Appl. Math. J. Chinese Univ.
2024, 39(1): 47-54

Higher-order expansions of powered extremes of logarithmic general error distribution

TAN Xiao-feng   LI Li-hui*

Abstract. In this paper, Let M_n denote the maximum of logarithmic general error distribution with parameter $v \geq 1$. Higher-order expansions for distributions of powered extremes M_n^p are derived under an optimal choice of normalizing constants. It is shown that M_n^p, when $v = 1$, converges to the Fréchet extreme value distribution at the rate of $1/n$, and if $v > 1$ then M_n^p converges to the Gumbel extreme value distribution at the rate of $(\log\log n)^2/(\log n)^{1-\frac{1}{v}}$.

§1 Introduction

General error distribution is an extension of the normal distribution. The probability density function (pdf) of the standardized general error distribution is

$$g_v(x) = \frac{v\exp(-(1/2)|x/\lambda|^v)}{2^{1+1/v}\lambda\Gamma(1/v)} \tag{1.1}$$

for $v > 0$ and $x \in \mathbb{R}$, where $\lambda = [2^{-2/v}\Gamma(1/v)/\Gamma(3/v)]^{1/2}$, and $\Gamma(\cdot)$ denotes the gamma function. Let $G_v(x) = \int_{-\infty}^{x} g_v(s)\,ds$ denote the cumulative distribution function (cdf) of the standardized general error distribution (denoted by GED(v)). Note that GED(2) is the standard normal distribution and GED(1) means the Laplace distribution.

Probability properties of GED(v) have been studied in recent years. Peng et al. (2009) established the Mills' inequalities and Mills' ratio of GED(v) as $v > 1$ and Peng et al. (2009) and Vasudeva et al. (2014) considered the limiting distribution of partial maximum from GED(v) random variables for $v > 0$. Vasudeva et al. (2014) further considered the density convergence of partial maximum; strong stability of partial maximum and asymptotic behaviors of near maximum and near-maxima sum. Uniform convergence rate of distribution of maximum to its extreme value distribution was derived by Peng et al. (2010) and Jia and Li (2014) showed the distributional expansions of partial maximum from GED(v) for $v > 0$. Li et al. (2018) considered the moment convergence of powered normal extremes. The work of Peng et al.

Received: 2019-07-12.   Revised: 2021-02-03.
MR Subject Classification: 60G70, 62E20.
Keywords: logarithmic general error distribution, convergence rate, higher-order expansion, powered extreme.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-024-3859-4.
*Corresponding author.

(2010) and Jia and Li (2014) showed that with an optimal normalizing constants convergence rate of distribution of partial maximum to its limit is proportional to $1/\log n$ as $v \neq 1$, similar to the case of GED(2), the standard normal case, studied by Hall (1979) and Nair (1981). In order to improve the convergence rate, Hall (1980) considered the distribution of powered extreme $|M_n|^p$ with M_n denoting the partial maximum of normal random variables, and showed that the distribution of normalized M_n^2 converges to the Gumble extreme value distribution at the rate of $1/(\log n)^2$ under optimal normalizing constants, while the convergence rates are still the order of $1/\log n$ for the case of $0 < p \neq 2$. For more details, see Hall (1980) and Leadbetter at al. (1983). Zhou and Ling (2016) established the higher-order expansions for distributions and densities of powered extremes from normal samples. Lu and Peng (2018) considered the higher-order expansions of distribution of powered order statistics of GED(v) random variables, and showed that the convergence rate can be improved under an optimal normalizing constants.

This short note is to consider the higher order expansions of distribution of powered extremes of logarithmic general error distribution (written as logGED) random variables. logGED defined by Liao et al. (2014) is a natural extension of the log-normal distribution. Let $\eta = e^{\xi}$ with ξ following the GED distribution with parameter $v > 0$, it is said that η follows the logGED with parameter v, written as $\eta \sim \text{logGED}(v)$.

Let $f_v(x)$ and $F_v(x)$ denote the pdf and cdf of η respectively. it is easy to check that $f_v(x)$ is given by

$$f_v(x) = \frac{vx^{-1}\exp(-\frac{1}{2}|\frac{\log x}{\lambda}|^v)}{2^{1+1/v}\lambda\Gamma(1/v)}, \quad x > 0, \tag{1.2}$$

where $\lambda = [2^{-2/v}\Gamma(1/v)/\Gamma(3/v)]^{1/2}$. Note that the logGED($v$) reduces to the logarithmic Laplace distribution for $v = 1$ and to the log-normal distribution for $v = 2$. For $v > 1$, by the following Mills' ratio of logGED(v) due to Liao et al.(2014),

$$\frac{1 - F_v(x)}{f_v(x)} \sim \frac{2\lambda^v}{v}(\log x)^{1-v}x \tag{1.3}$$

as $x \to \infty$, we can show

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{M_n - \beta_n}{\alpha_n} \leq x\right) = \Lambda(x) = \exp(-\exp(-x))$$

with normalizing constants α_n and β_n given by

$$\begin{cases} \alpha_n = \frac{2^{1/v}\exp(2^{1/v}\lambda(\log n)^{1/v})}{v(\log n)^{1-1/v}}, \\ \beta_n = \exp(2^{1/v}\lambda(\log n)^{1/v}) - \frac{2^{1/v}\lambda}{v}\frac{\exp(2^{1/v}\lambda(\log n)^{1/v})}{(\log n)^{1-1/v}}\left(\frac{v-1}{v}\log\log n + \log 2\Gamma(1/v)\right). \end{cases} \tag{1.4}$$

For recent work on logGED(v), Yang et al.(2016) studied the asymptotic expansion of the distribution of the normalized M_n; Chen and Du (2016) derived asymptotic expansions of density of normalized logGED(v) extremes; Yang and Li (2016) also considered expansions of distribution of maximum under power normalization. Other related work on extreme value distributions of given distributions and their associated uniform convergence rates, we refer to Peng et al. (2010), Liao and Peng (2012), Liao et al. (2013), Liao et al. (2014), Jia et al. (2015) and references therein.

The rest of this paper is organized as follows. Section 2 provides the main results. Some auxiliary results and the proofs of the main results are given in Section 3.

§2 Main results

In this section, we establish the distributions expansions of powered extremes of logarithmic general error distribution with power index $p > 0$. Noting the fact from Yang et al. (2016) that F_v doesn't belong to any domain of attraction of extreme value distribution as $0 < v < 1$, in this paper we only consider the case $v \geq 1$. Throughout the paper, constants α_n and β_n are given by (1.4).

Theorem 1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables with common distribution* logGED(v) *with* $v \geq 1$*. Let* $M_n = \max(X_k, 1 \leq k \leq n)$ *denote the partial maximum, then*

(i) for $v = 1$*, with normalizing constants* $a_n = (2/n)^{-p/\sqrt{2}}$*, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(M_n^p \leq a_n x\right) = \Phi_{\sqrt{2}/p}(x) = \exp(-x^{-\frac{\sqrt{2}}{p}}), \quad x > 0; \tag{2.1}$$

(ii) for $v > 1$*, with normalizing constants* $a_n = p\alpha_n\beta_n^{p-1}$*,* $b_n = \beta_n^p$ *we have*

$$\lim_{n\to\infty} \mathbb{P}\left(M_n^p \leq a_n x + b_n\right) = \Lambda(x) = \exp(-\exp(-x)). \tag{2.2}$$

Remark 1. Equation (2.2) notes that $\lim_{n\to\infty} \mathbb{P}\left(M_n \leq (a_n x + b_n)^{\frac{1}{p}}\right) = \Lambda(x)$, $-\infty < x < \infty$. This would mean that the limit law for (M_n) is Gumbel under a non-linear normalization. When $p = \frac{1}{2}$, the norming sequence reduces to $(a_n^2 x^2 + 2a_n b_n x + b_n^2)$, which is quadratic.

Theorem 2. *Let* $\{X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables with common distribution* logGED(v) *with* $v \geq 1$*. Let* $M_n = \max(X_k, 1 \leq k \leq n)$ *denote the partial maximum, then*

(i) for $v = 1$*, with normalizing constants* $a_n = (2/n)^{-p/\sqrt{2}}$*, for* $x > 0$ *we have*

$$\lim_{n\to\infty} n\Big\{n\left(\mathbb{P}(M_n^p \leq a_n x) - \Phi_{\sqrt{2}/p}(x)\right) + k_1(x)\Phi_{\sqrt{2}/p}(x)\Big\} = \omega_1(x)\Phi_{\sqrt{2}/p}(x), \tag{2.3}$$

where

$$k_1(x) = \frac{1}{2}x^{\frac{-2\sqrt{2}}{p}}, \quad \omega_1(x) = \frac{1}{8}x^{\frac{-4\sqrt{2}}{p}} - \frac{1}{3}x^{\frac{-3\sqrt{2}}{p}};$$

(ii) for $v > 1$*, with normalizing constants* $a_n = p\alpha_n\beta_n^{p-1}$*,* $b_n = \beta_n^p$ *we have*

$$\lim_{n\to\infty} \tfrac{(\log n)^{1-\frac{1}{v}}}{(\log\log n)^2}\Big\{\tfrac{(\log n)^{1-\frac{1}{v}}}{(\log\log n)^2}\left(\mathbb{P}\left(M_n^p \leq a_n x + b_n\right) - \Lambda(x)\right) + k_v(x)\Lambda(x)\Big\} = \omega_v(x)\Lambda(x), \tag{2.4}$$

where

$$k_v(x) = \frac{2^{\frac{1}{v}-1}\lambda(v-1)^2}{v^3}e^{-x}, \quad \omega_v(x) = \frac{2^{\frac{2}{v}-3}\lambda^2(v-1)^4(e^{-2x} - e^{-x})}{v^6}.$$

Remark 2. For $v = 1$, Theorem 2(i) shows that the convergence rate of $\mathbb{P}(M_n^p \leq a_n x)$ to the extreme value distribution $\Phi_{\sqrt{2}/p}(x)$ is proportional to $1/n$, while for the case of $v > 1$, Theorem 2(ii) shows that the convergence rate of $\mathbb{P}(M_n^p \leq a_n x + b_n)$ to the extreme value distribution $\Lambda(x)$ is proportional to $(\log\log n)^2/(\log n)^{1-1/v}$.

§3 Proofs

In order to prove the main results, we need some auxiliary lemmas.

Lemma 1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d.* logGED(v) *random variables with common distribution $F_v(x)$, $v \geq 1$. Let $F_{p,v}(x) = P(X_n^p \leq x)$, for large x we have*

(i) if $v = 1$, $1 - F_{p,1}(x) = \frac{1}{2} x^{-\sqrt{2}/p}$;

(ii) if $v > 1$, $1 - F_{p,v}(x) = c(x) \exp\big(- \int_e^x \frac{g(t)}{f(t)}\, dt \big)$, where

$$c(x) \to \frac{(\lambda p)^{v-1} e^{-\frac{1}{2(\lambda p)^v}}}{2^{1/v}\Gamma(1/v)}, f(x) = \frac{2x(\lambda p)^v}{v(\log x)^{v-1}} \quad \text{with} \quad f(x)' \to 0,$$
$$g(x) = 1 + \frac{2(v-1)(\lambda p)^v}{v(\log x)^v} \to 1$$

as $x \to \infty$.

Proof. For the case of $v = 1$, $\lambda = 2^{-1/2}$ and elementary calculation to get the result. For $v > 1$, $F_{p,v}(x) = F_v(x^{\frac{1}{p}})$ and apply (1.3) to get the distributional tail representation. The proof is complete. □

Lemma 2. *Let $F_v(x)$ and $f_v(x)$ respectively denote the cdf and pdf of* logGED(v) *with $v > 1$, with normalizing constants $a_n = p\alpha_n \beta_n^{p-1}$, $b_n = \beta_n^p$, for large x we have*

$$\begin{aligned}
&1 - F_v((a_n x + b_n)^{\frac{1}{p}}) \\
=& n^{-1} e^{-x} \Big(1 + \frac{(\frac{1}{v} - 1)(1 + x - \log 2\Gamma(\frac{1}{v}) + \frac{1}{2}(x - \log 2\Gamma(\frac{1}{v}))^2) + (1 - \frac{1}{v})^2 (1 + x - \log 2\Gamma(\frac{1}{v})) \log\log n}{\log n} \\
&- \frac{(\frac{v-1}{v})^3 (\log\log n)^2}{2\log n} + \frac{\frac{2\lambda(v-1)}{v^2}(\log 2\Gamma(\frac{1}{v}) - x)\log\log n + \frac{\lambda(v-1)^2}{v^3}(\log\log n)^2 + \frac{\lambda}{v}(px^2 + (\log 2\Gamma(\frac{1}{v}))^2)}{(2\log n)^{1-\frac{1}{v}}} \\
&+ \frac{2^{\frac{1}{v}} \lambda v^{-1}(2x - (2+x)\log 2\Gamma(\frac{1}{v}))}{(\log n)^{1-\frac{1}{v}}} + \frac{2^{\frac{2}{v}-3}\lambda^2 (v-1)^4 (\log\log n)^4}{v^6 (\log n)^{2-\frac{2}{v}}} + o(\frac{(\log\log n)^4}{(\log n)^{2-\frac{2}{v}}}) \Big).
\end{aligned} \tag{3.1}$$

Proof. Noting that $f_v(x) = g_v(\log x)/x, F_v(x) = G_v(\log x)$, by (3.1) of Lemma 1 in Jia and Li (2014), for $v > 1$ and large x we have

$$\begin{aligned}
1 - F_v(x) =& \frac{2\lambda^v}{v} \Big\{ 1 + 2(v^{-1} - 1)\lambda^v (\log x)^{-v} + 4(v^{-1} - 1)(v^{-1} - 2)\lambda^{2v}(\log x)^{-2v} \\
&+ 8(v^{-1} - 1)(v^{-1} - 2)(v^{-1} - 3)\lambda^{3v}(\log x)^{-3v} + O((\log x)^{-4v}) \Big\} (\log x)^{1-v} x f_v(x).
\end{aligned} \tag{3.2}$$

Let

$$B_{v,n} = \exp(2^{\frac{1}{v}} \lambda (\log n)^{\frac{1}{v}}), \quad A_{v,n} = \frac{v-1}{v}\log\log n + \log 2\Gamma(\frac{1}{v}), \quad C_{v,n} = 1 - \frac{2^{\frac{1}{v}}\lambda}{v(\log n)^{1-\frac{1}{v}}} A_{v,n},$$

then

$$\alpha_n = \frac{2^{\frac{1}{v}} \lambda B_{v,n}}{v(\log n)^{1-\frac{1}{v}}}, \quad \beta_n = B_{v,n} C_{v,n}.$$

By the Tailor's expansion, we can get

$$C_{v,n}^{-1} = 1 + \frac{2^{\frac{1}{v}} \lambda (v-1) \log\log n}{v^2 (\log n)^{1-\frac{1}{v}}} + \frac{2^{\frac{1}{v}} \lambda \log 2\Gamma(\frac{1}{v})}{v(\log)^{1-\frac{1}{v}}} + \frac{2^{\frac{2}{v}} \lambda^2 (v-1)^2 (\log\log n)^2}{v^4 (\log n)^{2-\frac{2}{v}}} + o(\frac{(\log\log n)^2}{(\log n)^{2-\frac{2}{v}}}), \tag{3.3}$$

$$C_{v,n}^{-2} = 1 + \frac{2^{1+\frac{1}{v}}\lambda(v-1)\log\log n}{v^2(\log n)^{1-\frac{1}{v}}} + \frac{2^{1+\frac{1}{v}}\lambda\log 2\Gamma(\frac{1}{v})}{v(\log)^{1-\frac{1}{v}}} + \frac{3\cdot 2^{\frac{2}{v}}\lambda^2(v-1)^2(\log\log n)^2}{v^4(\log n)^{2-\frac{2}{v}}} + o(\frac{(\log\log n)^2}{(\log n)^{2-\frac{2}{v}}}), \quad (3.4)$$

$$C_{v,n}^{-3} = 1 + \frac{3\cdot 2^{\frac{1}{v}}\lambda(v-1)\log\log n}{v^2(\log n)^{1-\frac{1}{v}}} + \frac{3\cdot 2^{\frac{1}{v}}\lambda\log 2\Gamma(\frac{1}{v})}{v(\log)^{1-\frac{1}{v}}} + \frac{6\cdot 2^{\frac{2}{v}}\lambda^2(v-1)^2(\log\log n)^2}{v^4(\log n)^{2-\frac{2}{v}}} + o(\frac{(\log\log n)^2}{(\log n)^{2-\frac{2}{v}}}), \quad (3.5)$$

and

$$\begin{aligned}&\log C_{v,n}\\ &= 2^{\frac{1}{v}}\lambda(\log n)^{\frac{1}{v}}\Big[-\frac{\log 2\Gamma(\frac{1}{v})}{v\log n} - \frac{(v-1)\log\log n}{v^2\log n} - \frac{2^{\frac{1}{v}-1}\lambda(v-1)^2(\log\log n)^2}{v^4(\log n)^{2-\frac{1}{v}}} - \frac{2^{\frac{1}{v}-1}\lambda(\log 2\Gamma(\frac{1}{v}))^2}{v^2(\log n)^{2-\frac{1}{v}}}\\ &- \frac{2^{\frac{1}{v}}\lambda(v-1)\log 2\Gamma(\frac{1}{v})\log\log n}{v^3(\log n)^{2-\frac{1}{v}}} + \frac{2^{\frac{2}{v}}\lambda^2(v-1)^3(\log\log n)^3}{3v^6(\log n)^{3-\frac{2}{v}}} + o\Big(\frac{(\log\log n)^3}{(\log n)^{3-\frac{2}{v}}}\Big)\Big]. \quad (3.6)\end{aligned}$$

Combining (3.3)–(3.5) we have

$$\begin{aligned}&\frac{1}{p}\log\left(1 + \frac{2^{\frac{1}{v}}\lambda px}{v(\log n)^{1-\frac{1}{v}}C_{v,n}}\right)\\ =\ & 2^{\frac{1}{v}}\lambda(\log n)^{\frac{1}{v}}\Big[\frac{x}{v\log n} + \frac{2^{\frac{1}{v}}\lambda(v-1)x\log\log n}{v^3(\log n)^{2-\frac{1}{v}}} + \frac{2^{\frac{1}{v}}\lambda\log 2\Gamma(\frac{1}{v})x - 2^{\frac{1}{v}-1}\lambda px^2}{v^2(\log n)^{2-\frac{1}{v}}}\\ +\ & \frac{2^{\frac{2}{v}}\lambda^2(v-1)^2x(\log\log n)^2}{v^5(\log n)^{3-\frac{2}{v}}} + o\Big(\frac{(\log\log n)^2}{(\log n)^{3-\frac{2}{v}}}\Big)\Big]. \quad (3.7)\end{aligned}$$

Let

$$D_{p,v}(x) = (a_nx+b_n)^{\frac{1}{p}} = B_{v,n}C_{v,n}\left(1 + \frac{2^{\frac{1}{v}}\lambda px}{v(\log n)^{1-\frac{1}{v}}C_{v,n}}\right)^{\frac{1}{p}}. \quad (3.8)$$

Applying (3.3) and (3.4), we can get

$$D_{p,v}(x) = B_{v,n}\Big[1 + \frac{2^{\frac{1}{v}}\lambda(x-\log 2\Gamma(\frac{1}{v}))}{v(\log n)^{1-\frac{1}{v}}} - \frac{2^{\frac{1}{v}}\lambda(v-1)\log\log n}{v^2(\log n)^{1-\frac{1}{v}}} + \frac{(1-p)2^{\frac{2}{v}-1}\lambda^2x^2}{v^2(\log n)^{2-\frac{2}{v}}} + o(\frac{1}{(\log n)^{2-\frac{2}{v}}})\Big]. (3.9)$$

Combining (3.6),(3.7) and (3.8), we have

$$\begin{aligned}&\log D_{p,v}(x)\\ &= 2^{\frac{1}{v}}\lambda(\log n)^{\frac{1}{v}}\Big[1 + \frac{x-\log 2\Gamma(\frac{1}{v})}{v\log n} - \frac{(v-1)\log\log n}{v^2\log n} + \frac{2^{\frac{1}{v}}\lambda(v-1)(x-\log 2\Gamma(\frac{1}{v}))\log\log n}{v^3(\log n)^{2-\frac{1}{v}}}\\ &- \frac{2^{\frac{1}{v}-1}\lambda(v-1)^2(\log\log n)^2}{v^4(\log n)^{2-\frac{1}{v}}} + D_1(x) + \frac{2^{\frac{2}{v}}\lambda^2(v-1)^3(\log\log n)^3}{3v^6(\log n)^{3-\frac{2}{v}}} + o(\frac{(\log\log n)^3}{(\log n)^{3-\frac{2}{v}}})\Big], \quad (3.10)\end{aligned}$$

where $D_1(x) = \frac{2^{\frac{1}{v}}\lambda\log 2\Gamma(\frac{1}{v})x - 2^{\frac{1}{v}-1}\lambda px^2 - 2^{\frac{1}{v}-1}\lambda(\log 2\Gamma(\frac{1}{v}))^2}{v^2(\log n)^{2-\frac{1}{v}}}$. Hence,

$$\begin{aligned}&(\log D_{p,v}(x))^{1-v}\\ &= 2^{\frac{1-v}{v}}\lambda^{1-v}(\log n)^{\frac{1-v}{v}}\Big[1 + \frac{(1-v)(x-\log 2\Gamma(\frac{1}{v}))}{v\log n} + \frac{(v-1)^2\log\log n}{v^2\log n} + \frac{2^{\frac{1}{v}}\lambda(v-1)^2(\log 2\Gamma(\frac{1}{v})-x)\log\log n}{v^3(\log n)^{2-\frac{1}{v}}}\\ &+ \frac{2^{\frac{1}{v}-1}\lambda(v-1)^3(\log\log n)^2}{v^4(\log n)^{2-\frac{1}{v}}} + (1-v)D_1(x) + \frac{(v-1)^3(\log\log n)^2}{2v^3(\log n)^2} + o(\frac{(\log\log n)^2}{(\log n)^2})\Big]. \quad (3.11)\end{aligned}$$

Further, applying (3.10) we have

$$1 + 2(v^{-1}-1)\lambda^v(\log D_{p,v}(x))^{-v} = 1 + \frac{v^{-1}-1}{\log n} + \frac{(v^{-1}-1)(v-1)\log\log n}{v(\log n)^2} + o(\frac{\log\log n}{(\log n)^2}) \quad (3.12)$$

and

$$\begin{aligned}
&f_v(D_{p,v}(x))\\
&=\frac{v}{2^{1+\frac{1}{v}}\lambda\Gamma(\frac{1}{v})}D_{p,v}(x)^{-1}\exp(-\frac{1}{2\lambda^v}(\log D_{p,v}(x))^v)\\
&=\frac{vB_{v,n}^{-1}}{2^{\frac{1}{v}}\lambda}n^{-1}e^{-x}(\log n)^{\frac{v-1}{v}}\Big[1+\frac{2^{\frac{1}{v}}\lambda(v-1)(1-x+\log 2\Gamma(\frac{1}{v}))\log\log n}{v^2(\log n)^{1-\frac{1}{v}}}+\frac{2^{\frac{1}{v}-1}\lambda(v-1)^2(\log\log n)^2}{v^3(\log n)^{1-\frac{1}{v}}}\\
&-D_2(x)-\frac{(v-1)^3(\log\log n)^2}{2v^3\log n}+\frac{(x-\log 2\Gamma(\frac{1}{v}))(v-1)^2\log\log n}{v^2\log n}-\frac{(v-1)(x-\log 2\Gamma(\frac{1}{v}))^2}{2v\log n}\\
&+\frac{2^{\frac{2}{v}-3}\lambda^2(v-1)^4(\log\log n)^4}{v^6(\log n)^{2-\frac{2}{v}}}+o(\frac{(\log\log n)^4}{(\log n)^{2-\frac{2}{v}}})\Big],
\end{aligned}\tag{3.13}$$

where

$$D_2(x)=\frac{2^{\frac{1}{v}}\lambda\log 2\Gamma(\frac{1}{v})x-2^{\frac{1}{v}-1}\lambda px^2-2^{\frac{1}{v}-1}\lambda(\log 2\Gamma(\frac{1}{v}))^2-2^{\frac{1}{v}}\lambda(x-\log 2\Gamma(\frac{1}{v}))}{v(\log n)^{1-\frac{1}{v}}}.$$

Combining (3.2), (3.9) and (3.11)-(3.13) we obtain (3.1). □

Proof of Theorem 1. (i) For $v=1$, Lemma 1(i) implies

$$\lim_{t\to\infty}\frac{1-F_{p,1}(tx)}{1-F_{p,1}(t)}=x^{\frac{-\sqrt{2}}{p}}$$

for $x>0$. Thus by Proposition 1.11 in Resnick (1987), we have $F_{p,1}(x)\in D(\Phi_{\sqrt{2}/p})$. By the definition of logGED(v) we know that $M_n^p=\max_{1\le k\le n}X_k^p$. With $a_n=(\frac{2}{n})^{-\frac{p}{\sqrt{2}}}$ we have

$$\lim_{n\to\infty}\mathbb{P}(M_n^p\le a_nx)=\Phi_{\frac{\sqrt{2}}{p}}(x).$$

(ii) For $v>1$, by Corollary 1.7 in Resnick (1987), Lemma 1(ii) showed that $F_{p,v}(x)\in D(\Lambda)$. Note $M_n^p=\max_{1\le k\le n}X_k^p$. Applying (3.1), we observe that

$$n\Big(1-F_{p,v}(a_nx+b_n)\Big)=n\Big(1-F_v((a_nx+b_n)^{\frac{1}{p}})\Big)\to e^{-x},$$

the desired result follows. □

Proof of Theorem 2. (i) For $v=1$, under normalizing constants $a_n=(\frac{2}{n})^{\frac{-p}{\sqrt{2}}}$, we have

$$\begin{aligned}
\mathbb{P}\Big(M_n^p\le a_nx\Big)-\Phi_{\frac{\sqrt{2}}{p}}(x) &= (1-\frac{1}{n}x^{-\frac{\sqrt{2}}{p}})^n-\Phi_{\frac{\sqrt{2}}{p}}(x)\\
&=\Big[\exp\Big(n\log(1-\frac{1}{n}x^{-\frac{\sqrt{2}}{p}})+x^{-\frac{\sqrt{2}}{p}}\Big)-1\Big]\Phi_{\frac{\sqrt{2}}{p}}(x)\\
&=\Big[\exp\Big(-\frac{1}{2n}x^{-\frac{2\sqrt{2}}{p}}-\frac{1}{3n^2}x^{-\frac{3\sqrt{2}}{p}}+o(\frac{1}{n^2})\Big)-1\Big]\Phi_{\frac{\sqrt{2}}{p}}(x)\\
&=\Big[-\frac{1}{2n}x^{-\frac{2\sqrt{2}}{p}}+\frac{1}{n^2}(\frac{1}{8}x^{-\frac{4\sqrt{2}}{p}}-\frac{1}{3}x^{-\frac{3\sqrt{2}}{p}})+o(\frac{1}{n^2})\Big]\Phi_{\frac{\sqrt{2}}{p}}(x).
\end{aligned}\tag{3.14}$$

Thus (2.3) follows from (3.14).

(ii) For $v>1$, under normalizing constants $a_n=p\alpha_n\beta_n^{p-1}$ and $b_n=\beta_n^p$, applying (3.1) we have

$$\begin{aligned}
&-n(1-F_v(D_{p,v}(x)))+e^{-x}\\
=&\Big(-\frac{2^{\frac{1}{v}-1}\lambda(v-1)^2(\log\log n)^2}{v^3(\log n)^{1-\frac{1}{v}}}-\frac{2^{\frac{2}{v}-3}\lambda^2(v-1)^4(\log\log n)^4}{v^6(\log n)^{2-\frac{2}{v}}}+o(\frac{(\log\log n)^4}{(\log n)^{2-\frac{2}{v}}})\Big)e^{-x}.
\end{aligned}$$

For sufficient large n, we conclude

$$
\begin{aligned}
&\mathbb{P}\left(M_n^p \le a_n x + b_n\right) - \Lambda(x)\\
&= F_v^n(D_{p,v}(x)) - \Lambda(x)\\
&= \Big[\exp(n\log F_v(D_{p,v}(x)) + e^{-x}) - 1\Big]\Lambda(x)\\
&= \Big[-n\Big(1-F_v(D_{p,v}(x))\Big)+e^{-x}+\frac{[-n\Big(1-F_v(D_{p,v}(x))\Big)+e^{-x}]^2}{2}+o\big(n^2(1-F_v(D_{p,v}(x)))^2\big)\Big]\Lambda(x)\\
&= \Big[\Big(-\frac{2^{\frac{1}{v}-1}\lambda(v-1)^2(\log\log n)^2}{v^3(\log n)^{1-\frac{1}{v}}}-\frac{2^{\frac{2}{v}-3}\lambda^2(v-1)^4(\log\log n)^4}{v^6(\log n)^{2-\frac{2}{v}}}+o\big(\frac{(\log\log n)^4}{(\log n)^{2-\frac{2}{v}}}\big)\Big)e^{-x}\\
&\quad+\frac{1}{2}\Big(-\frac{2^{\frac{1}{v}-1}\lambda(v-1)^2(\log\log n)^2}{v^3(\log n)^{1-\frac{1}{v}}}-\frac{2^{\frac{2}{v}-3}\lambda^2(v-1)^4(\log\log n)^4}{v^6(\log n)^{2-\frac{2}{v}}}+o\big(\frac{(\log\log n)^4}{(\log n)^{2-\frac{2}{v}}}\big)\Big)^2e^{-2x}\\
&\quad+o\big(\frac{(\log\log n)^4}{(\log n)^{2-\frac{2}{v}}}\big)\Big]\Lambda(x)\\
&= \Big[-\frac{2^{\frac{1}{v}-1}\lambda(v-1)^2e^{-x}(\log\log n)^2}{v^3(\log n)^{1-\frac{1}{v}}}+\frac{2^{\frac{2}{v}-3}\lambda^2(v-1)^4(e^{-2x}-e^{-x})(\log\log n)^4}{v^6(\log n)^{2-\frac{2}{v}}}+o\big(\frac{(\log\log n)^4}{(\log n)^{2-\frac{2}{v}}}\big)\Big]\Lambda(x). \qquad (3.15)
\end{aligned}
$$

Thus (2.4) follows from (3.15). □

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] S Q Chen, L L Du. *Asymptotic expansions of density of normalized extremes from logarithmic general error distribution*, Comm Statist Theory Methods, 2016, 7: 3459-3478.

[2] P Hall. *On the rate of convergence of normal extremes*, J Appl Probab, 1979, 16: 433-439.

[3] P Hall. *Estimating probabilities of normal extremes*, Adv Appl Probab, 1980, 12: 491-500.

[4] P Jia, T T Li. *Higher-order expansions for distribution of extrems from general error distribution*, J Inequal Appl, 2014, 2014(1): 1-8.

[5] P Jia, X Liao, Z X Peng. *Asymptotic expansions of the moments of extremes from general error distribution*, J Math Anal Appl, 2015, 2: 1131-1145.

[6] M R Leadbetter, G Lindgren, H Rootzén. *Extremes and related properties of random sequences and processes*, Springer Verlag, New York, 1983.

[7] T T Li, Z X Peng. *Moment convergence of powered normal extremes*, Comm Statist Theory Methods, 2018, 14: 3453-3463.

[8] X Liao, Z X Peng. *Convergence rates of limit distribution of maxima of lognormal samples*, J Math Anal Appl, 2012, 2: 643-653.

[9] X Liao, Z X Peng, S Nadarajah. *Tail properties and asymptotic expansions for the maximum of the logarithmic skew-normal distribution*, J Appl Probab, 2013, 3: 900-907.

[10] X Liao, Z X Peng, S Nadarajah. *Tail behavior and limit distribution of maximum of logarithmic general error distribution*, Comm Statist Theory Methods, 2014, 24: 5276-5289.

[11] Y Y Lu, Z X Peng. *Distributional expansions of powered order statistics from general error distribution*, 2019, http://arxiv.org/abs/1905.01397.

[12] K A Nair. *Asymptotic distribution and moments of normal extremes*, Annal Probab, 1981, 9: 150-153.

[13] Z X Peng, S Nadarajah, F Lin. *Convergence rate of extremes for the general error distribution*, J Appl Probab, 2010, 3: 668-679.

[14] Z X Peng, B Tong, S Nadarajah. *Tail behavior of the general error distribution*, Comm Statist Theory Methods, 2009, 38: 1884-1892.

[15] S I Resnick. *Extreme Values, Regular Variation, and Point Processes*, Springer Verlag, New York, 1987.

[16] R Vasudeva, J V Kumari, A Ravi. *On the asymptotic behaviour of extremes and near maxima of random observations from the general error distributions*, J Appl Probab, 2014, 51: 528-541.

[17] G Yang, T T Li. *Expansions on extremes from logarithmic general error distribution under power normalization*, Stat Pap, 2016, 3: 781-793.

[18] G Yang, X Liao, Z X Peng. *Distributional expansion of maximum from logarithmic general error distribution*, Appl Math J Chinese Univ, 2016, 2: 157-164.

[19] W Zhou, C X Ling. *Higher-order expansions of powered extremes of normal samples*, Statist Probab Lett, 2016, 111: 12-17.

School of Mathematics and Statistics, Southwest University, Chongqing 400715, China.

Emails: txfswu1995@163.com, llhlx@swu.edu.cn