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The tangential k-Cauchy–Fueter type operator and Penrose type integral formula on the generalized complex Heisenberg group

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Abstract. The tangential k-Cauchy–Fueter operator and k-CF functions are counterparts of the tangential Cauchy–Riemann operator and CR functions on the Heisenberg group in the theory of several complex variables, respectively. In this paper, we introduce a Lie group that the Heisenberg group can be imbedded into and call it generalized complex Heisenberg. We investigate quaternionic analysis on the generalized complex Heisenberg. We also give the Penrose integral formula for k-CF functions and construct the tangential k-Cauchy-Fueter complex.

§1 Introduction

The generalized complex Heisenberg group $\mathfrak{H}$ is the complex space $\mathbb{C}^8$ with the multiplication

$$(\mathbf{y},\mathbf{s})\cdot(\widehat{\mathbf{y}},\widehat{\mathbf{s}}) := \left(\mathbf{y}+\widehat{\mathbf{y}},\mathbf{s}+\widehat{\mathbf{s}}-2\mathbf{y}^T\mathbf{J}\widehat{\mathbf{y}}\right), \tag{1.1}$$

where

$$(\mathbf{y},\mathbf{s}) := \left(\begin{pmatrix} y_{00'} & y_{01'} \\ y_{10'} & y_{11'} \end{pmatrix}, \begin{pmatrix} s_{0'0'} & s_{0'1'} \\ s_{1'0'} & s_{1'1'} \end{pmatrix}\right) \in \mathbb{C}^8, \tag{1.2}$$

and $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In the classical flat case, by complexifying 4-D Minkowski space as $\mathbb{C}^4$, one can construct twistor space by using complex geometry method (cf. [6, 8, 14]). The Penrose transform was originally used by Eastwood, Penrose and Wells [7] to construct holomorphic solutions to massless field equations over the complexified Minkowski space, and was generalized by Baston [2] to complexified quaternionic Kähler manifolds. The twistor transformation for the Heisenberg group is given in [9] and we use the twistor method to study a horizontal ASD equations on the Heisenberg group [10].

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In this paper, we will use the twistor method to define the *α-planes* over $\mathfrak{H}$. The moduli space of all α-planes is the twistor space $\mathcal{P}$. Then we have the double fibration

$$\mathcal{P} \xleftarrow{\eta} \mathbb{C}^8 \times \mathbb{C}P^1 \xrightarrow{\tau} \mathbb{C}^8.$$

We will write down a local coordinate chart of the double fibration in this paper.

We can construct a differential complex, the *tangential k-Cauchy–Fueter complex*, with the first operator to be the tangential k-Cauchy–Fueter type operator $\mathscr{D}_0^{(k)}$:

$$0 \to C^\infty(\Omega, \mathscr{V}_0) \xrightarrow{\mathscr{D}_0^{(k)}} C^\infty(\Omega, \mathscr{V}_1) \xrightarrow{\mathscr{D}_1^{(k)}} C^\infty(\Omega, \mathscr{V}_2) \to 0, \tag{1.3}$$

for a domain Ω in $\mathfrak{H}$, where

$$\mathscr{V}_j := \odot^{k-j}\,\mathbb{C}^2 \otimes \wedge^j \mathbb{C}^2, \quad j = 0, 1 \tag{1.4}$$

for fixed $k = 0, 1, \cdots$. Here $\odot^p\mathbb{C}^2$ is the pth symmetric power of $\mathbb{C}^2$. See [11, Theorem 2.1] and [9, Theorem 1.3] for the similar construction of tangential k-Cauchy-Fueter operator on the right quaternionic Heisenberg group and Heisenberg group, respectively. It is a generalization of k-Cauchy-Fueter complex on the quaternionic space $\mathbb{H}^n$ (cf. [3–5, 12, 13] and reference therein).

In Section 2, we introduce the α-planes and the twistor space of the generalized complex Heisenberg group. In Section 3, we give the definition of the tangential k-Cauchy–Fueter type operators on the generalized complex Heisenberg group and their basic properties. We also prove the Penrose-type integral formula on the generalized complex Heisenberg group. In Section 4, we prove that (1.3) is a complex.

§2 α-planes and the twistor space of the generalized complex Heisenberg group

The *left translation* over $\mathfrak{H}$ is

$$\tau_{(\mathbf{y}',\mathbf{s}')} : (\mathbf{y}, \mathbf{s}) \mapsto (\mathbf{y}', \mathbf{s}') \cdot (\mathbf{y}, \mathbf{s}), \qquad (\mathbf{y}, \mathbf{s}) \in \mathfrak{H} \tag{2.1}$$

for fixed $(\mathbf{y}', \mathbf{s}') \in \mathfrak{H}$. A vector field V over $\mathfrak{H}$ is called *left invariant* if for any $(\mathbf{y}', \mathbf{s}') \in \mathfrak{H}$, we have

$$\tau_{(\mathbf{y}',\mathbf{s}')*} V = V,$$

where $\tau_{(\mathbf{y}',\mathbf{s}')}$ is the left translation in (2.1). Define

$$(V_{AA'}f)(\mathbf{y}, \mathbf{s}) := \frac{\mathrm{d}}{\mathrm{d}t} f((\mathbf{y}, \mathbf{s}) \cdot (te_{AA'}, \mathbf{0}))\bigg|_{t=0},$$

$$(S_{A'B'}f)(\mathbf{y}, \mathbf{s}) := \frac{\mathrm{d}}{\mathrm{d}t} f((\mathbf{y}, \mathbf{s}) \cdot (\mathbf{0}, te_{A'B'}))\bigg|_{t=0},$$

for $A, B = 0, 1$, where $e_{AB'}$ ($e_{A'B'}$) is a vector in $\mathbb{C}^4$ with all entries vanishing except for the (AB')-entry (($A'B'$)-entry) to be 1. We have left invariant vector fields on $\mathfrak{H}$:

$$\begin{aligned}
V_{00'} &:= \frac{\partial}{\partial y_{00'}} - 2y_{10'}S_{0'0'} - 2y_{11'}S_{1'0'}, \quad V_{01'} := \frac{\partial}{\partial y_{01'}} - 2y_{10'}S_{0'1'} - 2y_{11'}S_{1'1'},\\
V_{10'} &:= \frac{\partial}{\partial y_{10'}} - 2y_{00'}S_{0'0'} - 2y_{01'}S_{1'0'}, \quad V_{11'} := \frac{\partial}{\partial y_{11'}} - 2y_{00'}S_{0'1'} - 2y_{01'}S_{1'1'},\\
S_{A'B'} &:= \frac{\partial}{\partial s_{A'B'}},
\end{aligned} \tag{2.2}$$

where $A', B' = 0', 1'$. The left invariant horizontal vector fields satisfy the following proposition.

Proposition 2.1. *$\{V_{AA'}, S_{A'B'}\}$ spans a non-Abelian subalgebra, whose brackets satisfy*

$$[V_{00'}, V_{11'}] = [V_{10'}, V_{01'}] = -2S_{0'1'} + 2S_{1'0'}, \tag{2.3}$$

and all other brackets vanish.

Proof. By direct calculation, we have

$$\begin{aligned}[V_{00'}, V_{11'}] &= \left[\frac{\partial}{\partial y_{00'}} - 2y_{10'}S_{0'0'} - 2y_{11'}S_{1'0'}, \frac{\partial}{\partial y_{11'}} - 2y_{00'}S_{0'1'} - 2y_{01'}S_{1'1'}\right] \\ &= \frac{\partial}{\partial y_{00'}}(-2y_{00'}S_{0'1'}) - \frac{\partial}{\partial y_{11'}}(-2y_{11'}S_{1'0'}) = -2S_{0'1'} + 2S_{1'0'}.\end{aligned}$$

Similarly, we have $[V_{10'}, V_{01'}] = -2S_{0'1'} + 2S_{1'0'}$. □

Let 1-forms $\left\{\theta^{AA'}, \theta^{C'D'}\right\}$ be dual to left invariant vector fields $\{V_{AA'}, S_{C'D'}\}$ in (2.2) on $\mathfrak{H}$, i.e. $\theta^{AA'}(V_{BB'}) = \delta_{AB}\delta_{A'B'}$, $\theta^{AA'}(S_{C'D'}) = 0$, $\theta^{C'D'}(V_{BB'}) = 0$, $\theta^{A'B'}(S_{C'D'}) = \delta_{A'C'}\delta_{B'D'}$, where $A, B = 0, 1$ and $A', B', C', D' = 0', 1'$. Then for a function u on $\mathfrak{H}$, we have

$$\mathrm{d}u = \sum_{A,A'} V_{AA'}u \cdot \theta^{AA'} + \sum_{C',D'} S_{C'D'}u \cdot \theta^{C'D'}.$$

By the expression of $V_{AA'}$ in (2.2), we get that $\theta^{AA'} = \mathrm{d}y_{AA'}$ and

$$\theta^{0'0'} := \mathrm{d}s_{0'0'} + 2y_{10'}\mathrm{d}y_{00'} + 2y_{00'}\mathrm{d}y_{10'}, \quad \theta^{0'1'} := \mathrm{d}s_{0'1'} + 2y_{10'}\mathrm{d}y_{01'} + 2y_{00'}\mathrm{d}y_{11'},$$

$$\theta^{1'0'} := \mathrm{d}s_{1'0'} + 2y_{01'}\mathrm{d}y_{10'} + 2y_{11'}\mathrm{d}y_{00'}, \quad \theta^{1'1'} := \mathrm{d}s_{1'1'} + 2y_{11'}\mathrm{d}y_{01'} + 2y_{01'}\mathrm{d}y_{11'}.$$

Exterior differentiation gives us

$$\mathrm{d}\theta^{AA'} = 0, \quad \mathrm{d}\theta^{A'A'} = 0,$$

for $A = 0, 1, A' = 0', 1'$, and

$$\mathrm{d}\theta^{1'0'} = -2\theta^{00'} \wedge \theta^{11'} - 2\theta^{10'} \wedge \theta^{01'} = -\mathrm{d}\theta^{0'1'}.$$

For fixed $0 \neq (\pi_{0'}, \pi_{1'}) \in \mathbb{C}^2$, take

$$V_A := \pi_{0'}V_{A0'} - \pi_{1'}V_{A1'}, \quad A = 0, 1. \tag{2.4}$$

We have

$$[V_0, V_1] = 0,$$

by Proposition 2.1. Namely,

$$\mathrm{span}\{V_0, V_1\}$$

is an Abelian Lie subalgebra and an integrable distribution for fixed $0 \neq (\pi_{0'}, \pi_{1'}) \in \mathbb{C}^2$. Their integral surfaces are hyperplanes, which we also call *α-planes*. The *twistor space* $\mathcal{P}$ is the moduli space of all α-planes, which is a 7-D complex manifold. We have the double fibration over 8-D generalized complex Heisenberg group as follows

$$\mathcal{P} \xleftarrow{\eta} \mathcal{F} = \mathbb{C}^8 \times \mathbb{C}P^1 \xrightarrow{\tau} \mathfrak{H} \cong \mathbb{C}^8.$$

If we use the nonhomogeneous coordinates, $\zeta = \frac{\pi_{0'}}{\pi_{1'}}$, $\widetilde{\zeta} = \frac{\pi_{1'}}{\pi_{0'}}$, the vector field V_A in (2.4) can be rewritten as

$$V_A = \pi_{1'}V_A^{\zeta} = \pi_{0'}\widetilde{V}_A^{\widetilde{\zeta}},$$

where

$$V_A^{\zeta} = \zeta V_{A0'} - V_{A1'},$$

and

$$\widetilde{V}_A^{\widetilde{\zeta}} = V_{A0'} - \widetilde{\zeta}V_{A1'}.$$

We can describe α-planes, the integral surfaces of V_0, V_1, explicitly as follows. $\mathbb{C}^8 \times \mathbb{C}P^1$ is the complex manifold with two coordinate charts $\mathbb{C}^8 \times \mathbb{C}$ and $\mathbb{C}^8 \times \mathbb{C}$, glued by the mapping $\kappa : \mathbb{C}^8 \times \mathbb{C} \setminus \{0\} \to \mathbb{C}^8 \times \mathbb{C} \setminus \{0\}$ given by

$$(\mathbf{y}, \mathbf{s}, \zeta) \longmapsto \left(\mathbf{y}, \mathbf{s}, \zeta^{-1}\right).$$

Then if we use the nonhomogeneous coordinates, $\tau : \mathbb{C}^8 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}^8$ is given by $(\mathbf{y}, \mathbf{s}, \zeta) \longrightarrow (\mathbf{y}, \mathbf{s})$ and $\left(\mathbf{y}, \mathbf{s}, \widetilde{\zeta}\right) \longrightarrow (\mathbf{y}, \mathbf{s})$. Let us check that the integral surfaces of V_A^{ζ} and $V_A^{\widetilde{\zeta}}$ lifted to $\mathbb{C}^8 \times \mathbb{C}P^1$ by τ are the fiber of the mapping $\eta : \mathbb{C}^8 \times \mathbb{C} \to \mathbb{C}^7$ and $\widetilde{\eta} : \mathbb{C}^8 \times \mathbb{C} \to \mathbb{C}^7$, respectively.

Proposition 2.2. *One piece of the fiber of the mapping $\eta : \mathfrak{H} \times \mathbb{C} \to W \cong \mathbb{C}^7$ is given by*

$$\omega = \eta(\mathbf{y}, \mathbf{s}, \zeta) = \begin{pmatrix} \eta_0(\mathbf{y}, \mathbf{s}, \zeta) \\ \eta_1(\mathbf{y}, \mathbf{s}, \zeta) \\ \eta_2(\mathbf{y}, \mathbf{s}, \zeta) \\ \eta_3(\mathbf{y}, \mathbf{s}, \zeta) \\ \eta_4(\mathbf{y}, \mathbf{s}, \zeta) \\ \eta_5(\mathbf{y}, \mathbf{s}, \zeta) \\ \eta_6(\mathbf{y}, \mathbf{s}, \zeta) \end{pmatrix} = \begin{pmatrix} y_{00'} + \zeta y_{01'}, \\ y_{10'} + \zeta y_{11'}, \\ s_{0'0'} + \zeta s_{0'1'}, \\ s_{1'0'} + \zeta s_{1'1'}, \\ s_{0'0'} + \zeta\left(s_{0'1'} + s_{1'0'}\right) + \zeta^2 s_{1'1'}, \\ s_{0'0'} + s_{1'1'} + \langle \mathbf{y}_{0'}, \mathbf{y}_{0'} \rangle + \langle \mathbf{y}_{1'}, \mathbf{y}_{1'} \rangle, \\ \zeta, \end{pmatrix} \in W,$$

where

$$\langle y_{A'}, y_{B'} \rangle = y_{0A'} y_{1B'} + y_{1A'} y_{0B'},$$

for $A', B' = 0', 1'$. Then $\tau \circ \eta^{-1}(\omega)$ is a 4-D plane parameterized as

$$\begin{cases} y_{01'} = t_0, \\ y_{11'} = t_1, \\ s_{0'1'} = t_2, \\ s_{1'1'} = t_3, \\ y_{00'} = \omega_0 - \zeta t_0, \\ y_{10'} = \omega_1 - \zeta t_1, \\ s_{0'0'} = \omega_2 - \zeta t_2, \\ s_{1'0'} = \omega_3 - \zeta t_3, \end{cases}$$

with parameters $t_0, t_1, t_2, t_3 \in \mathbb{C}$. V_0^{ζ} and V_1^{ζ} are tangential to this plane, and so it is an α-plane.

Proof. For $A = 0$, we have

$$V_0^{\zeta} \omega_B = \left(\zeta V_{00'} - V_{01'}\right) \omega_B = \begin{cases} \zeta \delta_{0B} - \zeta \delta_{0B} = 0, & \text{when } B = 0, 1, \\ -2y_{10'}\zeta + 2y_{10'}\zeta = 0, & \text{when } B = 2, \\ -2y_{11'}\zeta + 2y_{11'}\zeta = 0, & \text{when } B = 3, \\ \zeta\left(-2y_{10'} - 2y_{11'}\zeta\right) - \left(-2y_{10'}\zeta - 2y_{11'}\zeta^2\right) = 0, & \text{when } B = 4. \end{cases} \tag{2.5}$$

Similarly, for $A = 1$, we have

$$V_1^{\zeta} \omega_B = \left(\zeta V_{10'} - V_{11'}\right) \omega_B = \begin{cases} \zeta \delta_{1B} - \zeta \delta_{1B} = 0, & \text{when } B = 0, 1, \\ -2y_{00'}\zeta + 2y_{00'}\zeta = 0, & \text{when } B = 2, \\ -2y_{01'}\zeta + 2y_{01'}\zeta = 0, & \text{when } B = 3, \\ \zeta\left(-2y_{00'} - 2y_{01'}\zeta\right) - \left(-2y_{00'}\zeta - 2y_{01'}\zeta^2\right) = 0, & \text{when } B = 4. \end{cases} \tag{2.6}$$

By expressions of $V_{AA'}$'s in (2.2), we have

$$\begin{aligned} V_{00'}(\omega_5) &= 2y_{10'} - 2y_{10'} = 0, \quad V_{01'}(\omega_5) = 2y_{11'} - 2y_{11'} = 0, \\ V_{10'}(\omega_5) &= 2y_{00'} - 2y_{00'} = 0, \quad V_{11'}(\omega_5) = 2y_{01'} - 2y_{01'} = 0. \end{aligned} \tag{2.7}$$

So we have

$$V_A^{\zeta}(\omega_5) = 0.$$

Note that for a fixed point $\omega = (\omega_0, \cdots, \omega_5, \zeta) \in W$, $\eta^{-1}(\omega)$ in $\mathbb{C}^8 \times \mathbb{C}$ has fixed last coordinate ζ. So $\tau \circ \eta^{-1}(\omega)$ is the plane determined by

$$\eta_0(\mathbf{y}, \mathbf{s}, \zeta) = \omega_0, \quad \cdots, \quad \eta_5(\mathbf{y}, \mathbf{s}, \zeta) = \omega_5.$$

The solutions of linear equations $\begin{cases} y_{00'} + \zeta y_{01'} = \omega_0, \\ y_{10'} + \zeta y_{11'} = \omega_1, \\ s_{0'0'} + \zeta s_{0'1'} = \omega_2, \\ s_{1'0'} + \zeta s_{1'1'} = \omega_3, \end{cases}$ are given by $y_{01'} = t_0$, $y_{00'} = \omega_0 - t_0\zeta$, $y_{11'} = t_1$, $y_{10'} = \omega_1 - t_1\zeta$, $s_{0'1'} = t_2$, $s_{0'0'} = \omega_2 - t_2\zeta$, $s_{1'1'} = t_3$, $s_{1'0'} = \omega_3 - t_3\zeta$ with parameters $t_0, t_1, t_2, t_3 \in \mathbb{C}$. □

On the other hand, if $\pi_{0'} \neq 0$, integral surfaces of $\widetilde{V}_0^{\widetilde{\zeta}}$ and $\widetilde{V}_1^{\widetilde{\zeta}}$ are fibers of the mapping $\widetilde{\eta} : \mathbb{C}^8 \times \mathbb{C} \longrightarrow \widetilde{W} \cong \mathbb{C}^7$ given by

$$\widetilde{\omega} = \widetilde{\eta}\left(\mathbf{y}, \mathbf{s}, \widetilde{\zeta}\right) = \begin{pmatrix} \widetilde{\eta}_0\left(\mathbf{y}, \mathbf{s}, \widetilde{\zeta}\right) \\ \widetilde{\eta}_1\left(\mathbf{y}, \mathbf{s}, \widetilde{\zeta}\right) \\ \widetilde{\eta}_2\left(\mathbf{y}, \mathbf{s}, \widetilde{\zeta}\right) \\ \widetilde{\eta}_3\left(\mathbf{y}, \mathbf{s}, \widetilde{\zeta}\right) \\ \widetilde{\eta}_4\left(\mathbf{y}, \mathbf{s}, \widetilde{\zeta}\right) \\ \widetilde{\eta}_5\left(\mathbf{y}, \mathbf{s}, \widetilde{\zeta}\right) \\ \widetilde{\eta}_6\left(\mathbf{y}, \mathbf{s}, \widetilde{\zeta}\right) \end{pmatrix} = \begin{pmatrix} \widetilde{\zeta} y_{00'} + y_{01'}, \\ \widetilde{\zeta} y_{10'} + y_{11'}, \\ \widetilde{\zeta} s_{0'0'} + s_{0'1'}, \\ \widetilde{\zeta} s_{1'0'} + s_{1'1'}, \\ \widetilde{\zeta}^2 s_{0'0'} + \widetilde{\zeta}\left(s_{0'1'} + s_{1'0'}\right) + s_{1'1'}, \\ s_{0'0'} + s_{1'1'} + \langle \mathbf{y}_{0'}, \mathbf{y}_{0'} \rangle + \langle \mathbf{y}_{1'}, \mathbf{y}_{1'} \rangle, \\ \widetilde{\zeta} \end{pmatrix} \in \widetilde{W}.$$

Then the transition function becomes

$$\begin{aligned} \Phi : \quad & W \setminus \{\zeta = 0\} \to \widetilde{W} \setminus \{\widetilde{\zeta} = 0\} \\ & (\omega_k, \omega_4, \omega_5, \zeta) \mapsto \left(\widetilde{\omega}_k, \widetilde{\omega}_4, \widetilde{\omega}_5, \widetilde{\zeta}\right) = \left(\zeta^{-1}\omega_k, \zeta^{-2}\omega_4, \omega_5, \zeta^{-1}\right), \end{aligned}$$

where $k = 0, 1, 2, 3$, which glues W and $\widetilde{W}$ to get a complex manifold $\mathcal{P}$. It is the moduli space of all α-planes, which is our twistor space.

§3 The Penrose integral formula

We denote by $\odot^p \mathbb{C}^2$ the p-th symmetric power of $\mathbb{C}^2$. Its element is denoted by a 2^p-tuple $(f_{A_1' \cdots A_p'})$, $A_1', \cdots, A_p' = 0', 1'$, which are invariant under permutations of subscripts. For $k = 1, 2, \cdots$, the tangential k-Cauchy–Fueter operator $\mathscr{D}_0^{(k)} : C^1\left(\mathfrak{H}, \odot^k \mathbb{C}^2\right) \to C^0\left(\mathfrak{H}, \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^2\right)$ is given by

$$\left(\mathscr{D}_0^{(k)} f\right)_{A_2' \cdots A_k' A} := \sum_{A_1' = 0', 1'} V_A^{A_1'} f_{A_1' A_2' \cdots A_k'}. \tag{3.1}$$

Here we use matrices

$$(\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \left(\varepsilon^{A'B'}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to raise or lower primed indices:

$$V_A^{A'} = \sum_{B'=0',1'} V_{AB'}\varepsilon^{B'A'},$$

i.e.

$$V_A^{0'} = V_{A1'}, \qquad V_A^{1'} = -V_{A0'}. \tag{3.2}$$

$\mathscr{D}_0^{(0)} : C^\infty(\mathfrak{H}, \mathbb{C}) \to C^\infty(\mathfrak{H}, \wedge^2\mathbb{C}^2)$ is given by

$$\left(\mathscr{D}_0^{(0)} f\right)_{[AB]} = \sum_{A'=0',1'} V_{A'[A}V_{B]}^{A'} f, \tag{3.3}$$

where $A, B = 0, 1$, and $A'_2, \cdots, A'_k = 0', 1'$. Here $2\phi_{[AB]} := \phi_{AB} - \phi_{BA}$ for $\phi \in C^\infty(\mathfrak{H}, \mathbb{C}^2 \otimes \mathbb{C}^2)$ is the skew-symmetrization, and we interchange the order of indices and write $V_{A'A} := V_{AA'}$, since it is convenient for taking skew-symmetrization. Therefore we have $\sum_{A'=0',1'} V_{A'[A}V_{B]}^{A'} = \sum_{A'=0',1'} \left(V_{AA'}V_B^{A'} - V_{BA'}V_A^{A'}\right)$.

A $\odot^k\mathbb{C}^2$-valued distribution f on $\Omega \subset \mathfrak{H}$ is called *k-CF* if $\mathscr{D}_0^{(k)} f = 0$ in the sense of distributions. For f holomorphic on $\mathbb{C}^7 \backslash \mathbb{C}^6$, where

$$\mathbb{C}^6 = \left\{\omega = (\omega_0, \omega_1, \cdots, \omega_6) \in \mathbb{C}^7 | \omega_6 = 0\right\},$$

we define the *Penrose-type* integral $\mathscr{P}_k f \in C^\infty(\mathfrak{H}, \odot^k\mathbb{C}^2)$ by

$$(\mathscr{P}_k f)_{A'_1 A'_2 \cdots A'_k}(\mathbf{y}, \mathbf{s}) := \oint_{|\zeta|=1} \zeta^{A'_1 + A'_2 + \cdots + A'_k} f(\omega_0, \omega_1, \cdots, \omega_6) \mathrm{d}\zeta. \tag{3.4}$$

where $A'_i = 0', 1'$. In fact, ω is the mapping from $\mathfrak{H} \times \mathbb{C}$ to the *twistor space* $\mathcal{P}$ of the generalized complex Heisenberg group $\mathfrak{H}$.

Theorem 3.1. *For $k = 0, 1, 2, \cdots$, $\mathscr{P}_k f$ is k-CF if f is holomorphic on $\mathbb{C}^7 \backslash \mathbb{C}^6$.*

Proof. (1) For $k = 1, 2, \cdots$, act $V_{AA'}$ on both sides of (3.4) and use the chain rule of derivatives and (2.7) to get,

$$\begin{aligned}
&V_{00'}(\mathscr{P}_k f)_{A'_1 \cdots A'_k}(\mathbf{y}, \mathbf{s}) \\
&= \oint_{|\zeta|=1} \zeta^{A'_1 + \cdots + A'_k} \left[\frac{\partial f}{\partial \omega_0}(\omega) - 2y_{10'}\left(\frac{\partial f}{\partial \omega_2}(\omega) + \frac{\partial f}{\partial \omega_4}(\omega)\right) - 2y_{11'}\left(\frac{\partial f}{\partial \omega_3}(\omega) + \zeta\frac{\partial f}{\partial \omega_4}(\omega)\right)\right] \mathrm{d}\zeta, \\
&V_{01'}(\mathscr{P}_k f)_{A'_1 \cdots A'_k}(\mathbf{y}, \mathbf{s}) = \oint_{|\zeta|=1} \zeta^{A'_1 + \cdots + A'_k + 1} \\
&\left[\frac{\partial f}{\partial \omega_0}(\omega) - 2y_{10'}\left(\frac{\partial f}{\partial \omega_2}(\omega) + \frac{\partial f}{\partial \omega_4}(\omega)\right) - 2y_{11'}\left(\frac{\partial f}{\partial \omega_3}(\omega) + \zeta\frac{\partial f}{\partial \omega_4}(\omega)\right)\right] \mathrm{d}\zeta.
\end{aligned} \tag{3.5}$$

So we have

$$V_{00'}(\mathscr{P}_k f)_{1' A'_2 \cdots A'_k}(\mathbf{y}, \mathbf{s}) = V_{01'}(\mathscr{P}_k f)_{0' A'_2 \cdots A'_k}(\mathbf{y}, \mathbf{s}),$$

which is equivalent to

$$\sum_{A'_1 = 0', 1'} V_0^{A'_1}(\mathscr{P}_k f)_{A'_1 A'_2 \cdots A'_k}(\mathbf{y}, \mathbf{s}) = 0,$$

by (3.2). Similarly, for $A=1$,

$$V_{10'}(\mathscr{P}_k f)_{A'_1\cdots A'_k}(\mathbf{y},\mathbf{s})$$
$$=\oint_{|\zeta|=1}\zeta^{A'_1+\cdots+A'_k}\left[\frac{\partial f}{\partial\omega_1}(\omega)-2y_{00'}\left(\frac{\partial f}{\partial\omega_2}(\omega)+\frac{\partial f}{\partial\omega_4}(\omega)\right)-2y_{01'}\left(\frac{\partial f}{\partial\omega_3}(\omega)+\zeta\frac{\partial f}{\partial\omega_4}(\omega)\right)\right]\mathrm{d}\zeta,$$
$$V_{11'}(\mathscr{P}_k f)_{A'_1\cdots A'_k}(\mathbf{y},\mathbf{s})=\oint_{|\zeta|=1}\zeta^{A'_1+\cdots+A'_k+1}$$
$$\left[\frac{\partial f}{\partial\omega_1}(\omega)-2y_{00'}\left(\frac{\partial f}{\partial\omega_2}(\omega)+\frac{\partial f}{\partial\omega_4}(\omega)\right)-2y_{01'}\left(\frac{\partial f}{\partial\omega_3}(\omega)+\zeta\frac{\partial f}{\partial\omega_4}(\omega)\right)\right]\mathrm{d}\zeta. \tag{3.6}$$

Then by (3.2) and (3.6), we have

$$\sum_{A'_1=0',1'}V_1^{A'_1}(\mathscr{P}_k f)_{A'_1A'_2\cdots A'_k}(\mathbf{y},\mathbf{s})=0.$$

Namely, $\mathscr{D}_0^{(k)}(\mathscr{P}_k f)(\mathbf{y},\mathbf{s})=0$ by (3.1).

(2) For $k=0$, by (3.2) and the brackets in (2.3), we have

$$\begin{aligned}2\sum_{A'=0',1'}V_{A'[A}V_{B]}^{A'}&=\sum_{A'=0',1'}V_{A'A}V_B^{A'}-\sum_{A'=0',1'}V_{A'B}V_A^{A'}\\&=V_{A0'}V_{B1'}-V_{A1'}V_{B0'}-V_{B0'}V_{A1'}+V_{B1'}V_{A0'}\\&=2(V_{A0'}V_{B1'}-V_{B0'}V_{A1'})-[V_{A0'},V_{B1'}]+[V_{B0'},V_{A1'}].\end{aligned}$$

So for $A=0$, $B=1$,

$$2\sum_{A'=0',1'}V_{A'[0}V_{1]}^{A'}=2(V_{00'}V_{11'}-V_{10'}V_{01'}).$$

By (3.5) and (3.6), we have

$$\begin{aligned}&V_{00'}V_{11'}(\mathscr{P}_k f)_{A'_1\cdots A'_k}(\mathbf{y},\mathbf{s})=\oint_{|\zeta|=1}\zeta^{A'_1+\cdots+A'_k+1}\left\{-2\left(\frac{\partial f}{\partial\omega_2}(\omega)+\frac{\partial f}{\partial\omega_4}(\omega)\right)\right.\\
&+\left[\frac{\partial^2 f}{\partial\omega_0\partial\omega_1}(\omega)-2y_{10'}\left(\frac{\partial^2 f}{\partial\omega_2\partial\omega_1}(\omega)+\frac{\partial^2 f}{\partial\omega_4\partial\omega_1}(\omega)\right)-2y_{11'}\left(\frac{\partial^2 f}{\partial\omega_3\partial\omega_1}(\omega)+\zeta\frac{\partial^2 f}{\partial\omega_4\partial\omega_1}(\omega)\right)\right]\\
&-2y_{00'}\left[\frac{\partial^2 f}{\partial\omega_0\partial\omega_2}(\omega)-2y_{10'}\left(\frac{\partial^2 f}{\partial\omega_2^2}(\omega)+\frac{\partial^2 f}{\partial\omega_4\partial\omega_2}(\omega)\right)-2y_{11'}\left(\frac{\partial^2 f}{\partial\omega_3\partial\omega_2}(\omega)+\zeta\frac{\partial^2 f}{\partial\omega_4\partial\omega_2}(\omega)\right)\right]\\
&-2y_{00'}\left[\frac{\partial^2 f}{\partial\omega_0\partial\omega_4}(\omega)-2y_{10'}\left(\frac{\partial^2 f}{\partial\omega_2\partial\omega_4}(\omega)+\frac{\partial^2 f}{\partial\omega_4^2}(\omega)\right)-2y_{11'}\left(\frac{\partial^2 f}{\partial\omega_3\partial\omega_4}(\omega)+\zeta\frac{\partial^2 f}{\partial\omega_4^2}(\omega)\right)\right]\\
&-2y_{01'}\left[\frac{\partial^2 f}{\partial\omega_0\partial\omega_3}(\omega)-2y_{10'}\left(\frac{\partial^2 f}{\partial\omega_2\partial\omega_3}(\omega)+\frac{\partial^2 f}{\partial\omega_4\partial\omega_3}(\omega)\right)-2y_{11'}\left(\frac{\partial^2 f}{\partial\omega_3^2}(\omega)+\zeta\frac{\partial^2 f}{\partial\omega_4\partial\omega_3}(\omega)\right)\right]\\
&\left.-2y_{01'}\zeta\left[\frac{\partial^2 f}{\partial\omega_0\partial\omega_4}(\omega)-2y_{10'}\left(\frac{\partial^2 f}{\partial\omega_2\partial\omega_4}(\omega)+\frac{\partial^2 f}{\partial\omega_4^2}(\omega)\right)-2y_{11'}\left(\frac{\partial^2 f}{\partial\omega_3\partial\omega_4}(\omega)+\zeta\frac{\partial^2 f}{\partial\omega_4^2}(\omega)\right)\right]\right\}\\
&\mathrm{d}\zeta,\end{aligned}$$

and

$$\begin{aligned}&V_{10'}V_{01'}(\mathscr{P}_k f)_{A'_1\cdots A'_k}(\mathbf{y},\mathbf{s})=\oint_{|\zeta|=1}\zeta^{A'_1+\cdots+A'_k+1}\left\{-2\left(\frac{\partial f}{\partial\omega_2}(\omega)+\frac{\partial f}{\partial\omega_4}(\omega)\right)\right.\\
&+\left[\frac{\partial^2 f}{\partial\omega_1\partial\omega_0}(\omega)-2y_{00'}\left(\frac{\partial^2 f}{\partial\omega_2\partial\omega_0}(\omega)+\frac{\partial^2 f}{\partial\omega_4\partial\omega_0}(\omega)\right)-2y_{01'}\left(\frac{\partial^2 f}{\partial\omega_3\partial\omega_0}(\omega)+\zeta\frac{\partial^2 f}{\partial\omega_4\partial\omega_0}(\omega)\right)\right]\end{aligned}$$

$$- 2y_{10'}\left[\frac{\partial^2 f}{\partial\omega_1\partial\omega_2}(\omega) - 2y_{00'}\left(\frac{\partial^2 f}{\partial\omega_2^2}(\omega) + \frac{\partial^2 f}{\partial\omega_4\partial\omega_2}(\omega)\right) - 2y_{01'}\left(\frac{\partial^2 f}{\partial\omega_3\partial\omega_2}(\omega) + \zeta\frac{\partial^2 f}{\partial\omega_4\partial\omega_2}(\omega)\right)\right]$$
$$- 2y_{10'}\left[\frac{\partial^2 f}{\partial\omega_1\partial\omega_4}(\omega) - 2y_{00'}\left(\frac{\partial^2 f}{\partial\omega_2\partial\omega_4}(\omega) + \frac{\partial^2 f}{\partial\omega_4^2}(\omega)\right) - 2y_{01'}\left(\frac{\partial^2 f}{\partial\omega_3\partial\omega_4}(\omega) + \zeta\frac{\partial^2 f}{\partial\omega_4^2}(\omega)\right)\right]$$
$$- 2y_{11'}\left[\frac{\partial^2 f}{\partial\omega_1\partial\omega_3}(\omega) - 2y_{00'}\left(\frac{\partial^2 f}{\partial\omega_2\partial\omega_3}(\omega) + \frac{\partial^2 f}{\partial\omega_4\partial\omega_3}(\omega)\right) - 2y_{01'}\left(\frac{\partial^2 f}{\partial\omega_3^2}(\omega) + \zeta\frac{\partial^2 f}{\partial\omega_4\partial\omega_3}(\omega)\right)\right]$$
$$-2y_{11'}\zeta\left[\frac{\partial^2 f}{\partial\omega_1\partial\omega_4}(\omega) - 2y_{00'}\left(\frac{\partial^2 f}{\partial\omega_2\partial\omega_4}(\omega) + \frac{\partial^2 f}{\partial\omega_4^2}(\omega)\right) - 2y_{01'}\left(\frac{\partial^2 f}{\partial\omega_3\partial\omega_4}(\omega) + \zeta\frac{\partial^2 f}{\partial\omega_4^2}(\omega)\right)\right]\Bigg\}$$

$d\zeta$.
So we have

$$\left(D_0^{(0)}\mathscr{P}_k f\right)_{[AB]} = \sum_{A'=0',1'} V_{A'[A}V_{B]}^{A'}(\mathscr{P}_k f) = 0.$$

The theorem is proved. □

§4 The tangential k-Cauchy-Fueter type complex over $\mathfrak{H}$

The second differential operator in (1.3) is as follows. $\mathscr{D}_1^{(k)} : C^\infty(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}) \to C^\infty(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2)$ is given by

$$\left(\mathscr{D}_1^{(k)} f\right)_{A_0A_1A_1'\cdots A_{k-2}'} = 2\sum_{A'=0',1'} V_{[A_0}^{A'} f_{A_1]A'A_1'\cdots A_{k-2}'}. \tag{4.1}$$

The following proposition holds directly by (2.3) and (3.2).

Proposition 4.1. $\left[V_A^{A'}, V_B^{B'}\right] = 0$ *except for*

$$\left[V_0^{1'}, V_1^{0'}\right] = \left[V_1^{1'}, V_0^{0'}\right] = 2S_{0'1'} - 2S_{1'0'}.$$

In particular, we have

$$\left[V_A^{0'}, V_B^{1'}\right] + \left[V_A^{1'}, V_B^{0'}\right] = 0, \tag{4.2}$$

for any $A, B = 0, 1$.

Lemma 4.1.

$$V_{[A}^{(A'}V_{B]}^{B')} = 0,$$

for any $A, B = 0, 1$ *and* $A', B' = 0', 1'$.

Proof. Note that

$$2V_{[A}^{A'}V_{B]}^{A'} = V_A^{A'}V_B^{A'} - V_B^{A'}V_A^{A'} = \left[V_A^{A'}, V_B^{A'}\right] = 0,$$

by Proposition 4.1, and

$$2V_{[A}^{0'}V_{B]}^{1'} + 2V_{[A}^{1'}V_{B]}^{0'} = V_A^{0'}V_B^{1'} - V_B^{0'}V_A^{1'} + V_A^{1'}V_B^{0'} - V_B^{1'}V_A^{0'} = \left[V_A^{0'}, V_B^{1'}\right] + \left[V_A^{1'}, V_B^{0'}\right] = 0,$$

by (4.2). The lemma is proved. □

Now we can check (1.3) to be a complex by direct calculation.

Theorem 4.1. (1.3) *is a complex, i.e.*

$$\mathscr{D}_1^{(k)} \circ \mathscr{D}_0^{(k)} = 0.$$

Proof. For $A, B = 0, 1$ and $A'_3, \cdots, A'_k = 0', 1'$, we have

$$\begin{aligned}\left(\mathscr{D}_1^{(k)} \circ \mathscr{D}_0^{(k)} f\right)_{ABA'_3\cdots A'_k} =& 2 \sum_{A'=0',1'} V^{A'}_{[A} \left(\mathscr{D}_0^{(k)} f\right)_{B]A'A'_3\cdots A'_k} \\ =& 2 \sum_{A',B'=0',1'} V^{A'}_{[A} V^{B'}_{B]} f_{B'A'A'_3\cdots A'_k} \\ =& 2 \sum_{A',B'=0',1'} V^{(A'}_{[A} V^{B')}_{B]} f_{B'A'A'_3\cdots A'_k} = 0,\end{aligned}$$

by Lemma 4.1 and $f_{B'A'A'_3\cdots A'_k} = f_{A'B'A'_3\cdots A'_k}$. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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