Appl. Math. J. Chinese Univ.
2023, 38(4): 604-613

The Gleason's problem on normal weight general function spaces in the unit ball of $\mathbb{C}^n$

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Abstract. In this paper, we first discuss the boundedness of certain integral operator T_t on the normal weight general function space $F(p,\mu,s)$ in the unit ball B_n of $\mathbb{C}^n$. As an application of this operator, we prove that the Gleason's problem is solvable on $F(p,\mu,s)$.

§1 Introduction

We call two quantities G and H are equivalent (denoted by "$G \asymp H$") if there are two constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 H \leq G \leq c_2 H$. If there exists a constant $c > 0$ such that $G \leq cH$ ($G \geq cH$), then we denote by "$G \lesssim H$" ("$G \gtrsim H$").

Let B_n denote the unit ball in the n-dimensional complex Euclidean space $\mathbb{C}^n$. For two points $w = (w_1, w_2, \cdots, w_n)$ and $z = (z_1, z_2, \cdots, z_n)$ in $\mathbb{C}^n$, let

$$\langle w, z\rangle = w_1\overline{z_1} + w_2\overline{z_2} \cdots + w_n\overline{z_n}.$$

The class of all holomorphic functions on B_n is denoted by $H(B_n)$. For $f \in H(B_n)$ and $z \in B_n$, the complex gradient of f is defined by

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), \cdots, \frac{\partial f}{\partial z_n}(z)\right).$$

Given $z \in B_n$, let φ_z be the automorphisms of B_n with $\varphi_z(z) = 0$, $\varphi_z(0) = z$ and $\varphi_a^{-1} = \varphi_a$. For $\rho > 0$ and $z \in B_n$, let $D(z,\rho) = \{w : w \in B_n \text{ and } \beta(z,w) < \rho\}$ denote the Bergman metric ball at z with radius ρ, where

$$\beta(z,w) = \frac{1}{2}\log\frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

Definition 1.1 A positive and continuous function μ on $[0,1)$ is called a normal function if there are constants $0 < a \leq b < \infty$ and $0 \leq r_0 < 1$ such that $\dfrac{\mu(r)}{(1-r^2)^b}$ is increasing on $[r_0, 1)$

Received: 2022-09-09.   Revised: 2023-05-26.
MR Subject Classification: 32A37, 47B33.
Keywords: Gleason's problem, solvability, $F(p,\mu,s)$ space, integral operator.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-023-4840-3.
Supported by the National Natural Science Foundation of China(11942109) and the Natural Science Foundation of Hunan Province in China(2022JJ30369).
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and $\dfrac{\mu(r)}{(1-r^2)^a}$ is decreasing on $[r_0, 1)$.

The following two functions are the examples of this kind of normal functions:

$$\mu_1(r) = (1-r^2)^\gamma \left(\log \frac{e}{1-r^2}\right)^\beta \left(\log\log \frac{e^2}{1-r^2}\right)^\alpha \quad (\gamma > 0,\ \alpha \text{ and } \beta \text{ real}),$$

$$\mu_2(r) = \begin{cases} \dfrac{(2n-2)!!}{(2n-1)!!}(1-r^2)^{\frac{1}{2}}, & \dfrac{n-1}{n} \le r^2 < \dfrac{2n^2-1}{2n(n+1)} \\ \dfrac{(2n)!!(n+1)}{(2n+1)!!}(1-r^2)^{\frac{3}{2}}, & \dfrac{2n^2-1}{2n(n+1)} \le r^2 < \dfrac{n}{n+1} \end{cases} \quad (n = 1, 2, \cdots).$$

For the convenience of proof, we let $r_0 = 0$ in this paper. The following spaces are several function spaces involved in this paper, and we give definitions respectively.

Definition 1.2 Let ν be a positive continuous function on $[0, 1)$ such that $\sup\limits_{0\le r<1} \nu(r) < \infty$. If $f \in H(B_n)$ and

$$||f||_{\mathcal{B}_\nu} = |f(0)| + \sup_{z\in B_n} \nu(|z|)|\nabla f(z)| < \infty,$$

then we say that f belongs to the ν-Bloch space $\mathcal{B}_\nu(B_n)$. In particular, if ν is a normal function on $[0, 1)$, then $\mathcal{B}_\nu(B_n)$ is called the normal weight Bloch space.

Definition 1.3 For $p > 0$ and a normal function μ on $[0, 1)$, if $f \in H(B_n)$ and

$$\int_{B_n} |\nabla f(z)|^p \frac{\mu^p(|z|)}{1-|z|^2}\, dv(z) < \infty,$$

then we say that f belongs to the normal weight Dirichlet type space $\mathcal{D}^p_\mu(B_n)$, where dv is the Lebesgue measure on B_n such that $v(B_n) = 1$. When $\mu(r) = (1-r^2)^{\frac{\alpha+1}{p}}$ $(\alpha > -1)$, the space $\mathcal{D}^p_\mu(B_n)$ is just the weighted Dirichlet type space $\mathcal{D}^p_\alpha(B_n)$.

For $p > 0$, $s \ge 0$, $q + n > -1$, $q + s > -1$, the general function space $F(p, q, s)$, consists of all $f \in H(B_n)$ and

$$||f||_{F(p,q,s)} = |f(0)| + \left\{\sup_{w\in B_n} \int_{B_n} |\nabla f(z)|^p\ (1-|z|^2)^q \log^s \frac{1}{|\varphi_w(z)|}\, dv(z)\right\}^{\frac{1}{p}} < \infty.$$

In [1], R H Zhao first introduced the space $F(p, q, s)$ on the unit disc. Soon, a lot of function spaces associated with $F(p, q, s)$ were studied, such as, [2]-[9] etc.

In [2], X J Zhang et al gave several equivalent characterizations of $F(p, q, s)$. For example,

$$||f||_{F(p,q,s)} \asymp |f(0)| + \left\{\sup_{w\in B_n} \int_{B_n} \frac{|\nabla f(z)|^p\ (1-|w|^2)^s}{|1-\langle z, w\rangle|^{2s}}(1-|z|^2)^{q+s}\, dv(z)\right\}^{\frac{1}{p}}.$$

The key measure in the above integral is $(1-|z|^2)^{q+s}\, dv(z)$. In order to study the general function spaces in a broader and more abstract perspective, it is meaningful to extend this measure $(1-|z|^2)^{q+s}\, dv(z)$ to a kind of abstract form. Recently, S L Li ([9]) extended $F(p, q, s)$ to a kind of abstract form as follows:

Definition 1.4 Let μ be a normal function on $[0, 1)$. For $p > 0$, the normal weight general function space, denoted by $F(p,\mu,s)$, consists of all $f\in H(B_n)$ and

$$||f||_{F(p,\mu,s)}=|f(0)|+\left\{\sup_{w\in B_n}\int_{B_n}\frac{|\nabla f(z)|^p(1-|w|^2)^s}{|1-\langle z,w\rangle|^{2s}}\frac{\mu^p(|z|)}{1-|z|^2}\,dv(z)\right\}^{\frac{1}{p}}<\infty.$$

In particular, $F(p,\mu,s)=F(p,q,s)$ when $\mu(r)=(1-r^2)^{\frac{q+s+1}{p}}$. If $s=0$, then $F(p,\mu,s)=\mathcal{D}^p_\mu(B_n)$. Therefore, $F(p,\mu,s)$ is not only a generalization of $F(p,q,s)$, but also a generalization of the weighted Dirichlet type space.

Definition 1.5 For $p>0$ and $\alpha>-1$, the weighted Bergman space $\mathcal{A}^p_\alpha(B_n)$ consists of holomorphic functions f in B_n and

$$||f||_{p,\alpha}=\left(\int_{B_n}|f(z)|^p\,dv_\alpha(z)\right)^{\frac{1}{p}}<\infty,$$

where $dv_\alpha(z)=c_\alpha(1-|z|^2)^\alpha\,dv(z)$, and the constant c_α such that $v_\alpha(B_n)=1$.

Using operators to study function spaces has a long history. There have been a large number of relevant literatures. In particular, Forelli-Rudin introduced the projection operator in [10]:

$$P_\tau f(z)=\int_{B_n}\frac{f(w)}{(1-\langle z,w\rangle)^{n+1+\tau}}\,dv_\tau(w)\quad(\tau>-1).$$

In order to solve the solvability of Gleason's problem, we first need to discuss the boundedness of kinds of Forelli-Rudin type operators. As for the research on Forelli-Rudin type operators, there has been a lot of work, such as [9]-[17], [32], [34] etc.

Let Y be a class of holomorphic functions in the domain $\Delta\subseteq\mathbb{C}^n$. Gleason's problem for Y, denoted by (Δ,β,Y), is the following: for any $\beta=(\beta_1,\beta_2,\ldots,\beta_n)\in\Delta$ and $h\in Y$ with $h(\beta)=0$, are there functions $h_1,h_2,\ldots,h_n\in Y$ such that

$$h(z)=\sum_{j=1}^n(z_j-\beta_j)\,h_j(z)\quad\text{for all } z=(z_1,z_2,\ldots,z_n)\in\Delta\ ?$$

The Gleason's problem originated from ball algebra([18]), and it has been a hot topic of research by mathematicians for decades. There are many references on this aspect, such as, [18]-[30], [33] etc. The key difficulty of the Gleason's problem depends on the domain Δ, the point $\beta\in\Delta$, and the function space Y. It is known that the Gleason's problem always has a solution in a function space with the multiple cylinder as the support domain, but it may not necessarily have a solution on a function space with the unit ball as the support domain. Therefore, the solvability needs to be discussed one by one. For abstract normal weight μ, is Gleason's problem solvable on $F(p,\mu,s)$? In this paper, we mainly solve this problem.

In this paper, we always let a and b be the two parameters in the definition of the normal function μ. Parameter ranges involving function space definitions are no longer repeated.

§2 Some Lemmas

In order to prove the main results of this paper, we give some lemmas first.

Lemma 2.1 ([19]) Let $\delta > -1$ and $t > \delta + n + 1$. Then

$$\int_{B_n} \frac{(1-|w|^2)^\delta}{|1-\langle z,w\rangle|^t}\, dv(w) \asymp \frac{1}{(1-|z|^2)^{t-\delta-n-1}} \quad \text{for all } z \in B_n.$$

Lemma 2.2 ([31]) Let μ be a normal function on $[0,1)$. For $w \in B_n$ and $r > 0$, there are the following properties:

(1) $\mu(|z|) \asymp \mu(|w|)$ for all $z \in D(w,r)$.

(2) $\displaystyle\frac{\mu(|z|)}{\mu(|w|)} \le \left(\frac{1-|z|^2}{1-|w|^2}\right)^a + \left(\frac{1-|z|^2}{1-|w|^2}\right)^b$ for all $z \in B_n$.

Lemma 2.3 Let μ be a normal function on $[0,1)$. If $f \in F(p,\mu,s)$, then

$$|\nabla f(w)| \lesssim \frac{||f||_{F(p,\mu,s)}}{\mu(|w|)(1-|w|^2)^{\frac{n-s}{p}}} \quad \text{for all } w \in B_n.$$

Moreover, if $s > n$, then $F(p,\mu,s) = \mathcal{B}_\nu(B_n)$, where $\nu(\rho) = (1-\rho^2)^{\frac{n-s}{p}}\ \mu(\rho)$.

Proof The results of the previous part come from Lemma 2.4 in [9]. As long as we take $g(z) = \dfrac{\partial f}{\partial z_l}(z)$ $(l \in \{1,2,\cdots n\},\ z \in B_n)$ and $\gamma = s/p$.

When $s > n$, for any $f \in \mathcal{B}_\nu(B_n)$ and $w \in B_n$, it follows from Lemma 2.1 that

$$\begin{aligned}
&\int_{B_n} \frac{(1-|w|^2)^s |\nabla f(z)|^p\ \mu^p(|z|)}{|1-\langle z,w\rangle|^{2s}(1-|z|^2)}\, dv(z)\\
&\le ||f||^p_{\mathcal{B}_\nu} \int_{B_n} \frac{(1-|w|^2)^s(1-|z|^2)^{s-n-1}}{|1-\langle z,w\rangle|^{2s}}\, dv(z)\\
&\asymp ||f||^p_{\mathcal{B}_\nu} \ \Rightarrow\ ||f||_{F(p,\mu,s)} \lesssim ||f||_{\mathcal{B}_\nu} \ \Rightarrow \mathcal{B}_\nu(B_n) \subseteq F(p,\mu,s).
\end{aligned}$$

This proof is completed. □

Note When $\lim\limits_{\rho\to 1^-}(1-\rho^2)^{\frac{n-s}{p}}\ \mu(\rho) = \infty$ (for example, $b + (n-s)/p < 0$), the space $\mathcal{B}_\nu(B_n)$ contains only constant valued functions by the maximum modulus principle. Therefore, $F(p,\mu,s)$ contains only constant valued functions in this case.

Lemma 2.4 ([32]) For $\delta > -1$ and $r \ge 0$, $t \ge 0$, let

$$I_{w,\eta} = \int_{B_n} \frac{(1-|z|^2)^\delta}{|1-\langle z,w\rangle|^t\ |1-\langle z,\eta\rangle|^r}\, dv(z) \quad (w,\eta \in B_n).$$

Then there are the following results.

(1) When $t - \delta > n+1 > r - \delta$,

$$I_{w,\eta} \asymp \frac{1}{(1-|w|^2)^{t-\delta-n-1}|1-\langle w,\eta\rangle|^r}.$$

(2) When $t - \delta > n+1$ and $r - \delta > n+1$,

$$I_{w,\eta} \asymp \frac{1}{(1-|w|^2)^{t-\delta-n-1}|1-\langle w,\eta\rangle|^r} + \frac{1}{(1-|\eta|^2)^{r-\delta-n-1}|1-\langle w,\eta\rangle|^t}.$$

(3) When $t - \delta > n+1 = r - \delta$,

$$I_{w,\eta} \asymp \frac{1}{(1-|w|^2)^{t-\delta-n-1}|1-\langle w,\eta\rangle|^r} + \frac{1}{|1-\langle w,\eta\rangle|^t}\log\frac{e}{1-|\varphi_w(\eta)|^2}.$$

§3 Main Results

Theorem 3.1 Let μ be a normal function on $[0,1)$. If t is large sufficiently, then $||T_t f||_{p,q,s} \lesssim ||f||_{p,q,s}$ for all $f \in F(p,\mu,s)$ when $0 \le s \le n$, where

$$T_t f(z) = \int_{B_n} \frac{|\nabla f(w)|\, dv_t(w)}{|1-\langle z,w\rangle|^{n+1+t}} \quad (z \in B_n).$$

Proof (1) Case $p > 1$.

It is clear that $\lim_{t\to\infty} \frac{(1+t)p-1}{pb+t} = p > 1$. Therefore, there exists a $t_0 > pb+n-s-1$ such that $pb+t < (1+t)p-1$ when $t > t_0$. We choose $t_0 < t < t_1 < pa+t$. This means that $p'(t-t_1/p) > -1$ (where $1/p+1/p'=1$) and $t_1 - t > 0$. By Hölder inequality and Lemma 2.1, we may obtain

$$\begin{aligned} |T_t f(z)|^p &\lesssim \left\{ \int_{B_n} \frac{(1-|w|^2)^{p'(t-\frac{t_1}{p})}\, dv(w)}{|1-\langle z,w\rangle|^{n+1+t}} \right\}^{\frac{p}{p'}} \int_{B_n} \frac{|\nabla f(w)|^p}{|1-\langle z,w\rangle|^{n+1+t}}\, dv_{t_1}(w) \\ &\asymp \frac{1}{(1-|z|^2)^{t_1-t}} \int_{B_n} \frac{|\nabla f(w)|^p}{|1-\langle z,w\rangle|^{n+1+t}}\, dv_{t_1}(w). \end{aligned} \tag{1}$$

For any $\xi \in B_n$, we first consider the integral

$$\int_{B_n} \frac{|\nabla f(w)|^p \mu^p(|w|)}{(1-|w|^2)^{pa}} \left\{ \int_{B_n} \frac{(1-|\xi|^2)^s (1-|z|^2)^{pa+t-t_1-1}\, dv(z)}{|1-\langle z,\xi\rangle|^{2s} |1-\langle z,w\rangle|^{n+1+t}} \right\} dv_{t_1}(w).$$

When $s=0$, by $pa-1 < t_0 < t_1 < pa+t$ and Lemma 2.1, we have

$$\begin{aligned} &\int_{B_n} \frac{|\nabla f(w)|^p \mu^p(|w|)}{(1-|w|^2)^{pa}} \left\{ \int_{B_n} \frac{(1-|z|^2)^{pa+t-t_1-1}\, dv(z)}{|1-\langle z,w\rangle|^{n+1+t}} \right\} dv_{t_1}(w) \\ &\asymp \int_{B_n} \frac{|\nabla f(w)|^p \mu^p(|w|)}{1-|w|^2}\, dv(w) \le ||f||^p_{F(p,\mu,s)}. \end{aligned}$$

When $2s - (pa+t-t_1-1) < n+1$, this conditions $t_0 < t < t_1 < pa+t$ show that $(n+1+t)-(pa+t-t_1-1) = (n+1)+1+t_1-pa > (n+1)+1+t_0-pa > n+1$, $pa+t-t_1-1 > -1$ and $(n+1+t)-(pa+t-t_1-1)-n-1 = 1+t_1-pa > 1+t_0-pa > 0$. It follows from Lemma 2.4(1) that

$$\begin{aligned} &\int_{B_n} \frac{|\nabla f(w)|^p \mu^p(|w|)}{(1-|w|^2)^{pa}} \left\{ \int_{B_n} \frac{(1-|\xi|^2)^s (1-|z|^2)^{pa+t-t_1-1}\, dv(z)}{|1-\langle z,\xi\rangle|^{2s} |1-\langle z,w\rangle|^{n+1+t}} \right\} dv_{t_1}(w). \\ &\asymp \int_{B_n} |\nabla f(w)|^p \frac{(1-|\xi|^2)^s}{|1-\langle w,\xi\rangle|^{2s}} \frac{\mu^p(|w|)}{1-|w|^2}\, dv(w) \le ||f||^p_{F(p,\mu,s)}. \end{aligned}$$

When $2s - (pa+t-t_1-1) > n+1$, the conditions $t_0 < t < t_1 < pa+t$ mean that $pa+t-t_1-1 > -1$ and $pa+t+n-s-t_1 > 0$, $t_1 - pa - n + s > -1$. By Lemma 2.1 and Lemma 2.3, Lemma 2.4(2), we have

$$\begin{aligned} &\int_{B_n} \frac{|\nabla f(w)|^p \mu^p(|w|)}{(1-|w|^2)^{pa}} \left\{ \int_{B_n} \frac{(1-|\xi|^2)^s (1-|z|^2)^{pa+t-t_1-1}\, dv(z)}{|1-\langle z,\xi\rangle|^{2s} |1-\langle z,w\rangle|^{n+1+t}} \right\} dv_{t_1}(w) \\ &\asymp \int_{B_n} \frac{|\nabla f(w)|^p \mu^p(|w|)}{(1-|w|^2)^{pa}} \frac{(1-|\xi|^2)^{pa+t+n-s-t_1}}{|1-\langle \xi,w\rangle|^{n+1+t}}\, dv_{t_1}(w) \end{aligned}$$

$$
\begin{aligned}
&+\int_{B_n}|\nabla f(w)|^p\frac{(1-|\xi|^2)^s}{|1-\langle w,\xi\rangle|^{2s}}\frac{\mu^p(|w|)}{1-|w|^2}\ dv(w)\\
&\lesssim ||f||^p_{F(p,\mu,s)}\int_{B_n}\frac{(1-|\xi|^2)^{pa+t+n-s-t_1}(1-|w|^2)^{t_1-pa-n+s}}{|1-\langle \xi,w\rangle|^{n+1+t}}\ dv(w)\\
&+\ ||f||^p_{F(p,\mu,s)}\asymp ||f||^p_{F(p,\mu,s)}.
\end{aligned}
$$

When $2s-(pa+t-t_1-1)=n+1$, the conditions $t_0<t<t_1<pa+t$ mean that we may take $0<\sigma_0<\min\{s,\ t_1+s+1-pa-n\}$ so that $pa+t-t_1-1>-1$ and $t_1-pa-n-\sigma_0+s>-1$, $s-\sigma_0>0$. By Lemma 2.1 and Lemma 2.3, Lemma 2.4(3), $\sup\limits_{0<x\le 1}x^{\sigma_0}\log\frac{e}{x}=\frac{e^{\sigma_0-1}}{\sigma_0}$, we get

$$
\begin{aligned}
&\int_{B_n}\frac{|\nabla f(w)|^p\mu^p(|w|)}{(1-|w|^2)^{pa}}\left\{\int_{B_n}\frac{(1-|\xi|^2)^s(1-|z|^2)^{pa+t-t_1-1}\ dv(z)}{|1-\langle z,\xi\rangle|^{2s}|1-\langle z,w\rangle|^{n+1+t}}\right\}dv_{t_1}(w)\\
&\asymp \int_{B_n}|\nabla f(w)|^p\frac{(1-|\xi|^2)^s}{|1-\langle w,\xi\rangle|^{2s}}\frac{\mu^p(|w|)}{1-|w|^2}\ dv(w)\\
&+\int_{B_n}\frac{|\nabla f(w)|^p\mu^p(|w|)}{(1-|w|^2)^{pa}}\frac{(1-|\xi|^2)^s}{|1-\langle \xi,w\rangle|^{n+1+t}}\log\frac{e}{1-|\varphi_w(\xi)|^2}\ dv_{t_1}(w)\\
&\lesssim ||f||^p_{F(p,\mu,s)}+||f||^p_{F(p,\mu,s)}\int_{B_n}\frac{(1-|w|^2)^{t_1-pa-n-\sigma_0+s}\ dv(w)}{(1-|\xi|^2)^{\sigma_0-s}|1-\langle \xi,w\rangle|^{n+1+t-2\sigma_0}}\asymp ||f||^p_{F(p,\mu,s)}.
\end{aligned}
$$

Similarly, if $t_0<t<t_1<pa+t$, then we may prove that

$$
\begin{aligned}
&\int_{B_n}\frac{|\nabla f(w)|^p\mu^p(|w|)}{(1-|w|^2)^{pb}}\left\{\int_{B_n}\frac{(1-|\xi|^2)^s(1-|z|^2)^{pb+t-t_1-1}\ dv(z)}{|1-\langle z,\xi\rangle|^{2s}|1-\langle z,w\rangle|^{n+1+t}}\right\}dv_{t_1}(w)\\
&\lesssim ||f||^p_{F(p,\mu,s)}.
\end{aligned}
$$

Therefore, (1) and Lemma 2.2(2) combined with the above discusses, we have

$$
\begin{aligned}
&\int_{B_n}|T_tf(z)|^p\frac{(1-|\xi|^2)^s}{|1-\langle z,\xi\rangle|^{2s}}\frac{\mu^p(|z|)}{1-|z|^2}\ dv(z)\\
&\lesssim \int_{B_n}|\nabla f(w)|^p\left\{\int_{B_n}\frac{(1-|\xi|^2)^s\mu^p(|z|)(1-|z|^2)^{t-t_1-1}\ dv(z)}{|1-\langle z,\xi\rangle|^{2s}|1-\langle z,w\rangle|^{n+1+t}}\right\}dv_{t_1}(w)\\
&\lesssim \int_{B_n}\frac{|\nabla f(w)|^p\mu^p(|w|)}{(1-|w|^2)^{pa}}\left\{\int_{B_n}\frac{(1-|\xi|^2)^s(1-|z|^2)^{pa+t-t_1-1}\ dv(z)}{|1-\langle z,\xi\rangle|^{2s}|1-\langle z,w\rangle|^{n+1+t}}\right\}dv_{t_1}(w)\\
&+\int_{B_n}\frac{|\nabla f(w)|^p\mu^p(|w|)}{(1-|w|^2)^{pb}}\left\{\int_{B_n}\frac{(1-|\xi|^2)^s(1-|z|^2)^{pb+t-t_1-1}\ dv(z)}{|1-\langle z,\xi\rangle|^{2s}|1-\langle z,w\rangle|^{n+1+t}}\right\}dv_{t_1}(w)\\
&\lesssim ||f||^p_{F(p,\mu,s)}.
\end{aligned}
$$

(2) Case $0<p\le 1$.

We choose $b+(n-s)/p-1<t=\frac{n+1+t'}{p}-n-1$ such that $t'>pb+n-s-1$.

For any $z\in B_n$ and $l\in\{1,2,\cdots,n\}$, we take

$$F_l(w)=\frac{\partial f}{\partial w_l}(w)\frac{1}{(1-\langle w,z\rangle)^{n+1+t}}\quad (w\in B_n).$$

It follows from Lemma 2.15 in [13] that

$$\int_{B_n}|F_l(w)|(1-|w|^2)^t\ dv(w)\lesssim\left\{\int_{B_n}|F_l(w)|^p\ dv_{t'}(w)\right\}^{\frac{1}{p}}.$$

This shows that

$$|T_t f(z)|^p \asymp \sum_{l=1}^{n}\left\{\int_{B_n}|F_l(w)|(1-|w|^2)^t\ dv(w)\right\}^p$$
$$\lesssim \sum_{l=1}^{n}\int_{B_n}|F_l(w)|^p\ dv_{t'}(w) \asymp \int_{B_n}\frac{|\nabla f(w)|^p\ dv_{t'}(w)}{|1-\langle w,z\rangle|^{n+1+t'}}.$$

Next, it is similar to the proof of case $p>1$ when $t'>pb+n-s-1$. We have

$$\int_{B_n}|T_t f(z)|^p\frac{(1-|\xi|^2)^s}{|1-\langle z,\xi\rangle|^{2s}}\frac{\mu^p(|z|)}{1-|z|^2}\ dv(z)$$
$$\lesssim \int_{B_n}\frac{|\nabla f(w)|^p\mu^p(|w|)}{(1-|w|^2)^{pa}}\left\{\int_{B_n}\frac{(1-|\xi|^2)^s(1-|z|^2)^{pa-1}\ dv(z)}{|1-\langle z,\xi\rangle|^{2s}|1-\langle z,w\rangle|^{n+1+t'}}\right\}dv_{t'}(w)$$
$$+\int_{B_n}\frac{|\nabla f(w)|^p\mu^p(|w|)}{(1-|w|^2)^{pb}}\left\{\int_{B_n}\frac{(1-|\xi|^2)^s(1-|z|^2)^{pb-1}\ dv(z)}{|1-\langle z,\xi\rangle|^{2s}|1-\langle z,w\rangle|^{n+1+t'}}\right\}dv_{t'}(w)$$
$$\lesssim ||f||^p_{F(p,\mu,s)}.$$

This proof is completed. □

As an application of Theorem 3.1, we prove the Gleason's problem is solvable on $F(p,\mu,s)$.

Theorem 3.2 Let μ be a normal function on $[0,1)$. For any integer $k\geq 1$ and $\beta\in B_n$, there exist bounded linear operators A_α $(|\alpha|=k)$ on $F(p,\mu,s)$ such that

$$f(z)=\sum_{|\alpha|=k}(z-\beta)^\alpha A_\alpha f(z)\ \text{ for any } f\in F(p,\mu,s)$$

with $D^\gamma f(\beta)=0$ $(|\gamma|=0,1,\cdots,k-1)$, where α and γ are multi-index.

Proof When $s>n$, it follows from Lemma 2.3 that $F(p,\mu,s)=\mathcal{B}_\nu(B_n)$. This result has been proved in [33]. We need only to consider the case $0\leq s\leq n$.

First, we consider the case $k=1$.

When $\beta=(0,\cdots,0)$, for any $f\in F(p,\mu,s)$ with $f(\beta)=0$ and $j\in\{1,\dots,n\}$, let

$$A_j f(z)=\int_0^1\frac{\partial f}{\partial z_j}(tz)\ dt.$$

Then each A_j is a linear operator and

$$\sum_{j=1}^{n}z_jA_jf(z)=\int_0^1\frac{Rf(tz)}{t}\ dt=f(z)-f(\beta)=f(z). \tag{2}$$

In the following, we prove that A_j is bounded on $F(p,\mu,s)$ for every $j\in\{1,2,\dots,n\}$.

It follows from Lemma 2.3 that

$$\left|\frac{\partial f}{\partial z_j}(z)\right|\leq|\nabla f(z)|\lesssim\frac{||f||_{F(p,\mu,s)}}{\mu(|z|)(1-|z|^2)^{\frac{n-s}{p}}}\leq\frac{||f||_{F(p,\mu,s)}}{\mu(0)(1-|z|^2)^{b+\frac{n-s}{p}}}.$$

Therefore, as long as $t-b-(n-s)/p>-1$, it is clear that

$$\int_{B_n}\left|\frac{\partial f}{\partial z_j}(z)\right|dv_t(z)\lesssim||f||_{F(p,\mu,s)}\int_{B_n}(1-|z|^2)^{t-b-\frac{n-s}{p}}\ dv(z)<\infty.$$

This shows that $\frac{\partial f}{\partial z_j} \in A_t^1(B_n)$. By Theorem 2.2 in [13], we have

$$\frac{\partial f}{\partial z_j}(z) = \int_{B_n} \frac{\partial f}{\partial w_j}(w) \frac{1}{(1-\langle z,w\rangle)^{n+1+t}}\ dv_t(w) \quad (z \in B_n). \tag{3}$$

By (2)-(3) and Fubini Theorem, it is clear that

$$A_j f(z) = \int_{B_n} \frac{\partial f}{\partial w_j}(w) \left\{ \int_0^1 \frac{d\rho}{(1-\rho\langle z,w\rangle)^{n+t+1}} \right\} dv_t(w).$$

This shows that

$$\begin{aligned}\nabla A_j f(z) &= \int_{B_n} \frac{\partial f}{\partial w_j}(w) \left\{ \int_0^1 \frac{(n+t+1)\rho\ d\rho}{(1-\rho\langle z,w\rangle)^{n+t+2}} \right\} \overline{w}\ dv_t(w) \\ &= \int_{B_n} \frac{\partial f}{\partial w_j}(w) \frac{\overline{w}\ Q(z,w)}{(1-\langle z,w\rangle)^{n+t+1}}\ dv_t(w),\end{aligned}$$

where $\overline{w}$ is a vector, and the integral

$$\int_{B_n} (\cdot)\ \overline{w}\ dv_t(w) = \left(\int_{B_n} (\cdot)\ \overline{w_1}\ dv_t(w), \cdots, \int_{B_n} (\cdot)\ \overline{w_n}\ dv_t(w) \right),$$

$$Q(z,w) = \frac{n+1+t}{(n+t)\langle z,w\rangle} + \frac{1}{n+t} \frac{(1-\langle z,w\rangle)^{n+t+1}-1}{\langle z,w\rangle^2}$$

when $\langle z,w\rangle \neq 0$ or $Q(z,w) = (n+1+t)/2$ when $\langle z,w\rangle = 0$. Otherwise,

$$\lim_{y\to 0} \left\{ \frac{n+1+t}{(n+t)y} + \frac{1}{n+t} \frac{(1-y)^{n+t+1}-1}{y^2} \right\} = \frac{n+1+t}{2}.$$

This means that $|Q(z,w)| \lesssim 1$. Therefore,

$$|\nabla A_j f(z)| \lesssim \int_{B_n} \frac{|\nabla f(w)|}{|1-\langle z,w\rangle|^{n+1+t}}\ dv_t(w) = T_t f(z). \tag{4}$$

As long as t sufficiently large, it follows from (4) and Theorem 3.1 that

$$\sup_{\xi\in B_n} \int_{B_n} |\nabla A_j f(z)|^p \frac{(1-|\xi|^2)^s}{|1-\langle z,\xi\rangle|^{2s}} \frac{\mu^p(|z|)}{1-|z|^2}\ dv(z) \lesssim ||f||^p_{F(p,\mu,s)}.$$

On the other hand, it follows from Lemma 2.3 that

$$|A_j f(0,\cdots,0)| = \left| \int_{B_n} \frac{\partial f}{\partial w_j}(w)\ dv_t(w) \right| \lesssim ||f||_{F(p,\mu,s)}.$$

In a word, we have $||A_j f||_{F(p,\mu,s)} \lesssim ||f||_{F(p,\mu,s)}$ for all cases.

When $\beta = (\beta_1,\cdots,\beta_n) \neq (0,\cdots,0)$, it is clear that

$$\begin{aligned}f(z) = f(z) - f(\beta) &= \int_0^1 \left\{ \frac{d}{dt} f[\beta + t(z-\beta)] \right\} dt \\ &= \sum_{j=1}^n (z_j - \beta_j) \int_0^1 D_j f[\beta + t(z-\beta)]\ dt = \sum_{j=1}^n (z_j-\beta_j) A_j f(z).\end{aligned}$$

It is similar to the proof of (4). We may obtain

$$\begin{aligned}|\nabla A_j f(z)| &\lesssim \int_{B_n} \frac{|\nabla f(w)|}{|1-\langle \beta,w\rangle|^{n+2+t}|1-\langle z,w\rangle|^{n+1+t}}\ dv_t(w) \\ &\le \frac{1}{(1-|\beta|)^{n+2+t}} \int_{B_n} \frac{|\nabla f(w)|}{|1-\langle z,w\rangle|^{n+1+t}}\ dv_t(w) \asymp Tf(z).\end{aligned}$$

It follows from Theorem 3.1 that A_j is bounded on $F(p,\mu,s)$.

For $k \geq 2$, the proof is the same as that of Theorem 5 in [20].

This proof is completed. $\square$

Declarations

Conflict of interest The authors declare no conflict of interest.

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