} = \alpha_1^2QP^\pi QP^D + (Q^DP^\pi)^4P^2Q^\pi QP^\pi QP^D,$$

where $\alpha_1 = \sum_{i=0}^{\infty}(Q^D)^{i+1}(-P)^iQQ^DP^\pi$. Also, $Q_3(P_1+Q_1)^D = Q_3P_1^{-1}$. Therefore,

$$\begin{aligned} (P+Q)^D &= \alpha+\beta+((I-(P+Q)P^\pi\beta)\sum_{k=0}^{\infty}(P+Q)^kP^\pi Q(P^D)^{k+2} \\ &- \alpha_1^2QP^\pi QP^D-(Q^DP^\pi)^4P^2Q^\pi QP^\pi QP^D. \end{aligned}$$

□

If $[P,Q]=0$, then $Q_3=0,\ N_{12}=0,\ X_3=0$ in Theorem 4.1, whence $\beta=\sum_{i=0}^{\infty}(Q^D)^{i+1}(-P)^i P^\pi$ in (52). So, we obtain the result as follows.

Corollary 4.2. *Let $P,\ Q \in \mathcal{B}(\mathcal{H})$ is Drazin invertible. If $[P,Q]=0$, then $P+Q$ is Drazin invertible if and only if $I+P^DQ$ is Drazin invertible. In this case,*

$$(P+Q)^D = \alpha+\beta = P^D(I+P^DQ)QQ^D + \sum_{i=0}^{\infty}(P^D)^{i+1}(-Q)^iQ^\pi + \sum_{i=0}^{\infty}(Q^D)^{i+1}(-P)^iP^\pi.$$

Declarations

Conflict of interest The authors declare no conflict of interest.

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