Appl. Math. J. Chinese Univ.
2023, 38(4): 591-603

Least square method based on Haar wavelet to solve multi-dimensional stochastic Itô-Volterra integral equations

JIANG Guo* KE Ting DENG Meng-ting

Abstract. This paper proposes a method combining blue the Haar wavelet and the least square to solve the multi-dimensional stochastic Itô-Volterra integral equation. This approach is to transform stochastic integral equations into a system of algebraic equations. Meanwhile, the error analysis is proven. Finally, the effectiveness of the approach is verified by two numerical examples.

§1 Introduction

In practical applications, on the one hand, complex systems in engineering and physics are affected by stochastic factors such as stochastic disturbances, stochastic environments, stochastic boundary conditions, stochastic inputs, and stochastic initial conditions; on the other hand, people ignore the processes whose physical reasons are temporarily unclear. Both of these cases can be described by stochastic integral equation or approximated by stochastic process. The research field of SDE is very broad, for example, engineering, the nonlinear age-structured population model and mathematical finance and so on ([1,2]).

However, many stochastic integral equations can not be solved explicitly. Hence, it is of great significance to study a convenient and accurate numerical algorithm. Different orthogonal basis functions or polynomials, for instance, Iterative technique, Walsh functions, block pulse functions(BPFs), Chebyshev polynomials, Legendre polynomials and Fourier series were applied to solve different Volterra integral equations. Here, we only mentioned the references such as the papers ([3-13]) and other relevant literatures. In recent years, as a powerful tool, Haar wavelet has been widely applied in numerical analysis. Many scholars have used it to solve stochastic equations ([14-17]).

In paper ([11]), authors utilized BPFs to gain the numerical solution of the linear stochastic Itô-Volterra integral equations (SIVIEs). The authors ([18]) put forward a computational

Received: 2022-05-18. Revised: 2022-07-11.
MR Subject Classification: 60H20, 45D99, 65C30.
Keywords: least squares method, Haar wavelet, Itô-Volterra integral equations, integration operational matrix.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-023-4748-y.
Supported by the NSF of Hubei Province(2022CFD042).
*Corresponding author.

method to solve SIVIEs by the least squares method and BPFs. Moreover, the paper ([19]) solved the m-dimensional SIVIEs based on BPFs. In paper ([14]), the author derived the approximate solution of linear SIVIEs by Haar wavelets (HWs). The paper ([16]) proposed an effective numerical method for solving nonlinear SIVIEs based on HWs.

In this article, we study the following multi-dimensional linear SIVIEs

$$V(t) = g(t) + \int_0^t \widetilde{M}(\tau,t)V(\tau)d\tau + \sum_{r=1}^{q}\int_0^t \widehat{M_r}(\tau,t)V(\tau)dB_r(\tau), t \in [0,T), \tag{1}$$

where $r = 1, 2, \cdots, q$ and $0 < \tau < t$. $V(t)$ is an unknown stochastic process and $B_r(\tau)$ are Brownian motions. They are all denoted on the same probability space of $(\Omega, \mathcal{F}, P)$. $\widetilde{M}(\tau,t)$ and $\widehat{M_r}(\tau,t)$ are kernel functions, $g(t)$ is an initial function. Moreover, $\int_0^t \widehat{M_r}(\tau,t)V(\tau)dB_r(\tau)$ are the Itô integrals.

Compared with the above papers ([11,18,19]), the difference of this paper is to discuss the numerical solution of the linear SIVIEs with respect to multiple independent Brownie motions using the Least square method and Haar wavelet. The method combines the least square method with the haar wavelet to obtain a more accurate numerical solution. The most significant innovation is that SIVIEs are converted into a system of algebraic equations by this approach. According to the simulation results and numerical results of numerical examples in Section 5, it can be found that the error of this method is relatively small, and the approximate solution is closer to the exact solution than the literature ([19]).

In section 2, we give some fundamental theories about BPFs and HWs. The relationship between BPFs and HWs is presented in section 3. In section 4, the computational method is given concretely. In section 5, the error analysis is acquired. Two numerical examples are used to confirm the availability of the approach in section 6.

§2 Preliminaries

BPFs and HWs are recommended in this section, the details see literatures ([4,10,11,17,19]).

2.1 Block Pulse Functions

The definition of BPFs is as follows

$$\xi_l(t) = \begin{cases} 1, & lh \le t < (l+1)h, \\ 0, & otherwise. \end{cases} \tag{2}$$

where $l = 0, 1, \cdots, m-1$, $h = \frac{T}{m}$ and $t \in [0,T)$.

They have the following properties

i) Orthogonality:

$$\int_0^T \xi_l(t)\xi_j(t)dt = h\delta_{lj}. \tag{3}$$

where δ_{lj} is Kronecker delta.

ii) Disjointness:

$$\xi_l(t)\xi_j(t) = \delta_{lj}\xi_l(t), \quad l, j = 0, 1, \cdots, m-1, \tag{4}$$

iii) Completeness: for every $f(t) \in L^2([0,T))$, then

$$\int_0^T f^2(t)dt = \sum_{l=1}^{\infty} f_l^2 \|\xi_l(t)\|^2, \tag{5}$$

where $f_l = \frac{1}{h}\int_0^T f(t)\xi_l(t)dt$.

The vector form of BPFs is given

$$\Lambda_m(t) = (\xi_0(t), \xi_1(t), \cdots, \xi_{m-1}(t))^T. \tag{6}$$

From the above description,

$$\Lambda_m(t)\Lambda_m^T(t) = \begin{pmatrix} \xi_0(t) & 0 & \cdots & 0 \\ 0 & \xi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \xi_{m-1}(t) \end{pmatrix}_{m\times m},$$

$$\Lambda_m^T(t)\Lambda_m(t) = 1,$$

Suppose that a m-vector $F_m = (f_0, f_1, \cdots, f_{m-1})^T$, then

$$\Lambda_m(t)\Lambda_m^T(t)F_m = \mathbf{D}_{F_m}\Lambda_m(t),$$

where $\mathbf{D}_F$ is a diagonal matrix whose diagonal entries are the vector F_m.

If $\mathbf{K}$ is an $m \times m$ matrix, then

$$\Lambda_m^T(t)\mathbf{K}\Lambda_m(t) = \widehat{K}^T\Lambda_m(t),$$

where $\widehat{K}$ is a m-vector composed of the diagonal entries of $\mathbf{K}$.

Any function $V(t) \in L^2([0,T))$ can be approximated as

$$V(t) \simeq V_m(t) = \sum_{i=0}^{m-1} v_i\xi_i(t) = \hat{V}_m^T\Lambda_m(t) = \Lambda_m^T(t)\hat{V}_m, \tag{7}$$

where

$$\hat{V}_m = (v_0, v_1, \cdots, v_{m-1})^T. \tag{8}$$

For any $M(\tau,t) \in L^2([0,T_1) \times [0,T_2))$, it can also be expanded as

$$M(\tau,t) = \Lambda_{m_1}^T(\tau)\mathbf{M}\Lambda_{m_2}(t) = \Lambda_{m_2}^T(t)\mathbf{M}^T\Lambda_{m_1}(\tau),$$

where $\mathbf{M} = (\hat{m}_{ij})_{m_1\times m_2}$,

$$\hat{m}_{ij} = \frac{1}{h_1h_2}\int_0^{T_1}\int_0^{T_2} M(\tau,t)\xi_i(\tau)\xi_j(t)d\tau dt, \tag{9}$$

and $h_1 = \frac{T_1}{m_1}, h_2 = \frac{T_2}{m_2}$.

2.2 Haar wavelets

HWs are denoted as ([14])

$$h_i(t) = 2^{\frac{j}{2}}h(2^jt - z), \quad j \geq 0, 0 \leq z < 2^j, i = 2^j + z, i,j,z \in \mathbb{N},$$

where $h_0(t) = 1, t \in [0,1)$, and

$$h(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1. \end{cases} \tag{10}$$

The $h_i(t)$ are pairwise orthonormal and

$$\int_0^1 h_i(t)h_j(t)dt = \delta_{ij}. \tag{11}$$

Any squared integrable function $V(t)$ on $[0,1)$ can be approximated by HWs

$$V(t) = c_0h_0(t) + \sum_{i=1}^{\infty} c_ih_i(t), \tag{12}$$

where

$$c_i = \int_0^1 V(t)h_i(t)dt,$$

and $i = 0$ or $i = 2^j + z$.

It can be found that when $m = 2^J$ (J is the wavelet resolution), (12) can be rewritten as

$$V(t) = c_0h_0(t) + \sum_{i=1}^{m-1} c_ih_i(t), \quad i = 2^j + z, \quad j = 0, 1, \cdots, J-1.$$

or

$$V(t) \simeq C_m^T H_m(t) = H_m^T(t)C_m, \tag{13}$$

where $H_m(t) = (h_0(t), h_1(t), \cdots, h_{m-1}(t))^T$ and $C_m = (c_0, c_1, \cdots, c_{m-1})^T$.

Any $M(\tau, t) \in L^2([0,1) \times [0,1))$ can be expanded as

$$M(\tau, t) = H_{m_1}^T(\tau)\mathbf{M}H_{m_2}(t),$$

where $\mathbf{M} = (\hat{m}_{il})_{m_1\times m_2}$,

$$\hat{m}_{il} = \int_0^1\int_0^1 M(\tau, t)h_i(\tau)h_l(t)d\tau dt, \tag{14}$$

and $i, l = 0, 1, \cdots, m-1$. For convenience, let $m_1 = m_2 = m$ in the following sections.

§3 HWs and BPFs

This section introduces the relationship between HWs and BPFs and the related Lemmas.We let $T = 1$ in BPFs in this section.

Lemma 1. *Let $H_m(t)$ and $\Lambda_m(t)$ are HWs and BPFs vector respectively ([14]),*

$$H_m(t) = \mathbf{P}\Lambda_m(t), \quad m = 2^J, \tag{15}$$

where $\mathbf{P} = (P_{ij})_{m\times m}$ and

$$P_{ij} = 2^{\frac{j}{2}}h_{i-1}(\frac{2j-1}{2m}), \quad i-1 = 2^j + z, 0 \le z < 2^j, i, j = 1, \cdots, m.$$

Remark 1. *According to the definition of $\mathbf{P}$ in (15), we have ([14])*

$$\mathbf{P}^{-1} = \frac{1}{m}\mathbf{P}^T.$$

Remark 2. *For any an m-vector U, then ([14])*

$$H_m(t)H_m^T(t)U = \tilde{\mathbf{U}}H_m(t),$$

where $\tilde{\mathbf{U}} = \mathbf{P}\bar{\mathbf{U}}\mathbf{P}^{-1}$ is an matrix and $\bar{\mathbf{U}}$ is a diagonal matrix whose diagonal entries are the vector $\mathbf{P}^TU$.

Remark 3. *Suppose that* $\mathbf{S}$ *is an* $m \times m$ *matrix, we get ([14])*

$$H_m^T(t)\mathbf{S}H_m(t) = \widehat{S}^T H_m(t),$$

where $\widehat{S}^T = N\mathbf{P}^{-1}$ *is a vector and the entries of the vector* N *are the diagonal entries of* $\mathbf{P^T S P}$.

Lemma 2. *Suppose that* $\Lambda_m(t)$ *is defined in (6), there are ([11,14])*

$$\int_0^t \Lambda_m(\tau)d\tau \simeq \mathbf{Q}\Lambda_m(t), \tag{16}$$

where

$$\mathbf{Q} = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m\times m}.$$

Lemma 3. *Suppose that* $\Lambda_m(t)$ *is defined in (6), then ([11,14])*

$$\int_0^t \Lambda_m(\tau)dB(\tau) \simeq \mathbf{Q}_B\Lambda_m(t), \tag{17}$$

where

$$\mathbf{Q}_B = \begin{pmatrix} B_{\frac{h}{2}} & B_h & B_h & \cdots & B_h \\ 0 & B_{\frac{3h}{2}} - B_h & B_{2h} - B_h & \cdots & B_{2h} - B_h^H \\ 0 & 0 & B_{\frac{5h}{2}} - B_{2h} & \cdots & B_{3h} - B_{2h} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_{\frac{(2m-1)h}{2}} - B_{(m-1)h} \end{pmatrix}_{m\times m},$$

Lemma 4. *Suppose that* $H_m(t)$ *is given in (13), there are ([11,14])*

$$\int_0^t H_m(\tau)d\tau \simeq \frac{1}{m}\mathbf{P}\mathbf{Q}\mathbf{P}^T H_m(t) = \mathbf{\Upsilon} H_m(t),$$

where $\mathbf{P}$ *and* $\mathbf{Q}$ *are the same as (15) and Lemma 2 respectively,* $\mathbf{\Upsilon} = \frac{1}{m}\mathbf{P}\mathbf{Q}\mathbf{P}^T$.

Lemma 5. *Suppose that* $H_m(t)$ *is given in (13), there are ([11,14])*

$$\int_0^t H_m(\tau)dB(\tau) \simeq \frac{1}{m}\mathbf{P}\mathbf{Q}_B\mathbf{P}^T H_m(t) = \mathbf{\Upsilon}_B H_m(t),$$

where $\mathbf{P}$ *and* $\mathbf{Q}_B$ *are the same as (15) and Lemma 3 respectively,* $\mathbf{\Upsilon}_B = \frac{1}{m}\mathbf{P}\mathbf{Q}_B\mathbf{P}^T$.

§4 Method description

First we give the following operator

$$L(V(t)) := V(t) - \int_0^t \widetilde{M}(\tau,t)V(\tau)d\tau - \sum_{r=1}^{q}\int_0^t \widehat{M}_r(\tau,t)V(\tau)dB_r(\tau). \tag{18}$$

where L is linear operator.

If $V(t)$ is the precise solution to (1), we have

$$\| L(V(t)) - g(t) \|_{L^2} = 0.$$

Suppose $\varepsilon > 0$, $V_\varepsilon(t)$ is the numerical solution, then the residual norm of $V_\varepsilon(t)$ is as follows

$$\| L(V_\varepsilon(t)) - g(t) \|_{L^2} < \varepsilon. \tag{19}$$

In terms of (19), we now propose a method to obtain an approximate solution, $V_m(t)$ is a linear combination of HWs,

$$V_m(t) = \sum_{i=0}^{m-1} c_i h_i(t),$$

where c_i is the unknown.

Next, we discuss the following problem

$$\min_{c_0,\cdots,c_{m-1}} \| L(V_m(t)) - g(t) \|_{L^2} . \tag{20}$$

A set of values of $\widehat{c}_0, \cdots, \widehat{c}_{m-1}$ is got by minimizing (20), then $\sum_{i=1}^{m} \hat{c}_i h_i(t)$ is an approximate solution of (1), and

$$\mathbb{E}\left[\min_{c_0,\cdots,c_{m-1}} \|L(V_m(t)) - g(t)\|_{L^2}\right] = \mathbb{E}\left[\left\| L(\sum_{i=0}^{m-1} \widehat{c}_i h_i) - g(t)\right\|_{L^2}\right] \to 0, \tag{21}$$

where $m \to \infty$ (or $h \to 0$).

For obtaining the minimum value of (20), we must take the partial derivative of c_i,

$$\frac{\partial}{\partial c_i} \int_0^T \left(\sum_{j=0}^{m-1} c_j L(h_j(t)) - g(t) \right)^2 dt = 0, i = 0, 1, \cdots, m-1,$$

then

$$\int_0^T \sum_{j=0}^{m-1} c_j L(h_j(t)) L(h_i(t)) dt = \int_0^T g(t) L(h_i(t)) dt, \tag{22}$$

and

$$\sum_{j=0}^{m-1} c_j \langle L(h_i(t)), L(h_j(t))\rangle = \langle L(h_i(t)), g(t))\rangle,$$

where $\langle L(h_i(t)), L(h_j(t))\rangle = \int_0^T L(h_i(t)) L(h_j(t)) dt$.

We define the following matrix

$$C = \mathbf{A}^{-1} W, \tag{23}$$

where $W = (w_0, w_1, \cdots, w_{m-1})^T \in \mathbb{R}^m$, $w_i = \langle L(h_i(t)), g(t)\rangle$, $C = (c_0, c_1, \cdots, c_{m-1})^T \in \mathbb{R}^m$, and $\mathbf{A} = (\eta_{ij}) \in \mathbb{R}^{m\times m}$, $\eta_{ij} = \langle L(h_i(t)), L(h_j(t))\rangle, i, j = 0, 1, \cdots, m-1$.

According to the previous content, we know that $V(t), g(t), \widetilde{M}(u,t)$ and $\widehat{M}(u,t)$ can be expanded as

$$V(t) \simeq V_m(t) = C_m^T H_m(t) = H_m^T(t) C_m, \tag{24}$$

$$g(t) \simeq g_m(t) = G_m^T H_m(t) = H_m^T(t) G_m, \tag{25}$$

$$\widetilde{M}(\tau,t) \simeq \widetilde{M}_m(\tau,t) = H_m^T(\tau)\mathbf{M}_1 H_m(t) = H_m^T(t)\mathbf{M}_1^T H_m(\tau), \tag{26}$$

$$\widehat{M}(\tau,t) \simeq \widehat{M}_m(\tau,t) = H_m^T(\tau)\mathbf{M}_2 H_m(t) = H_m^T(t)\mathbf{M}_2^T H_m(\tau), \tag{27}$$

where $\mathbf{M}_1$and $\mathbf{M}_2$ are HWs coefficients matrices, C_m and G_m are HWs coefficients vector.

Now, for solving η_{ij} and w_i. By (18), (24)-(27), we get

$$\begin{aligned} L(h_i(t)) &= h_i(t) - \int_0^t \widetilde{M}(\tau,t)h_i(\tau)d\tau - \sum_{r=1}^{q}\int_0^t \widehat{M}_r(\tau,t)h_i(\tau)dB_r(\tau) \\ &= I_iH_m(t) - H_m^T(t)\mathbf{M}_1^T\int_0^t H_m(\tau)H_m^T(\tau)I_id\tau \\ &- H_m^T(t)\sum_{r=1}^{q}\mathbf{M}_2^T\int_0^t H_m(\tau)H_m^T(\tau)I_idB_r(\tau), \end{aligned} \tag{28}$$

where I_i is a vector which is the ith row of an $m\times m$ identity matrix.

According to Remark 2,

$$L(h_i(t)) = I_iH_m(t) - H_m^T(t)\mathbf{M}_1^T\int_0^t \tilde{\mathbf{I}}_iH_m(\tau)d\tau - H_m^T(t)\sum_{r=1}^{q}\mathbf{M}_2^T\int_0^t \tilde{\mathbf{I}}_iH_m(\tau)dB_r(\tau).$$

By Lemma 4, Lemma 5, and Remark 3,

$$\begin{aligned} L(h_i(t)) &= I_iH_m(t) - H_m^T(t)\mathbf{M}_1^T\tilde{\mathbf{I}}_i\boldsymbol{\Upsilon}H_m(t) - H_m^T(t)\sum_{r=1}^{q}\mathbf{M}_2^T\tilde{\mathbf{I}}_i\boldsymbol{\Upsilon}_{B_r}H_m(t) \\ &= \left(I_i - H_m^T(t)\mathbf{M}_1^T\tilde{\mathbf{I}}_i\boldsymbol{\Upsilon} - H_m^T(t)\sum_{r=1}^{q}\mathbf{M}_2^T\tilde{\mathbf{I}}_i\boldsymbol{\Upsilon}_{B_r}\right)H_m(t) \\ &= I_iH_m(t) - H_m^T(t)\alpha H_m(t) - H_m^T(t)\beta H_m(t) \\ &= (I_i - \hat{\alpha} - \hat{\beta})H_m(t), \end{aligned} \tag{29}$$

where matrices $\alpha = \mathbf{M}_1^T\tilde{\mathbf{I}}_i\boldsymbol{\Upsilon}$, $\beta = \sum_{r=1}^{q}\mathbf{M}_2^T\tilde{\mathbf{I}}_i\boldsymbol{\Upsilon}_{B_r}$, and vector $\hat{\alpha}$ and $\hat{\beta}$ can be obtained by Remark 3.

By (29) and Remark 2,

$$\begin{aligned} \eta_{ij} &= \langle L(h_i(t)), L(h_j(t))\rangle \\ &= \int_0^T L(h_i(t))L(h_j(t))dt \\ &= \int_0^T (I_i - \hat{\alpha} - \hat{\beta})H_m(t)(I_j - \hat{\alpha} - \hat{\beta})H_m(t)dt \\ &= X_1\int_0^T H_m(t)H_m^T(t)X_2^Tdt \\ &= X_1\int_0^T \tilde{\mathbf{X}}_2H_m(t)dt \\ &= X_1\tilde{\mathbf{X}}_2\boldsymbol{\Upsilon}H_m(T), \end{aligned} \tag{30}$$

where vectors $X_1 = (I_i - \hat{\alpha} - \hat{\beta})$, $X_2 = (I_j - \hat{\alpha} - \hat{\beta})$, and matrix $\tilde{\mathbf{X}}_2$ can be obtained by Remark 2.

$$w_i = \langle L(h_i(t)), g(t)\rangle$$

$$
\begin{aligned}
&= \int_0^T L(h_i(t))g(t)dt \\
&\simeq \int_0^T X_1 H_m(t) G_m^T H_m(t)dt \\
&= X_1 \int_0^T H_m(t) H_m^T(t) G_m dt \\
&= X_1 \int_0^T \tilde{\mathbf{G}}_m H_m(t)dt \\
&= X_1 \tilde{\mathbf{G}}_m \boldsymbol{\Upsilon} H_m(T),
\end{aligned} \tag{31}
$$

where matrix $\tilde{\mathbf{G}}_m$ can be obtained by Remark 2.

§5 Error analysis

We firstly give two Lemmas before discussing error analysis.

Lemma 6. *(Continuous module) The continuity module $\omega(g, \theta)$ of the general function g with respect to θ on $[0, T]$ is denoted as ([18])*

$$\omega(g, \theta) = \sup\{|g(x) - g(y)||x, y \in [0, T], |x - y| \leq \theta, \theta > 0\}. \tag{32}$$

If and only if $\lim_{\theta \to 0} \omega(g, \theta) = 0$, $g(t)$ is uniformly continuous over $[0, T]$ (see the reference ([20])).

Lemma 7. *For any function $g \in C[0, T]$, we have ([18])*

$$\|g - R_h\|_\infty \leq \omega(g, h).$$

where $R_h = \sum_{l=1}^m y_l h_l(t)$, $y_l = g(\frac{t_{l-1}+t_l}{2})$, $t_l = lh$, $h = \frac{T}{m}$, $l = 1, \cdots, m$.

Theorem 1. *Assume that $\widetilde{V}_m(t) = \sum_{i=1}^m e_i h_i(t)$ where $e_i = V(\frac{t_{i-1}+t_i}{2})$, $\widetilde{M}(\tau, t)$ and $\widehat{M}_r(\tau, t)$ are known functions, $\|\widetilde{M}(\tau, t)\|_\infty \leq \zeta, \|\widehat{M}_r(\tau, t)\|_\infty \leq \zeta$, where ζ is a positive constant. Then when $h \to 0$, we get*

(i)

$$\|\widetilde{V}_m(t) - V(t)\|_{\infty, E} = \mathbb{E}\left[\sup_{t \in [0, T)} |\widetilde{V}_m(t) - V(t)|\right] \to 0.$$

(ii)

$$\mathbb{E}\left[\min_{c_1, \cdots, c_m} \left\| V_m(t) - g(t) - \int_0^t \widetilde{M}(\tau, t) V_m(\tau) d\tau - \sum_{r=1}^q \int_0^t \widehat{M}_r(\tau, t) V_m(\tau) dB_r(\tau) \right\|_{L^2}^2 \right] \to 0.$$

Proof. (i) According to Lemma7, we have

$$\|\widetilde{V}_m(t) - V(t)\|_{\infty, E} = \mathbb{E}\left[\sup_{t \in [0, T)} |\widetilde{V}_m(t) - V(t)|\right] \leq \mathbb{E}[\omega(V, h)] \to 0. \tag{33}$$

(ii)

$$
\begin{aligned}
&\mathbb{E}\left[\min_{c_1,\cdots,c_m}\left\|V_m(t)-g(t)-\int_0^t \widetilde{M}(\tau,t)V_m(\tau)d\tau-\sum_{r=1}^{q}\int_0^t \widehat{M}_r(\tau,t)V_m(\tau)dB_r(\tau)\right\|_{L^2}^2\right]\\
&\le \mathbb{E}\left[\left\|\widetilde{V}_m(t)-g(t)-\int_0^t \widetilde{M}(\tau,t)\widetilde{V}_m(\tau)d\tau-\sum_{r=1}^{q}\int_0^t \widehat{M}(\tau,t)\widetilde{V}_m(\tau)dB_r(\tau)\right\|_{L^2}^2\right]\\
&\le T^2\left\|\widetilde{V}_m(t)-g(t)-\int_0^t \widetilde{M}(\tau,t)\widetilde{V}_m(\tau)d\tau-\sum_{r=1}^{q}\int_0^t \widehat{M}(\tau,t)\widetilde{V}_m(\tau)dB_r(\tau)\right\|_{\infty,E}^2\\
&= T^2\\
&\left\|\widetilde{V}_m(t)-V(t)+\int_0^t \widetilde{M}(\tau,t)(V(\tau)-\widetilde{V}_m(\tau))d\tau+\sum_{r=1}^{q}\int_0^t \widehat{M}(\tau,t)(V(\tau)-\widetilde{V}_m(\tau))dB_r(\tau)\right\|_{\infty,E}^2\\
&\le 3T^2\left(\|\widetilde{V}_m(t)-V(t)\|_{\infty,E}^2+\left\|\int_0^t \widetilde{M}(\tau,t)(V(\tau)-\widetilde{V}_m(\tau))d\tau\right\|_{\infty,E}^2\right.\\
&\left.+\left\|\sum_{r=1}^{q}\int_0^t \widehat{M}(\tau,t)(V(\tau)-\widetilde{V}_m(\tau))dB_r(\tau)\right\|_{\infty,E}^2\right)\\
&\le 3T^2\|\widetilde{V}_m(t)-V(t)\|_{\infty,E}^2+3T^2\zeta^2\|V(t)-\widetilde{V}_m(t)\|_{\infty,E}^2\\
&+3T^2\left\|\sum_{r=1}^{q}\int_0^t \widehat{M}(\tau,t)(V(\tau)-\widetilde{V}_m(\tau))dB_r(\tau)\right\|_{\infty,E}^2.
\end{aligned}
\tag{34}
$$

On the basis of Doob's inequality and isometry property, we have

$$
\begin{aligned}
&\left\|\sum_{r=1}^{q}\int_0^t \widehat{M}(\tau,t)(V(\tau)-\widetilde{V}_m(\tau))dB_r(\tau)\right\|_{\infty,E}^2\\
&= \mathbb{E}\left[\sup_{0\le\tau\le t}\left|\sum_{r=1}^{q}\int_0^t \widehat{M}(\tau,t)(V(\tau)-\widetilde{V}_m(\tau))dB_r(\tau)\right|^2\right]\\
&\le 4\mathbb{E}\left[\left|\sum_{r=1}^{q}\int_0^t \widehat{M}(\tau,t)(V(\tau)-\widetilde{V}_m(\tau))dB_r(\tau)\right|^2\right]\\
&\le 4\mathbb{E}q\left[\sum_{r=1}^{q}\left|\int_0^t \widehat{M}(\tau,t)(V(\tau)-\widetilde{V}_m(\tau))dB_r(\tau)\right|^2\right]\\
&= 4\mathbb{E}q^2\left[\int_0^t |\widehat{M}(\tau,t)|^2|V(\tau)-\widetilde{V}_m(\tau)|^2d\tau\right]\\
&\le 4T\zeta^2q^2\|V(t)-\widetilde{V}_m(t)\|_{\infty,E}^2.
\end{aligned}
\tag{35}
$$

By (33)-(35), we get

$$\begin{aligned}&\mathbb{E}\left[\min_{c_1,\cdots,c_m}\left\|V_m(t)-g(t)-\int_0^t\widetilde{M}(\tau,t)V_m(\tau)d\tau-\sum_{r=1}^{q}\int_0^t\widehat{M_r}(\tau,t)V_m(\tau)dB_r(\tau)\right\|_{L^2}^2\right]\\&\leq\left(3T^2+3T^4\zeta^2+12T^3\zeta^2q^2\right)\|\widetilde{V}_m(t)-V(t)\|_{\infty,E}^2\\&\leq(3T^2+3T^4\zeta^2+12T^3\zeta^2q^2)\mathbb{E}[\omega(V,h)]\to 0,\end{aligned}\tag{36}$$

The proof is accomplished. □

§6 Numerical examples

Two numerical examples are given to confirm the effectiveness of the approach. All computations are run using MATLAB R2016a software on a Core(TM) i5 PC Laptop with 2.20 GHz of CPU and 4 GB of RAM.

Example 6.1. *The following linear SIVIE is considered ([19])*

$$V(t)=V_0+\int_0^t kV(\tau)d\tau+\sum_{r=1}^{4}\int_0^t a_rV(\tau)dB_r(\tau),\quad \tau,t\in[0,1).\tag{37}$$

Where $V(t)=V_0e^{(k-\frac{1}{2}\sum_{r=1}^4 a_r^2)t+\sum_{r=1}^4 a_rB_r(t)}$. Let $V_0=\frac{1}{200}, k=\frac{1}{20}, a_1=\frac{1}{50}, a_2=\frac{2}{50}, a_3=\frac{4}{50}$ and $a_4=\frac{9}{50}$, the error means E_m, error standard deviations E_s and confidence intervals for different time t of this example are shown in Table 1 and Table 2, where n is the number of trajetories. The simulation results are exhibited Figure 1 and Figure 2.

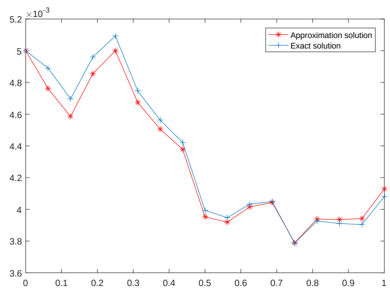

Figure 1. $m=2^4$, the approximate solution and exact solution for Example 6.1, the CPU time of the provided method is 19.84 seconds.

Table 1. When $m=2^4, n=100$, the numerical results are shown as follows.

t	E_m	E_s	95% confidence interval for error mean	
			Lower	Upper
1/16	1.31837963E-04	6.59189815E-05	2.6367592E-06	2.6103916E-04
3/16	1.06003128E-04	5.30015641E-05	2.1200625E-06	2.0988619E-04
5/16	7.97155665E-05	3.98577832E-05	1.5943113E-06	1.5783682E-04
7/16	5.05665735E-05	2.52832867E-05	1.0113314E-06	1.0012181E-04
9/16	1.42429556E-05	7.12147783E-06	2.8485911E-07	2.8201052E-05

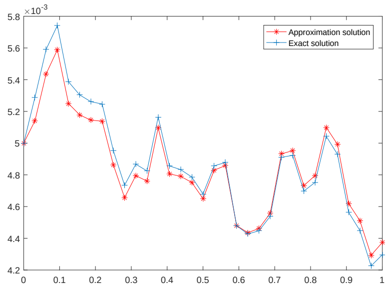

Figure 2. $m = 2^5$, the approximate solution and exact solution for Example 6.1, the CPU time of the provided method is 43.71 seconds.

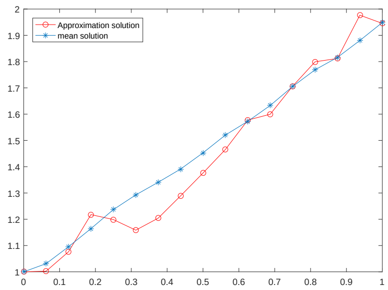

Figure 3. $m = 2^4$, the approximate solution and mean solution for Example 6.2, the CPU time of the provided method is 178.07 seconds.

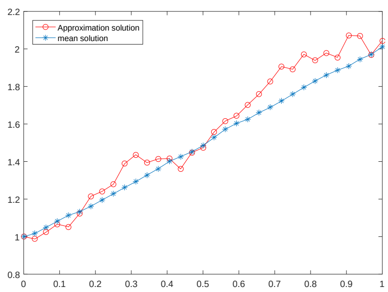

Figure 4. $m = 2^5$, the approximate solution and mean solution for Example 6.2, the CPU time of the provided method is 841.32 seconds.

Table 2. When $m = 2^5, n = 100$, the numerical results are shown as follows.

t	E_m	E_s	95% confidence interval for error mean	
			Lower	Upper
1/32	1.41125138E-04	7.05625694E-05	2.8225027E-06	2.7942777E-04
7/32	7.47392791E-05	3.73696395E-05	1.4947855E-06	1.4798377E-04
13/32	3.96747456E-05	1.98373728E-05	7.9349491E-07	7.8555996E-05
19/32	1.87498023E-05	9.37490116E-06	3.7499604E-07	3.7124608E-05
25/32	1.88988202E-05	9.44941014E-06	3.7797640E-07	3.7419664E-05

From Figure 1, Figure 2, Table 1, and Table 2, it can be show that the error is relatively small. Moreover, this method is more precise than the method in paper ([19]).

Example 6.2. *The following linear SIVIE is considered ([16]).*

$$V(t) = V_0 + \int_0^t e^{-(t-\tau)} V(\tau) d\tau + \sum_{r=1}^{4} \int_0^t a_r e^{-(t-\tau)} V(\tau) dB_r(\tau), \quad \tau, t \in [0,1), \tag{38}$$

Let $V_0 = 1, a_1 = \frac{1}{50}, a_2 = \frac{2}{50}, a_3 = \frac{4}{50}$ and $a_4 = \frac{9}{50}$, the simulation results are exhibited in Figure 3 and Figure 4.

The Figure 3 and Figure 4 exhibit that the approximate solution undulates around the mean orbit, where the mean solution is acquired by 100 trajectories.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] Z Avazzadeh, M Heydari. *Haar wavelet method for solving nonlinear age-structured population models*, Int J Biomath, 2017, 10(8): 1750114, https://doi.org/10.1142/S1793524517501145.

[2] C Zeng, Q Yang, Y Q Chen. *Solving nonlinear stochastic differential equations with fractional Brownian motion using reducibility approach*, Nonlinear Dynam, 2012, 67(4): 2719-2726.

[3] F Mirzaee, N Samadyar. *Numerical solution of nonlinear stochastic Itô-Volterra integral equations driven by fractional Brownian motion*, Math Meth Appl Sci, 2018, 41(4): 1410-1423.

[4] X Y Sang, G Jiang, J H Wu, Y Y Lu. *Numerical solution of nonlinear stochastic Itô-Volterra integral equations by Block pulse functions*, J Math Appl, 2019, 32(4): 935-946. (in Chinese)

[5] K Maleknejad, S Sohrabi, Y Rostami. *Numerical solution of nonliear Volterra integral equations of the second kind by using Chebyshev polynomials*, J Appl Math Comput, 2007, 188(1): 123-128.

[6] M Asgari, E Hashmemizadeh, M Khodabin, et al. *Numerical solution of nonlinear stochastic integral equation by stochastic operational matrix based on Bernstein polynomials*, J B Math Soc Sci Math, 2014, 75(1): 3-12.

[7] P E Kloeden, E Platen. *Numerical solution of stochastic differential equations*, M Berlin: Springer-Verlag, 1992.

[8] W F Blyth, R L May, P Widyaningsih. *Volterra integral equations solved in Fredholm form using Walsh functions*, Anziam J, 2003, 45: 269-282.

[9] K Maleknejad, B Basirat, E Hashemizadeh. *Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations*, J Comput Math Appl, 2011, 61(1): 2821-2828.

[10] K Maleknejad, M Shahrezaee, H Khatami. *Numerical solution of integral equations system of the second kind by Block Pulse functions*, J Comput Math Appl, 2005, 166(1): 15-24.

[11] K Maleknejad, M Khodabin, M Rostami. *Numerical solution of stochastic Volterra integral equation by a stochastic operational matrix based on Block pulse function*, J Math Comput Model, 2012, 55(3): 791-800.

[12] M Saffarzadeh, M Heydari, G B Loghmani. *Convergence analysis of an iterative numerical algorithm for solving nonlinear stochastic Itô-Volterra integral equations with m-dimensional Brownian motion*, Appl Numer Math, 2019, 146: 182-198.

[13] M Saffarzadeh, G B Loghmani, M Heydari. *An iterative technique for the numerical solution of nonlinear stochastic ItôCVolterra integral equations*, J Comput Appl Math, 2018, 333: 74-86.

[14] F Mohammadi. *Numerical solution of stochastic Itô-Volterra integral equations using Haar wavelets*, J Numer Math Theory Methods Appl, 2016, 9(3): 416-431.

[15] U Lepik. *Numerical solution of differential equations using Haar wavelets*, J Math Comput Simulat, 2005, 68: 127-143.

[16] J H Wu, G Jiang, X Y Sang. *Numerical solution of nonlinear stochastic Itô-Volterra integral equations based on Haar wavelets*, J Adv Differ Equ-Ny, 2019, 503: 1-12.

[17] G Jiang, X Y Sang, J H Wu, B W Li. *Numerical solution of two-dimensional nonlinear stochastic Itô-Volterra integral equations by applying Block pulse function*, J Adv Pure Math, 2019, 9(2): 53-66.

[18] M Ahmadinia, H AfshariA, M Heydari. *Numerical solution of Itô-Volterra integral equation by least squares method*, J Numer Algorithms, 2020, 84(89): 591-602.

[19] K Maleknejad, M Khodabin, M Rostami. *A numerical method for solving m-dimensional stochastic Itô-Volterra integral equations by stochastic operational matrix*, J Comput Math Appl, 2012, 63: 133-143.

[20] T J Rivlin. *An introduction to the approximation of functions*, Dover Publications, New York, 1981.

School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China.
Email: gjiang@hbnu.edu.cn