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Radial solution of the Logarithmic Laplacian system

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Abstract. The paper generalizes the direct method of moving planes to the Logarithmic Laplacian system. Firstly, some key ingredients of the method are discussed, for example, *Narrow region principle and Decay at infinity.* Then, the radial symmetry of the solution of the Logarithmic Laplacian system is obtained.

§1 Introduction

In recent years, the research on boundary value of linear and nonlinear integro differential operators has attracted extensive attention, among which the more popular topic is the fractional Laplacian. The fractional Laplacian equations or systems can be used to model diverse physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics and relativistic quantum mechanics of stars (see [1]-[3] and the references therein). A series of excellent results about the fractional Laplacian have been obtained [4]-[7].

In 2019, Chen and Weth [8] introduced the concept of a new fractional Laplacian operator called Logarithmic Laplacian. They first discussed the origin of the Logarithmic Laplacian and derived its related properties. Then they studied the functional analytic framework of Dirichlet problems involving the Logarithmic Laplacian on bounded domains and characterized the asymptotics of principal Dirichlet eigenvalues and eigenfunctions of $(-\Delta)^s$ as $s \to 0$. The integral representation of the Logarithmic Laplacian is as follows and the space $L_0^1(\mathbb{R}^N)$ denotes the space of locally integrable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\|u\|_{L_0^1} := \int_0^{\infty} \frac{|u(x)|}{(1+|x|)^N} ds < \infty.$$

Definition 1.1. *[8] If $u \in L_0^1(\mathbb{R}^N)$ is Dini continuous at some point $x \in \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$ is an open subset and $x \in \Omega$, then*

$$(-\Delta)^{\mathcal{L}} u(x) = C_N \int_{\Omega} \frac{u(x)-u(y)}{|x-y|^N} dy - C_N \int_{\mathbb{R}^N \setminus \Omega} \frac{u(y)}{|x-y|^N} dy + [h_{\Omega}(x) + \rho_N] u(x), \tag{1}$$

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where

$$h_\Omega(x) = C_N \int_{B_1(x)\setminus\Omega} \frac{1}{|x-y|^N} dy - C_N \int_{\Omega\setminus B_1(x)} \frac{1}{|x-y|^N} dy,$$

$C_N = \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2})$ *and* $\rho_N = 2\log 2 + g(\frac{N}{2}) - \gamma$, *where* Γ *is the Gamma function,* $g = \frac{\Gamma'}{\Gamma}$ *is the Digamma function,* $\gamma = -\Gamma'(1)$ *is the Euler Mascheroni constant and* $B_1(x)$ *denotes the open ball of radius 1 centered at* $x \in \mathbb{R}^N$.

In this paper, we investigate the radial symmetry of positive solution of the Logarithmic Laplacian system. Since the radial symmetry property is essential for the development of symmetrization techniques for fractional elliptic and parabolic differential equations [9]-[11], in the study of the fractional Laplacian, the radial symmetry of solutions has been a hot topic. Zhuo et al. [12] considered the following system of pseudo-differential nonlinear equations in $\mathbb{R}^N$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u_i(x) = f_i(u_1(x), \cdots, u_m(x)), & i = 1, \cdots, m, \\ u_i \geq 0, & i = 1, \cdots, m, \end{cases}$$

where $\alpha \in (0, 2)$. They obtained radial symmetry in the critical case and nonexistence in the subcritical case of positive solutions. Li [13] obtained symmetry of positive solutions in the situations where the initial values are symmetric. Jarohs and Weth [14] established asymptotic symmetry of weak solutions for a class of nonlinear fractional reaction-diffusion equations in bounded domains. However, there is no result about the radial symmetry of the solution of the logarithmic Laplacian system. We are interested in the radial symmetry of the positive solution of the following Logarithmic Laplacian system

$$\begin{cases} (-\Delta)^{\mathcal{L}} \omega(x) = \varphi(\omega, \varpi), & x \in \mathbb{R}^N, \\ (-\Delta)^{\mathcal{L}} \varpi(x) = \psi(\omega, \varpi), & x \in \mathbb{R}^N, \\ \omega(x) > 0, \quad \varpi(x) > 0, & x \in \mathbb{R}^N, \end{cases} \tag{2}$$

where $(-\Delta)^{\mathcal{L}}$ is the Logarithmic Laplacian operator and $\varphi, \psi \in C^1([0, +\infty) \times [0, +\infty), \mathbb{R})$.

The direct method of moving planes is an important method to obtain the radial symmetry of the solution of equations and systems. It is developed on the basis of the method of moving plane. The method of moving planes was introduced by Alexanderoff in the early 1950s. Caffarelli and Silvestre [15] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions in order to overcome the difficulties brought by fractional Laplacian nonlocality. In [16] and [17], by applying the method of moving planes in integral forms, the authors obtained the radial symmetry in the critical case and nonexistence in the subcritical case for positive solutions of the following integral equation

$$u(x) = C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{n-\alpha}} u^p(y) dy.$$

However, either by extension method or by the method of moving planes in integral forms, one needs to impose extra conditions on the solutions, which makes our research difficult. To overcome the shortcomings of the two methods above, Chen and Li [18] developed a direct method of moving planes for the fractional Laplacian, later, Liu and Ma [19] generalized the method of moving planes for the fractional Laplacian to the following system

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(u, v), & x \in \mathbb{R}^N, \\ (-\Delta)^{\frac{\alpha}{2}} v(x) = g(u, v), & x \in \mathbb{R}^N, \\ u(x) > 0, \quad v(x) > 0, & x \in \mathbb{R}^N, \end{cases}$$

where $\alpha \in (0,2)$ and $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian operator. The method can work directly on the nonlocal operator for nonlocal problems either on bounded or unbounded domains. We have also done some work in applying the method to study the fractional Laplacian [20]-[26]. As an extension of our previous work, we generalize the method to the Logarithmic Laplacian system (2) and investigate the radial symmetry of the solution of (2) in this paper. To carry out this work smoothly, we develop *Narrow region principle* and *Decay at infinity* about the Logarithmic Laplacian system. Then the radial symmetry of the solution of the Logarithmic Laplacian system is obtained by the direct method of moving planes.

§2 Preliminaries

In what follows, in order to prove main theorem smoothly, we restrict $\omega(x)$ and $\varpi(x)$ as follows.

$$(a)\ \omega(x) \le \frac{1}{|x|^{\alpha}}, \qquad \varpi(x) \le \frac{1}{|x|^{\beta}}, \qquad as\ |x| \to \infty;$$
$$(b)\ \frac{\partial \varphi}{\partial \omega}(\omega,\varpi) \le \omega^s \varpi^t,\ \ \frac{\partial \psi}{\partial \varpi}(\omega,\varpi) \le \omega^k \varpi^l,\ \ as\ (\omega,\varpi) \to (0^+,0^+);$$
$$(c)\ \frac{\partial \varphi}{\partial \varpi}(\omega,\varpi) \le \omega^t \varpi^s,\ \ \frac{\partial \psi}{\partial \omega}(\omega,\varpi) \le \omega^l \varpi^k,\ \ as\ (\omega,\varpi) \to (0^+,0^+);$$
$$(d)\ \frac{\partial \varphi}{\partial \varpi}(\omega,\varpi) > 0, \qquad \frac{\partial \psi}{\partial \omega}(\omega,\varpi) > 0,$$

where $\alpha, \beta > 0;\ t,k > 0;\ s,l \ge 0$.

Next, we introduce some symbols. Let x be a point in $\mathbb{R}^N$, choose any direction to be the x_1-direction. T_ν denotes a hyperplane in $\mathbb{R}^N$. Assume that

$$T_\nu = \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 = \nu,\ for\ some\ \nu \in \mathbb{R}\}.$$

Let

$$x^\nu = (2\nu - x_1, x_2, \cdots, x_N)$$

be the reflection of x about the plane T_ν. Define

$$\Sigma_\nu = \{x \in \mathbb{R}^N \mid x_1 < \nu\},$$

and

$$\omega_\nu(x) = \omega(x^\nu),\ \ \varpi_\nu(x) = \varpi(x^\nu).$$

Let

$$W_\nu(x) = \omega_\nu(x) - \omega(x),\ \ M_\nu(x) = \varpi_\nu(x) - \varpi(x).$$

Define

$$\Sigma_\nu^{W-} = \{x \in \Sigma_\nu | W_\nu(x) < 0\},\ \ \ \Sigma_\nu^{M-} = \{x \in \Sigma_\nu | M_\nu(x) < 0\}.$$

According to (1), for $x \in \Sigma_\nu$,

$$\begin{aligned}(-\Delta)^{\mathcal{L}} W_\nu(x) &= (-\Delta)^{\mathcal{L}} \omega(x^\nu) - (-\Delta)^{\mathcal{L}} \omega(x) \\ &= \varphi(\omega(x^\nu), \varpi(x^\nu)) - \varphi(\omega(x), \varpi(x)) \\ &= \frac{\partial \varphi}{\partial \omega}(\xi_1(x,\nu), \varpi(x)) W_\nu(x) + \frac{\partial \varphi}{\partial \varpi}(\omega_\nu(x), \eta_1(x,\nu)) M_\nu(x),\end{aligned} \tag{3}$$

and

$$(-\Delta)^{\mathcal{L}} M_\nu(x) = \frac{\partial \psi}{\partial \omega}(\xi_2(x,\nu), \varpi_\nu(x)) W_\nu(x) + \frac{\partial \psi}{\partial \varpi}(\omega(x), \eta_2(x,\nu)) M_\nu(x), \tag{4}$$

where $\xi_i(x,\nu)$ is between $\omega_\nu(x)$ and $\omega(x)$, $\eta_i(x,\nu)$ is between $\varpi_\nu(x)$ and $\varpi(x)$, $i=1,2$.

In addition, we also assume that $h_{\Sigma_\nu}(x)+\rho_N\ge 0$, $\varphi,\psi\in C^1([0,+\infty)\times[0,+\infty),\mathbb{R})$ and ω,ϖ are continuous on $\overline{\Sigma_\nu}$, Dini continuous in Σ_ν throughout the paper.

Next, we introduce the following two Theorems, which play an important role in the process of moving planes.

Theorem 2.1. *(Decay at infinity) Let $(\omega,\varpi)\in(L_0^1(\mathbb{R}^N))^2$ be a positive solution of (2). Under assumptions (a) − (d), for Eq.(3) and Eq.(4) with all $\nu\le 0$, there exists a constant $\mathcal{R}_0>0$,*
(1) if there is $x^*\in\Sigma_\nu,\ |x^*|>\mathcal{R}_0$ such that $W_\nu(x^*)=\min_{x\in\overline{\Sigma_\nu}}W_\nu(x)<0$, then

$$M_\nu(x^*)<c_0W_\nu(x^*)<0\ (c_0>1);$$

(2) if there is $y^*\in\Sigma_\nu,\ |y^*|>\mathcal{R}_0$ such that $M_\nu(y^*)=\min_{y\in\overline{\Sigma_\nu}}M_\nu(y)<0$, then

$$W_\nu(y^*)<c_0M_\nu(y^*)<0\ (c_0>1).$$

Proof. Assume that there is $x^*\in\Sigma_\nu$ such that

$$W_\nu(x^*)=\min_{x\in\overline{\Sigma_\nu}}W_\nu(x)<0. \tag{5}$$

From (3) and anti-symmetry property of $W_\nu(x)$, we have

$$\begin{aligned}
&(-\Delta)^{\mathcal{L}}W_\nu(x^*)\\
&=C_N\int_{\Sigma_\nu}\frac{W_\nu(x^*)-W_\nu(y)}{|x^*-y|^N}dy-C_N\int_{\mathbb{R}^N\setminus\Sigma_\nu}\frac{W_\nu(y)}{|x^*-y|^N}dy+[h_{\Sigma_\nu}(x^*)+\rho_N]W_\nu(x^*)\\
&=C_N\int_{\Sigma_\nu}\frac{W_\nu(x^*)-W_\nu(y)}{|x^*-y|^N}dy-C_N\int_{\Sigma_\nu}\frac{W_\nu(y^\nu)}{|x^*-y^\nu|^N}dy+[h_{\Sigma_\nu}(x^*)+\rho_N]W_\nu(x^*)\\
&=C_N\int_{\Sigma_\nu}\frac{W_\nu(x^*)-W_\nu(y)}{|x^*-y|^N}dy+C_N\int_{\Sigma_\nu}\frac{W_\nu(y)}{|x^*-y^\nu|^N}dy+[h_{\Sigma_\nu}(x^*)+\rho_N]W_\nu(x^*)\\
&\le C_N\int_{\Sigma_\nu}[\frac{W_\nu(x^*)-W_\nu(y)}{|x^*-y^\nu|^N}+\frac{W_\nu(y)}{|x^*-y^\nu|^N}]dy+[h_{\Sigma_\nu}(x^*)+\rho_N]W_\nu(x^*)\\
&=C_N\int_{\Sigma_\nu}\frac{W_\nu(x^*)}{|x^*-y^\nu|^N}dy+[h_{\Sigma_\nu}(x^*)+\rho_N]W_\nu(x^*).
\end{aligned}\tag{6}$$

Combining the above result and (3), we obtain

$$\frac{\partial\varphi}{\partial\varpi}(\omega_\nu(x^*),\eta_1(x^*,\nu))M_\nu(x^*)\le a_1(x^*,\nu)W_\nu(x^*), \tag{7}$$

where

$$a_1(x^*,\nu)=C_N\int_{\Sigma_\nu}\frac{1}{|x^*-y|^N}dy+h_{\Sigma_\nu}(x^*)+\rho_N-\frac{\partial\varphi}{\partial\omega}(\xi_1(x^*,\nu),\varpi(x^*)).$$

For each fixed $\nu\le 0$, when $x_1^*<\nu$, let $x^1=(2|x^*|+x_1^*,(x^*)')$, we have $B_{|x^*|}(x^1)\subset\mathbb{R}^N\setminus\Sigma_\nu$ and let $\omega_N=|B_1(0)|$ *in* R^N. Then

$$\begin{aligned}
\int_{\Sigma_\nu}\frac{1}{|x^*-y^\nu|^N}dy&=\int_{\mathbb{R}^N\setminus\Sigma_\nu}\frac{1}{|x^*-y|^N}dy\\
&\ge\int_{B_{|x^*|}(x^1)}\frac{1}{|x^*-y|^N}dy
\end{aligned}$$

$$\geq \int_{B_{|x^*|}(x^1)} \frac{1}{3^N|x^*|^N}dy$$
$$= \frac{\omega_N}{3^N},$$

Therefore,

$$a_1(x^*,\nu) \geq \frac{\omega_N C_N}{3^N} + h_{\Sigma_\nu}(x^*) + \rho_N - \frac{\partial\varphi}{\partial\omega}(\xi_1(x^*,\nu),\varpi(x^*)). \tag{8}$$

Owing to assumption (b), there exists $\delta > 0$ small enough such that when $\omega + \varpi < \delta$ and $\omega, \varpi > 0$,

$$\frac{\partial\varphi}{\partial\omega}(\omega,\varpi) \leq \omega^s\varpi^t.$$

For this particular δ, because of assumption (a), we choose $\mathcal{R}_1$ such that when $|x^*| > \mathcal{R}_1$,

$$0 < \omega(x^*) < \frac{1}{|x^*|^\alpha}, \quad 0 < \varpi(x^*) < \frac{1}{|x^*|^\beta} \quad and \quad \omega(x^*) + \varpi(x^*) < \delta.$$

Owning to $x^* \in \Sigma_\nu^{W-}$,

$$0 < \varphi_\nu(x^*) < \xi_1(x^*,\nu) < \varphi(x^*).$$

Then for all $|x^*| > \mathcal{R}_1$,

$$\frac{\partial\varphi}{\partial\omega}(\xi_1(x^*,\nu),\varpi(x^*)) \leq \xi_1(x^*,\nu)^s\varpi(x^*)^t < \omega(x^*)^s\varpi(x^*)^t < \frac{1}{|x^*|^{s\alpha+t\beta}}. \tag{9}$$

Putting (9) into (8), we can choose $\mathcal{R}_2 > \mathcal{R}_1$, such that for $|x^*| > \mathcal{R}_2$,

$$a_1(x^*,\nu) \geq c > 0. \tag{10}$$

Therefore, combining (10) with assumption (d), we derive from (7) that

$$M_\nu(x^*) \leq \frac{a_1(x^*,\nu)}{\frac{\partial\varphi}{\partial\varpi}(\omega_\nu(x^*),\eta_1(x^*,\nu))}W_\nu(x^*) < 0,$$

which implies $x^* \in \Sigma_\nu^{M-}$.

Now we just need to show that

$$\frac{a_1(x^*,\nu)}{\frac{\partial\varphi}{\partial\varpi}(\omega_\nu(x^*),\eta_1(x^*,\nu))} > c_0, \ when\ |x^*|\ sufficiently\ large.$$

According to assumption (c), there is δ' small enough, such that when $\omega+\varpi < \delta'$ and $\omega,\varpi > 0$,

$$\frac{\partial\varphi}{\partial\varpi}(\omega,\varpi) \leq \omega^t\varpi^s.$$

Because of (a), we can choose $\mathcal{R}_3 > \mathcal{R}_2$ so that for all $|x^*| > \mathcal{R}_3$,

$$0 < \omega(x^*) < \frac{1}{|x^*|^\alpha}, \quad 0 < \varpi(x^*) < \frac{1}{|x^*|^\beta}, \quad and \quad \omega(x^*) + \varpi(x^*) < \delta'.$$

Since $x^* \in \Sigma_\nu^{M-}$,

$$0 < \varpi_\nu(x^*) < \eta_1(x^*,\nu) < \varpi(x^*),$$

it follows that for $|x^*| > \mathcal{R}_3$,

$$\frac{\partial\varphi}{\partial\varpi}(\omega_\nu(x^*),\eta_1(x^*,\nu)) \leq \omega_\nu(x^*)^t\eta_1(x^*,\nu)^s < \omega_\nu(x^*)^t\varpi(x^*)^s < \frac{1}{|x^*|^{t\alpha+s\beta}}. \tag{11}$$

In view of (10) and (11), we obtain for $|x^*| > \mathcal{R}_3$,

$$\frac{a_1(x^*,\nu)}{\frac{\partial\varphi}{\partial\varpi}(\omega_\nu(x^*),\eta_1(x^*,\nu))} > c|x^*|^{t\alpha+s\beta} \to \infty, \ as\ |x^*| \to \infty.$$

Therefore, there is $\mathcal{R}_0 > \mathcal{R}_3$, for all $|x^*| > \mathcal{R}_0$,

$$\frac{a_1(x^*,\nu)}{\frac{\partial\varphi}{\partial\varpi}(\omega_\nu(x^*),\eta_1(x^*,\nu))} > c_0.$$

In summary, conclusion (1) holds. Applying a proof similar to conclusion (1), we get conclusion (2).

Theorem 2.2. (*Narrow Region Principle*) *Let* $(\omega, \varpi) \in (L^1_0(\mathbb{R}^N))^2$ *be a positive solution of system (2). Assume that* $\omega(x)$ *and* $\varpi(x)$ *satisfy*

$$(a') \lim_{|x|\to\infty} \omega(x) = 0, \qquad \lim_{|x|\to\infty} \varpi(x) = 0$$

and (d). *Then there exists* $l_0 > 0$ *such that for each* $l \in (0, l_0]$, $M_\nu(x)$ *and* $W_\nu(x)$ *satisfy*
(3) if there is $x^* \in \Theta_{\nu,l} := \{x \in \Sigma_\nu | \nu - l < x_1 < \nu\}$ *such that* $W_\nu(x^*) = \min_{x\in\overline{\Sigma_\nu}} W_\nu(x) < 0$, *then*

$$M_\nu(x^*) < c_0 W_\nu(x^*) < 0 \ (c_0 > 1);$$

(4) if there is $y^* \in \Theta_{\nu,l} := \{x \in \Sigma_\nu | \nu - l < x_1 < \nu\}$ *such that* $M_\nu(y^*) = \min_{x\in\overline{\Sigma_\nu}} M_\nu(x) < 0$, *then*

$$W_\nu(y^*) < c_0 M_\nu(y^*) < 0 \ (c_0 > 1).$$

Proof. Assume that

$$x^* \in \Theta_{\nu,l} \text{ and } W_\nu(x^*) = \min_{x\in\overline{\Sigma_\nu}} W_\nu(x) < 0$$

for some $l > 0$.

From (6) and (7), we have

$$(-\Delta)^{\mathcal{L}} W_\nu(x^*) = C_N \int_{\Sigma_\nu} \frac{W_\nu(x^*)}{|x^* - y^\nu|^N} dy + [h_{\Sigma_\nu}(x^*) + \rho_N] W_\nu(x^*),$$

and

$$\frac{\partial\varphi}{\partial\varpi}(\omega_\nu(x^*), \eta_1(x^*, \nu)) M_\nu(x^*) \le a_1(x^*, \nu) W_\nu(x^*), \tag{12}$$

where

$$a_1(x^*, \nu) = C_N \int_{\Sigma_\nu} \frac{1}{|x^* - y|^N} dy + h_{\Sigma_\nu}(x^*) + \rho_N - \frac{\partial\varphi}{\partial\omega}(\xi_1(x^*, \nu), \varpi(x^*)).$$

Let $D = \{y \mid l < y_1 - x^0_1 < 1, |y' - (x^0)'| < 1\}, s = y_1 - x^0_1, \tau = |y' - (x^0)'|$ *and* $\omega_{N-2} = |B_1(0)|$ *in* $\mathbb{R}^{N-2}$. Then

$$\begin{aligned}
\int_{\Sigma_\nu} \frac{1}{|x^* - y^\nu|^N} dy &= \int_{\mathbb{R}^N\setminus\Sigma_\nu} \frac{1}{|x^* - y|^N} dy \\
&\ge \int_D \frac{1}{|x^* - y|^N} dy \\
&= \int_l^1 \int_0^1 \frac{\omega_{N-2}\tau^{N-2} d\tau}{(s^2 + \tau^2)^{\frac{N}{2}}} ds \\
&= \int_l^1 \frac{1}{s} \int_0^{\frac{1}{s}} \frac{\omega_{N-2} t^{N-2} dt}{(1 + t^2)^{\frac{N}{2}}} ds \\
&\ge \int_l^1 \frac{1}{s} \int_0^1 \frac{\omega_{N-2} t^{N-2} dt}{(1 + t^2)^{\frac{N}{2}}} ds \\
&\ge C \int_l^1 \frac{1}{s} ds \to \infty. \quad (l \to 0)
\end{aligned}$$

According to assumption (a'), $\omega(x)$ and $\varpi(x)$ are bounded on $\mathbb{R}^N$, so $\xi_1(x, \nu)$ is also bounded. On account of $\varphi \in C^1([0, +\infty) \times [0, +\infty), \mathbb{R})$, there exists some $c > 0$ such that

$$|\frac{\partial\varphi}{\partial\omega}(\xi_1(x^*, \nu), \varpi(x^*))| < c, \quad for\ all\ \nu.$$

Hence there is $l_1 > 0$ such that for all $0 < l \leq l_1$ if $x^* \in \Theta_{\nu,l}$, $a_1(x^*, \nu) > 0$. Actually, we have

$$\lim_{l \to 0^+} a_1(x^*, \nu) = +\infty. \tag{13}$$

Combining the above result and (12), we derive that for all $0 < l < l_1$,

$$M_\nu(x^*) \leq \frac{a_1(x^*, \nu)}{\frac{\partial \varphi}{\partial \varpi}(\omega_\nu(x^*), \eta_1(x^*, \nu))} W_\nu(x^*). \tag{14}$$

Note that $\frac{\partial \varphi}{\partial \varpi}(\omega_\nu(x^*), \eta_1(x^*, \nu))$ is uniformly bounded with respect to ν, i.e. there is $c' > 0$ such that

$$0 < \frac{\partial \varphi}{\partial \varpi}(\omega_\nu(x^*), \eta_1(x^*, \nu) < c', \quad \forall \nu \in \mathbb{R}. \tag{15}$$

Using (13), (14) and (15), we can choose $l_0 \in (0, l_1)$ such that for all $l \in (0, l_0)$,

$$\frac{a_1(x^*, \nu)}{\frac{\partial \varphi}{\partial \varpi}(\omega_\nu(x^*), \eta_1(x^*, \nu))} > c_0. \tag{16}$$

Putting (16) into (14), we obtain conclusion (3).

Similar to the proof of conclusion (3), we get conclusion (4).

§3 Radial solution of the Logarithmic Laplacian system

Theorem 3.1. *Let $(\omega, \varpi) \in (L^1_0(\mathbb{R}^N))^2$ be a positive solution of (2). Under assumptions $(a) - (d)$, then there exists a point $x_0 \in \mathbb{R}^N$ such that*

$$\omega(x) = \omega(|x - x_0|), \quad \varpi(x) = \varpi(|x - x_0|).$$

Proof. We prove it by two steps.

Step 1. We show that there exists $\nu^* < -\mathcal{R}_0$ such that

$$W_\nu(x) \geq 0 \ \ and \ \ M_\nu(x) \geq 0, \quad \forall x \in \Sigma_\nu, \tag{17}$$

for all $\nu \leq \nu^*$, where $\mathcal{R}_0$ is chosen as in Theorem 2.1.

If (17) does not hold, without loss of generality, there exist $\nu \leq \nu^*$ and $x^* \in \Sigma_\nu$ such that

$$W_\nu(x^*) = \min_{x \in \overline{\Sigma_\nu}} W_\nu(x) < 0.$$

Since $\nu < \nu^* < -\mathcal{R}_0$, $|x^*| > \mathcal{R}_0$. According to Theorem 2.1,

$$M_\nu(x^*) < c_0 W_\nu(x^*) < 0, \tag{18}$$

where c_0 is given in Theorem 2.1.

From assumption (a), it holds that $\lim_{|x| \to \infty} M_\nu(x) = 0$. Moreover, $M_\nu(x) = 0$, for $x \in T_\nu$. Hence, there exists a point $y^* \in \Sigma_\nu$ such that

$$M_\nu(y^*) = \min_{x \in \overline{\Sigma_\nu}} M_\nu(x) < 0.$$

Owing to $|y^*| > \mathcal{R}_0$, it follows from Theorem 2.1 that

$$W_\nu(y^*) < c_0 M_\nu(y^*) < 0. \tag{19}$$

Combining (18) and (19),

$$M_\nu(x^*) < c_0 W_\nu(x^*) < c_0 W_\nu(y^*) < 2c_0 M_\nu(y^*) < 2c_0 M_\nu(x^*),$$

then it is a contradiction due to $c_0 > 1$ and $M_\nu(x^*) < 0$. Therefore, (17) holds.

We now move the hyperplane T_ν to right to its limiting position as long as (17) holds. Define

$$\nu_0 = \sup\{\nu \le 0 \mid W_\mu(x) \ge 0, M_\mu(x) \ge 0, \forall x \in \Sigma_\mu, \forall \mu \le \nu\}.$$

If $W_{\nu_0}(x) \equiv 0$, then (3) becomes

$$0 = (-\Delta)^{\mathcal{L}} W_{\nu_0}(x) = \frac{\partial \varphi}{\partial \varpi}(\omega_{\nu_0}(x), \eta_1(x, \nu_0)) M_{\nu_0}(x)$$

Combining assumption (d), we have $M_{\nu_0}(x) \equiv 0$. Similarly, if $M_{\nu_0}(x) \equiv 0$, then $W_{\nu_0}(x) \equiv 0$.

If $W_{\nu_0}(x) \not\equiv 0$, $\forall x \in \Sigma_{\nu_0}$, then $W_{\nu_0}(x) > 0$, $\forall x \in \Sigma_{\nu_0}$. If not, there is some point $x^* \in \Sigma_{\nu_0}$ such that

$$W_{\nu_0}(x^*) = 0.$$

Then

$$\begin{aligned}
&(-\Delta)^{\mathcal{L}} W_{\nu_0}(x^*) \\
&= C_N \int_{\Sigma_{\nu_0}} \frac{W_{\nu_0}(x^*) - W_{\nu_0}(y)}{|x^* - y|^N} dy - C_N \int_{\mathbb{R}^N \setminus \Sigma_{\nu_0}} \frac{W_{\nu_0}(y)}{|x^* - y|^N} dy + [h_{\Sigma_{\nu_0}}(x^*) + \rho_N] W_{\nu_0}(x^*) \\
&= C_N \int_{\Sigma_{\nu_0}} \frac{-W_{\nu_0}(y)}{|x^* - y|^N} dy - C_N \int_{\mathbb{R}^N \setminus \Sigma_{\nu_0}} \frac{W_{\nu_0}(y)}{|x^* - y|^N} dy \\
&= C_N \int_{\mathbb{R}^N} \frac{-W_{\nu_0}(y)}{|x^* - y|^N} dy.
\end{aligned}$$

In fact, $W_{\nu_0}(x) \ge 0$, $\forall x \in \Sigma_{\nu_0}$, it turns out that

$$(-\Delta)^{\mathcal{L}} W_{\nu_0}(x^*) < 0, \tag{20}$$

which is contradict to

$$(-\Delta)^{\mathcal{L}} W_{\nu_0}(x^*) = \frac{\partial \varphi}{\partial \varpi}(\omega_{\nu_0}(x^*), \eta_1(x^*, \nu_0)) M_{\nu_0}(x^*) \ge 0.$$

Consequently, we obtain $W_{\nu_0}(x) > 0$, $\forall x \in \Sigma_{\nu_0}$. By a similar progress, then one can show that if $M_{\nu_0}(x) \not\equiv 0$, $\forall x \in \Sigma_{\nu_0}$, $M_{\nu_0}(x) > 0$, $\forall x \in \Sigma_{\nu_0}$.

Step 2. We show that if $\nu_0 < 0$, then

$$W_{\nu_0}(x) \equiv 0, \quad M_{\nu_0}(x) \equiv 0, \quad \forall x \in \Sigma_{\nu_0}.$$

Suppose that $W_{\nu_0}(x) > 0, \forall x \in \Sigma_{\nu_0}$, from the previous discussion, we know $M_{\nu_0}(x) > 0, \forall x \in \Sigma_{\nu_0}$.

From the definition of ν_0 (< 0), there exist sequences $\{\nu_m\}_{m=1}^\infty$ and $\{x^m\}_{m=1}^\infty$ satisfying

$$\nu_0 < \nu_{m+1} < \nu_m < \cdots < \nu_1 < 0, \quad m = 1, 2, \cdots; \quad \lim_{m \to \infty} \nu_m = \nu_0; \tag{21}$$

$x^m \in \Sigma_{\nu_m}$ and $W_{\nu_m}(x^m) < 0$. Let x^m be the minimum point, that is

$$W_{\nu_m}(x^m) = \min_{x \in \overline{\Sigma_{\nu_m}}} W_{\nu_m}(x) < 0, \quad m = 1, 2, \cdots. \tag{22}$$

For sequence $\{x^m\}_{m=1}^\infty$, we discuss it in two cases.

Case 1. The sequence $\{x^m\}_{m=1}^\infty$ is bounded, that is

$$\lim_{m \to \infty} x^m = x^*. \tag{23}$$

According to (21) and (23), for $l_0 > 0$ defined by Theorem 2.2, we can choose $M_1 > 0$ such that if $m > M_1$, $\nu_0 < \nu_m < \nu_0 + l_0/2$ and $x^m \in \Omega_{\nu_m + l_0/2, l_0}$. Applying Theorem 2.2, we have

$$M_{\nu_m}(x^m) < c_0 W_{\nu_m}(x^m) < 0. \tag{24}$$

It follows that there is $y^m \in \Sigma_{\nu_m}$, such that

$$M_{\nu_m}(y^m) = \min_{x \in \overline{\Sigma_{\nu_m}}} M_{\nu_m}(x) < 0, \quad m = M_1 + 1, M_1 + 2, \cdots.$$

For sequence $\{y^m\}_{m=1}^{\infty}$, we also discuss it in two cases.

Case 1.1 $\{y^m\}_{M_1+1}^{\infty}$ has a bounded subsequence, that is

$$\lim_{m\to\infty} y^m = y^*.$$

Similarly, there is $M > M_1$, for all $m > M$, $y^m \in \Omega_{\nu_m+l_0/2,l_0}$. Applying Theorem 2.2, we have

$$W_{\nu_m}(y^m) < c_0 M_{\nu_m}(y^m) < 0. \tag{25}$$

Combining (24) and (25), for all $m > M$,

$$W_{\nu_m}(y^m) < c_0 M_{\nu_m}(y^m) \le c_0 M_{\nu_m}(x^m) < 2c_0 W_{\nu_m}(x^m) \le 2c_0 W_{\nu_m}(y^m),$$

then this is a contradiction due to $c_0 > 1$ and $W_{\nu_m}(y^m) < 0$.

Case 1.2 $\lim\limits_{m\to\infty} |y^m| = \infty$.

Obviously, there is $M > M_1$, such that for all $m > M$, $|y^m| > \mathcal{R}_0$. According to Theorem 2.1,

$$W_{\nu_m}(y^m) < c_0 M_{\nu_m}(y^m) < 0, \quad \forall m > M. \tag{26}$$

Combining (24) and (26), for all $m > M$,

$$W_{\nu_m}(y^m) < c_0 M_{\nu_m}(y^m) \le c_0 M_{\nu_m}(x^m) < 2c_0 W_{\nu_m}(x^m) \le 2c_0 W_{\nu_m}(y^m).$$

then this is a contradiction due to $c_0 > 1$ and $W_{\nu_m}(y^m) < 0$.

Case 2. $\lim\limits_{m\to\infty} |x^m| = \infty$.

There is $M_2 > 0$ such that for all $m > M_2$, $|x^m| > \mathcal{R}_0$. Because of Theorem 2.1,

$$M_{\nu_m}(x^m) < c_0 W_{\nu_m}(x^m) < 0, \quad \forall m > M_2. \tag{27}$$

Hence, for each $m > M_2$, there is $y^m \in \Sigma_{\nu_m}$ satisfying

$$M_{\nu_m}(y^m) = \min_{x\in\overline{\Sigma_{\nu_m}}} M_{\nu_m}(x) < 0.$$

Case 2.1 $\{y^m\}$ is bounded.

Similar to Case 1, we can also derive a contradiction.

Case 2.2 $\lim\limits_{m\to\infty} |y^m| = \infty$.

We can choose $M > M_2$, such that for all $m > M$, $|y^m| > \mathcal{R}_0$. From Theorem 2.1, we obtain

$$W_{\nu_m}(y^m) < c_0 M_{\nu_m}(y^m) < 0, \quad \forall m > M. \tag{28}$$

Combining (27) and (28), for all $m > M$,

$$W_{\nu_m}(y^m) < c_0 M_{\nu_m}(y^m) \le c_0 M_{\nu_m}(x^m) < 2c_0 W_{\nu_m}(x^m) \le 2c_0 W_{\nu_m}(y^m),$$

then this is a contradiction due to $c_0 > 1$ and $W_{\nu_m}(y^m) < 0$.

Combining with the above cases, if $\nu_0 < 0$,

$$W_{\nu_0}(x) \equiv 0, \quad M_{\nu_0}(x) \equiv 0. \quad \forall x \in \Sigma_{\nu_0}.$$

Similarly, if $\nu_0' > 0$, moving the hyperplane T_ν to left, one can show that

$$W_{\nu_0'}(x) \equiv 0, \quad M_{\nu_0'}(x) \equiv 0. \quad \forall x \in \Sigma_{\nu_0'}.$$

Therefore, (ω, ϖ) is radially symmetric about some point owing to the arbitrary choice of x_1-direction.

Remark 3.1. *The Logarithmic Laplacian operator arose as formal derivative $\partial_s|_{s=0}(-\Delta)^s$ of the fractional Laplacian at $s = 0$ [8]. It is different from the fractional Laplacian, which brings new challenges and many difficulties. Therefore, our current work is a meaningful contribution to the field of Logarithmic Laplacian operator.*

Declarations

Conflict of interest The authors declare no conflict of interest.

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