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Maximal operators of pseudo-differential operators with rough symbols

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Abstract. Consider a pseudo-differential operator

$$T_a f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \widehat{f}(\xi)\, \mathrm{d}\xi$$

where the symbol a is in the rough Hörmander class $L^\infty S^m_\rho$ with $m \in \mathbb{R}$ and $\rho \in [0,1]$. In this note, when $1 \le p \le 2$, if $m < \frac{n(\rho-1)}{p}$ and $a \in L^\infty S^m_\rho$, then for any $f \in S(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we prove that

$$M(T_a f)(x) \le C\,(M(|f|^p)(x))^{\frac{1}{p}}$$

where M is the Hardy-Littlewood maximal operator. Our theorem improves the known results and the bound on m is sharp, in the sense that $\frac{n(\rho-1)}{p}$ can not be replaced by a larger constant.

§1 Introduction

Pseudo-differential operators are extensively used in the theory of partial differential equations and quantum field theory. A systematic study of these operators was initiated by Kohn and Nirenberg [12] and Hörmander [5]. For a f in the Schwartz class $S(\mathbb{R}^n)$, the pseudo-differential operator T_a is defined by

$$T_a f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\widehat{f}(\xi) d\xi$$

where $\widehat{f}$ is the Fourier transform of f and the symbol a belongs to a certain symbol class. One of the most important symbol classes is the Hörmander class $S^m_{\rho,\delta}$ introduced in Hörmander [6]. A function $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ belongs to $S^m_{\rho,\delta}$ $(m \in \mathbb{R}, 0 \le \rho, \delta \le 1)$, if it satisfies

$$\sup_{x,\xi\in\mathbb{R}^n} (1+|\xi|)^{-m+\rho|\alpha|-\delta|\beta|} \left|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)\right| < +\infty$$

for all multi-indices α and β.

For pseudo-differential operators, an important problem is whether they are bounded on the Lebesgue space $L^p(\mathbb{R}^n)$. This problem has been extensively studied and there are numerous

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results. In particular, if $a \in S^m_{\rho,\delta}$ with $\delta < 1$, $m \le \min\{0, \frac{n(\rho-\delta)}{2}\}$, then T_a is bounded on L^2. One can see Hörmander [7], Hounie [8], etc. For $a \in S^m_{\rho,1}$ Rodino [14] proved that T_a is bounded on L^2 if $m < \frac{n(\rho-1)}{2}$ and constructed a symbol $a \in S^{\frac{n(\rho-1)}{2}}_{\rho,1}$ such that T_a is unbounded on L^2. (For the failure of L^2 boundedness for symbols in $S^0_{1,1}$, one can also see Ching [3] and Stein [15, P. 272].) For endpoint estimates, in some unpublished lecture notes, Stein showed that if $a \in S^{\frac{n(\rho-1)}{2}}_{\rho,\delta}$ and either $0 \le \delta < \rho = 1$ or $0 < \delta = \rho < 1$, then T_a is of weak type $(1,1)$ and bounded from H^1 to L^1. These results were extended in Álvarez and Hounie [1] to $0 < \rho \le 1, 0 \le \delta < 1$. The L^p boundedness follows from an interpolation argument.

In [10], Kenig and Staubach defined and studied the following class of rough symbols which behave in the spatial variable x like an L^∞ function. A function a, which is smooth in the frequency variable ξ and bounded measurable in the spatial variable x, belongs to the rough Hörmander class $L^\infty S^m_\rho$ $(m \in \mathbb{R}, 0 \le \rho \le 1)$, if it satisfies

$$\sup_{\xi \in \mathbb{R}^n} (1+|\xi|)^{-m+\rho|\alpha|} \left\|\partial^\alpha_\xi a(\cdot,\xi)\right\|_{L^\infty(\mathbb{R}^n)} < +\infty$$

for all multi-indices α. It is easy to see that $S^m_{\rho,1} \subset L^\infty S^m_\rho$.

When $a \in L^\infty S^m_\rho$ and $1 \le p \le 2$, Kenig and Staubach showed that the operator T_a is bounded on L^p if $m < \frac{n(\rho-1)}{p}$ ([10] Proposition 2.3).

It is interesting to further investigate the pointwise estimates for the pseudo-differential operators. Let us recall some notations. For a locally integrable function f on $\mathbb{R}^n$, the Hardy-Littlewood maximal operator and sharp maximal function of f are defined by

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)|dy,$$

$$f^\#(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - f_B|dy$$

where the supremum is taken over all balls B containing x and $f_B = \frac{1}{|B|}\int_B f(x)dx$ is the mean value of f on the ball B. It is obviously that $f^\#(x) \le 2Mf(x)$. Besides, it is well-known that $f^\# \in L^\infty$ characterizes the BMO space, i.e.

$$BMO = \{f \in L^1_{loc}(\mathbb{R}^n) : f^\# \in L^\infty\}.$$

We define the general Hardy-Littlewood maximal operator of f as

$$M_p f(x) = (M(|f|^p)(x))^{\frac{1}{p}} = \sup_{x \in B} \left(\frac{1}{|B|}\int_B |f(y)|^p dy\right)^{1/p}.$$

When $p = 1$, it is the standard Hardy-Littlewood maximal operator. As M is bounded on $L^q(1 < q \le \infty)$, it is easy to see that M_p is bounded on L^q ($q \ge p$ and $q > 1$).

The estimates for the sharp maximal functions of the pseudo-differential operators have been studied extensively.

1. Miller [13] showed that for $a \in S^0_{1,0}$ and $1 < p < \infty$, there holds
$$(T_a f)^\#(x) \le CM_p f(x), \forall x \in \mathbb{R}^n. \tag{1}$$
It was extended for $a \in S^0_{1,\delta}(0 \le \delta < 1)$ by Journé [9].

2. For $a \in S^{\frac{n(\rho-1)}{2}}_{\rho,\delta}$ $(0 \le \delta < \rho \le 1)$, Chanillo and Torchinsky [2] proved that
$$(T_a f)^\#(x) \le CM_2 f(x), \forall x \in \mathbb{R}^n \tag{2}$$

and asked whether $p = 2$ was the smallest value such that $(T_a f)^{\#}(x) \leq CM_p f(x)$.
In the same paper, they obtained the inequality (1) for $a(x,\xi) = \sigma(x,\xi)e^{i|\xi|^b}, \sigma \in S_{1,0}^{-\frac{nb}{2}}, 0 < b < 1$ and this result was improved in [17].

3. For $a \in S^m_{\rho,\delta}$ $(0 < \rho \leq 1, 0 \leq \delta < 1)$, Álvarez and Hounie ([1] Theorem 4.1) proved the inequality (1) if $m \leq n(\rho - 1) - \mu$ where
$$2\mu = 1 + n(\rho + \lambda) - \sqrt{\{1 + n(\rho + \lambda)\}^2 - 4n\lambda}$$
and $\lambda = \max(0, \frac{\delta-\rho}{2})$. Here, it is easy to check that $\mu \geq 0$ and $\mu = 0$ only if $\delta \leq \rho$.

4. Kim and Shin [11] generalized the results in [2] to $0 < \rho \leq \delta \leq 1$.
For $a \in S_{\rho,\delta}^{\frac{n(2\rho-\delta-1)}{2}}$ $(0 < \rho \leq \delta < 1)$ or $a \in S^m_{\rho,1}$ $(0 < \rho < 1,\ m < (\frac{n}{2} + 1)(\rho - 1))$, they proved the inequality (2).

5. Recently, for $a \in L^\infty S^m_\rho, m < n(\rho - 1)$, in [16] Wang and Chen used the following inequality to get the weighted inequalities for rough pseudo-differential operators,
$$|T_a f(x)| \leq CMf(x).$$

In this note, we consider the pseudo-differential operator T_a for $f \in S(\mathbb{R}^n), a \in L^\infty S^m_\rho$ and obtain the following theorem.

Theorem 1.1. *When $1 \leq p \leq 2$, if $m < \frac{n(\rho-1)}{p}$ and $a \in L^\infty S^m_\rho$, then for any $f \in S(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, there holds*
$$M(T_a f)(x) \leq C(M(|f|^p)(x))^{\frac{1}{p}}. \tag{3}$$

As a direct corollary, take $p = 2$, then we have

Corollary 1.1. *When $a \in L^\infty S^m_\rho$ with $m < \frac{n(\rho-1)}{2}$, we have*
$$M(T_a f)(x) \leq C(M(|f|^2)(x))^{\frac{1}{2}} \leq C\|f\|_\infty. \tag{4}$$

Remark 1. As $f^{\#}(x) \leq 2Mf(x)$ and $S^m_{\rho,1} \subset L^\infty S^m_\rho$, our theorem is a strict improvement of the corresponding results. For example, as a direct corollary, we generate the results of Miller [13] and Journé [9] to $a \in S^0_{1,1}$. For the theorem of Álvarez and Hounie, we remove the restriction on δ and improve the bound of m from $n(\rho - 1) - \mu$ to $n(\rho - 1)$ (note that $\mu > 0$ when $\rho < \delta$). For the theorem of Kim and Shin, we improve the bound of m from $(\frac{n}{2} + 1)(\rho - 1)$ to $\frac{n(\rho-1)}{2}$.
Remark 2. Furthermore, the upper bound on m is sharp in our theorem, because in general T_a is not bounded on L^p for some $m > \frac{n(\rho-1)}{p}$ and $a \in L^\infty S^m_\rho$. One can see [14] for $p = 2$ and [4] for $1 \leq p < 2$.
Remark 3. By the similar computations, we can get some corresponding results for the Fourier integral operators. But the results that we can get now are not satisfactory.

Our result can be used in some weighted norm inequalities, but here we omit the details. Throughout this note, without furthermore illustrations, the letter C is used to denote a positive constant that may depend on n, m, ρ, p and the quasi-norms of $a \in L^\infty S^m_\rho$. Furthermore, it may vary in different occurrences.

§2 Proof of the main result

Here we use $B)_r$ to denote the ball centered at origin with radius r. At first, we recall the Littlewood-Paley theory. Take $\eta \in C_c^\infty(B_2)$ such that $\eta = 1$ in B_1. Then for any $\xi \in \mathbb{R}^n$ there holds

$$\eta(\xi) + \sum_{j=1}^{\infty}[\eta(2^{-j}\xi) - \eta(2^{1-j}\xi)] = 1.$$

Set $a_0(x,\xi) = \eta(\xi)a(x,\xi)$ and $a_j(x,\xi) = [\eta(2^{-j}\xi) - \eta(2^{1-j}\xi)]a(x,\xi)\,(j > 0)$. It is easy to see that $a_j(x,\cdot)$ is supported in $\{\xi : |\xi| < 2^{1+j}\}$ and

$$|\partial_\xi^\alpha a_j(x,\xi)| \le C2^{j(m-\rho|\alpha|)}$$

for all multi-indices α.

Now, by the standard arguments, we can decompose T_a as

$$\begin{aligned} T_a f(x) &= \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\widehat{f}(\xi)d\xi \\ &= \sum_{j=0}^{\infty}\int_{\mathbb{R}^n} e^{ix\cdot\xi} a_j(x,\xi)\widehat{f}(\xi)d\xi \\ &= \sum_{j=0}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_j(x,\xi)f(y)dyd\xi \\ &= \sum_{j=0}^{\infty}\int_{\mathbb{R}^n} k_j(x,x-y)f(y)dy \end{aligned}$$

where $k_j(x,z) = \int_{\mathbb{R}^n} e^{iz\cdot\xi}a_j(x,\xi)d\xi$.

Set $\sigma_j(z) = 2^{-\frac{jn\rho}{p}}(1 + 2^{j\rho}|z|)^{n+1}$. For $1 \le p \le 2$, we have $p' = \frac{p}{p-1} \ge 2$. Now, for any $x \in \mathbb{R}^n$, by some simple computations and Hausdorff-Young inequality, we can yield that

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} |k_j(x,x-y)\sigma_j(x-y)|^{p'}dy\right)^{\frac{1}{p'}} \\ &= 2^{-\frac{jn\rho}{p}}\left(\int_{\mathbb{R}^n} |k_j(x,z)(1+2^{j\rho}|z|)^{n+1}|^{p'}dz\right)^{\frac{1}{p'}} \\ &\le C2^{-\frac{jn\rho}{p}}\sum_{|\alpha|\le n+1}\left(\int_{\mathbb{R}^n} |2^{j\rho|\alpha|}z^\alpha k_j(x,z)|^{p'}dz\right)^{\frac{1}{p'}} \\ &= C2^{-\frac{jn\rho}{p}}\sum_{|\alpha|\le n+1}\left(\int_{\mathbb{R}^n} \left|2^{j\rho|\alpha|}z^\alpha \int_{\mathbb{R}^n} e^{iz\cdot\xi}a_j(x,\xi)d\xi\right|^{p'}dz\right)^{\frac{1}{p'}} \\ &= C2^{-\frac{jn\rho}{p}}\sum_{|\alpha|\le n+1}\left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} e^{iz\cdot\xi}\left(2^{j\rho|\alpha|}\partial_\xi^\alpha a_j(x,\xi)\right)d\xi\right|^{p'}dz\right)^{\frac{1}{p'}} \\ &\le C2^{-\frac{jn\rho}{p}}\sum_{|\alpha|\le n+1}\left(\int_{\mathbb{R}^n} |2^{j\rho|\alpha|}\partial_\xi^\alpha a_j(x,\xi)|^p d\xi\right)^{\frac{1}{p}} \\ &\le C2^{-\frac{jn\rho}{p}}\sum_{|\alpha|\le n+1}\left(\int_{|\xi|<2^{1+j}} 2^{jmp}d\xi\right)^{\frac{1}{p}} \\ &\le C2^{j(m-\frac{n(\rho-1)}{p})}. \end{aligned} \tag{5}$$

Here the assumption $1 \le p \le 2$ is necessary to use Hausdorff-Young inequality.

So, from (5) we can get that

$$\begin{aligned}
|T_a f(x)| &= \left|\sum_{j=0}^{\infty}\int_{\mathbb{R}^n} k_j(x,x-y)f(y)dy\right| \\
&\le \sum_{j=0}^{\infty}\int_{\mathbb{R}^n}\left|k_j(x,x-y)\sigma_j(x-y)\frac{f(y)}{\sigma_j(x-y)}\right|dy \\
&\le \sum_{j=0}^{\infty}\left(\int_{\mathbb{R}^n}|k_j(x,x-y)\sigma_j(x-y)|^{p'}dy\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n}\frac{|f(y)|^p}{\sigma_j^p(x-y)}dy\right)^{\frac{1}{p}} \\
&\le C\sum_{j=0}^{\infty}2^{j(m-\frac{n(\rho-1)}{p})}\left(\int_{\mathbb{R}^n}\frac{|f(y)|^p}{\sigma_j^p(x-y)}dy\right)^{\frac{1}{p}}. \qquad (6)
\end{aligned}$$

In the second inequality we use the Hölder inequality.

When $m<\frac{n(\rho-1)}{p}$, for any $x\in\mathbb{R}^n$ and $r>0$, from (6) we have

$$\begin{aligned}
\int_{B(x,r)}|T_a f(z)|dz &\le C\sum_{j=0}^{\infty}2^{j(m-\frac{n(\rho-1)}{p})}\int_{B(x,r)}\left(\int_{\mathbb{R}^n}\frac{|f(y)|^p}{\sigma_j^p(z-y)}dy\right)^{\frac{1}{p}}dz \\
&\le C\sup_j\int_{B(x,r)}\left(\int_{\mathbb{R}^n}\frac{|f(y)|^p}{\sigma_j^p(z-y)}dy\right)^{\frac{1}{p}}dz \\
&\le Cr^{n(1-\frac{1}{p})}\sup_j\left(\int_{B(x,r)}\int_{\mathbb{R}^n}\frac{|f(y)|^p}{\sigma_j^p(z-y)}dydz\right)^{\frac{1}{p}} \\
&= Cr^{n(1-\frac{1}{p})}\sup_j\left(\int_{\mathbb{R}^n}\int_{B(x,r)}\frac{2^{jn\rho}}{(1+2^{j\rho}|z-y|)^{(n+1)p}}dz|f(y)|^p dy\right)^{\frac{1}{p}}. \qquad (7)
\end{aligned}$$

When $|y-x|<2r$, it is easy to check that

$$\begin{aligned}
\int_{B(x,r)}\frac{2^{jn\rho}}{(1+2^{j\rho}|z-y|)^{(n+1)p}}dz &\le C\int_{B(y,3r)}\frac{2^{jn\rho}}{(1+2^{j\rho}|z-y|)^{(n+1)p}}dz \\
&\le C\int_{|w|<3\cdot 2^{j\rho}r}\frac{1}{(1+|w|)^{(n+1)p}}dw \\
&\le C\min\{1,2^{jn\rho}r^n\}\le C. \qquad (8)
\end{aligned}$$

On the other hand, when $|y-x|\ge 2r$, one can easily get that

$$\begin{aligned}
\int_{B(x,r)}\frac{2^{jn\rho}}{(1+2^{j\rho}|z-y|)^{(n+1)p}}dz &\le C\int_{B(x,r)}\frac{2^{jn\rho}}{(1+2^{j\rho}|x-y|)^{(n+1)p}}dz \\
&\le \frac{C2^{jn\rho}r^n}{(1+2^{j\rho}|x-y|)^{(n+1)p}}. \qquad (9)
\end{aligned}$$

Set $\Phi(w)=(1+|w|)^{-(n+1)p}$. When $p\ge 1$, using the well-known maximal function estimate for approximations of the identity, we have

$$\sup_j\int_{\mathbb{R}^n}2^{jn\rho}(1+2^{j\rho}|y-x|)^{-(n+1)p}|f(y)|^p dy$$

$$
\begin{aligned}
&= \sup_j 2^{jn\rho} \int_{\mathbb{R}^n} \Phi(2^{j\rho}(x-y))|f(y)|^p dy \\
&\leq \|\Phi\|_1 M(|f|^p)(x) = CM(|f|^p)(x). \qquad (10)
\end{aligned}
$$

Therefore, if $m < \frac{n(\rho-1)}{p}$, for any $x \in \mathbb{R}^n$ and $r > 0$, from (7), (8), (9) and (10), we can obtain that

$$
\begin{aligned}
& r^{-n} \int_{B(x,r)} |T_a f(z)| dz \\
\leq\ & Cr^{-\frac{n}{p}} \sup_j \left(\int_{\mathbb{R}^n} \int_{B(x,r)} \frac{2^{jn\rho}}{(1+2^{j\rho}|z-y|)^{(n+1)p}} dz |f(y)|^p dy \right)^{\frac{1}{p}} \\
\leq\ & C \sup_j \left(r^{-n} \int_{|y-x|<2r} |f(y)|^p dy + r^{-n} \int_{|y-x|\geq 2r} \frac{2^{jn\rho} r^n}{(1+2^{j\rho}|x-y|)^{(n+1)p}} |f(y)|^p dy \right)^{\frac{1}{p}} \\
\leq\ & C \left(M(|f|^p)(x) + \sup_j \int_{\mathbb{R}^n} 2^{jn\rho}(1+2^{j\rho}|y-x|)^{-(n+1)p} |f(y)|^p dy \right)^{\frac{1}{p}} \\
\leq\ & C(M(|f|^p)(x))^{\frac{1}{p}}.
\end{aligned}
$$

Thus we prove that

$$
M(T_a f)(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |T_a f(z)| dz \leq C(M(|f|^p)(x))^{\frac{1}{p}}.
$$

Declarations

Conflict of interest The authors declare no conflict of interest.

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