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Positive definiteness of fourth-order partially symmetric tensors

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Abstract. In this paper, we consider the positive definiteness of fourth-order partially symmetric tensors. First, two analytically sufficient and necessary conditions of positive definiteness are provided for fourth-order two dimensional partially symmetric tensors. Then, we obtain several sufficient conditions for rank-one positive definiteness of fourth-order three dimensional partially symmetric tensors.

§1 Introduction

Let $\mathbb{R}^n$ be the set of all n dimensional real vectors, and $[n] = \{1, 2, ..., n\}$, $\mathbb{R}$ be the set of all real numbers. A fourth-order n dimensional real tensor $\mathcal{A}$ consists of n^4 entries in the real field $\mathbb{R}$, i.e.

$$\mathcal{A} = (a_{ijkl}),\ a_{ijkl} \in \mathbb{R},\ i, j, k, l \in [n].$$

$\mathcal{A} = (a_{ijkl})$ is called fourth-order partially symmetric tensor, if the entries of $\mathcal{A}$ satisfy the following symmetry condition

$$a_{ijkl} = a_{ijlk} = a_{klij},\ i, j, k, l \in [n]. \tag{1.1}$$

In the theory of elasticity [3], for a linearly anisotropic elastic solid the components c_{ijkl} of the tensor of elastic moduli satisfy

$$c_{ijkl} = c_{ijlk} = c_{klij},\ i, j, k, l \in [3].$$

That is, the tensor of elastic moduli for elastic materials is exactly partially symmetric.

We say that the elasticity modulus tensor $\mathcal{C} = (c_{ijkl})$ is strongly elliptic if and only if

$$f(\mathbf{x}, \mathbf{y}) = \mathcal{C}\mathbf{xyxy} = \sum_{i,j,k,l=1}^{3} c_{ijkl} x_i y_j x_k y_l > 0 \tag{1.2}$$

for all nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

In 2009, Qi et al.[1] presented that the strong ellipticity condition holds if and only if the partially symmetric tensor $\mathcal{A}$ is positive definite.

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Definition 1.1. [1] Suppose that $\mathcal{A} = (a_{ijkl})$ is a fourth-order real partially symmetric tensor where

$$a_{ijkl} = a_{ijlk} = a_{klij},\ i, k \in [m],\ j, l \in [n]. \tag{1.3}$$

Let

$$f(\mathbf{x}, \mathbf{y}) = \mathcal{A}\mathbf{xyxy} = \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l \tag{1.4}$$

If $f(\mathbf{x}, \mathbf{y}) > 0$ for all $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} \neq 0$, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \neq 0$, then we say that $\mathcal{A}$ is positive definite.

The elasticity tensor $\mathcal{A}$ is called rank-one positive definite [2] if for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$,

$$f(\mathbf{x}, \mathbf{x}) = \mathcal{A}\mathbf{x}^4 = \mathcal{A}\mathbf{xxxx} = \sum_{i,j,k,l=1}^{n} a_{ijkl} x_i x_j x_k x_l > 0 \tag{1.5}$$

Clearly, if the strong ellipticity holds, then $\mathcal{A}$ is positive definite, and vice versa. However, if the strong ellipticity holds, then $\mathcal{A}$ is rank-one positive definite, and the reverse conclusion is not true.

In recent decades, many efforts have been made to judge strong ellipticity (see, Refs.[1-9]), and recently, Qi, Dai and Han [1] studied the strong ellipticity condition via M-eigenvalues. The strong ellipticity condition holds if and only if the smallest M-eigenvalue of the elasticity tensor is positive. Han, Dai and Qi [2] provided a necessary and sufficient condition for the strong ellipticity condition of anisotropic elastic materials. The authors link the condition to the rank-one positive definiteness of three fourth-order tensors and a sixth-order tensor. The positive definiteness of tensors and partially symmetric tensors has been an important issue in many areas, and has been discussed extensively [10-17]. However, few papers discuss the analytical expression of positive definiteness of partially symmetric tensor. Therefore, it is necessary to find the analytical expressions of the positive definiteness and rank-one positive definiteness of the fourth-order partially symmetric tensors.

In this paper, we first derive two necessary and sufficient conditions for the positive definiteness of fourth-order two dimensional partially symmetric tensors in Section 2. However, it is still difficult to find the analytical expression of the necessary and sufficient conditions for the positive definiteness of fourth-order three dimensional partially symmetric tensors. In Section 3, we obtain several sufficient conditions for the rank-one positive definiteness of fourth-order three dimensional partially symmetric tensor.

Next, we give some lemmas that need to be used in the following conclusions. The first lemma should be well-known, and, was shown hundreds of years ago.

Lemma 1.1. *Let $f(t)$ be a quadratic polynomial,*

$$f(t) = at^2 + bt + c$$

with $a > 0$. Then $f(t) > 0(\geq 0)$ for all $t \in \mathbb{R}$ if and only if $4ac - b^2 > 0(\geq 0)$.

Lemma 1.2. [18][19] *A real symmetric matrix is positive definite if and only if all its leading principal minors are positive. A real symmetric matrix is positive semidefinite if and only if all its principal minors are nonnegative.*

Lemma 1.3. [10] *Let $g(u)$ be a quartic polynomial with $a > 0$ and $e > 0$,*

$$g(u) = au^4 + bu^3 + cu^2 + du + e.$$

Then $g(u) \geq 0$ *for all* u *if and only if* $\Delta \geq 0, |b\sqrt{e} - d\sqrt{a}| \leq 4\sqrt{ace + 2ae\sqrt{ae}}$ *and either*
(i) $-2\sqrt{ae} \leq c \leq 6\sqrt{ae}$; *or*
(ii) $c > 6\sqrt{ae}$ *and* $|b\sqrt{e} + d\sqrt{a}| \leq 4\sqrt{ace - 2ae\sqrt{ae}}$.
Furthermore, $g(u) > 0$ *for all* u *if and only if*
(1) $\Delta = 0$, $b\sqrt{e} = d\sqrt{a}$, $b^2 + 8a\sqrt{ae} = 4ac < 24a\sqrt{ae}$;
(2) $\Delta > 0$, $|b\sqrt{e} - d\sqrt{a}| \leq 4\sqrt{ace + 2ae\sqrt{ae}}$ *and either*
(i) $-2\sqrt{ae} \leq c \leq 6\sqrt{ae}$, *or (ii)* $c > 6\sqrt{ae}$ *and* $|b\sqrt{e} + d\sqrt{a}| \leq 4\sqrt{ace - 2ae\sqrt{ae}}$.
Where $\Delta = 4(12ae - 3bd + c^2)^3 - (72ace + 9bcd - 2c^3 - 27ad^2 - 27b^2e)^2$.

§2 Positive definiteness of fourth-order two dimensional partially symmetric tensor

Let $\mathcal{A}$ be a fourth-order two dimensional partially symmetric tensor defined in (1.1). Then for two vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x} = (x_1, x_2)^\top$, $\mathbf{y} = (y_1, y_2)^\top$,

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) =& \mathcal{A}\mathbf{xyxy} \\ =& a_{1111}x_1^2y_1^2 + 2a_{1112}x_1^2y_1y_2 + 2a_{1121}x_1x_2y_1^2 + 2(a_{1122} + a_{1221})x_1x_2y_1y_2 \\ &+ a_{1212}x_1^2y_2^2 + 2a_{1222}x_1x_2y_2^2 + a_{2121}x_2^2y_1^2 + 2a_{2122}x_2^2y_1y_2 + a_{2222}x_2^2y_2^2. \end{aligned} \tag{2.1}$$

Theorem 2.1. *Let* $\mathcal{A}$ *be a fourth-order two dimensional partially symmetric tensor with* $a_{1111} > 0$, $a_{2222} > 0$, $a_{1212} > 0$ *and let*

$$\begin{aligned} M =& 12(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2) - 3(2a_{1111}a_{2122} - 2a_{1121}a_{1122})(2a_{1112}a_{2222} - \\ & 2a_{1222}a_{1122}) + (a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})^2, \end{aligned}$$

$$\begin{aligned} N =& 72(a_{1111}a_{2121} - a_{1121}^2)(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})(a_{2222}a_{1212} - a_{1222}^2) \\ &+ 9(2a_{1111}a_{2122} - 2a_{1121}a_{1122})(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221}) \\ &(2a_{1112}a_{2222} - 2a_{1222}a_{1122}) - 2(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})^3 \\ &- 27(a_{1111}a_{2121} - a_{1121}^2)(2a_{1112}a_{2222} - 2a_{1222}a_{1122})^2 \\ &- 27(2a_{1111}a_{2122} - 2a_{1121}a_{1122})^2(a_{2222}a_{1212} - a_{1222}^2). \end{aligned}$$

Then $\mathcal{A}$ *is positive definite if and only if* $a_{1111}a_{1212} - a_{1112}^2 > 0$, $a_{2222}a_{2121} - a_{2122}^2 > 0$ *and*

$$\begin{aligned} &(1) 4M^3 - N^2 = 0,\ (a_{1111}a_{2122} - a_{1121}a_{1122})\sqrt{a_{2222}a_{1212} - a_{1222}^2} = \\ &(a_{1112}a_{2222} - a_{1222}a_{1122})\sqrt{a_{1111}a_{2121} - a_{1121}^2},\ (a_{1111}a_{2122} - a_{1121}a_{1122})^2 + \\ &2(a_{1111}a_{2121} - a_{1121}^2)\sqrt{(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2)} = \\ &(a_{1111}a_{2121} - a_{1121}^2)(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221}) < \\ &6(a_{1111}a_{2121} - a_{1121}^2)\sqrt{(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2)}; \end{aligned}$$

$$\begin{aligned} &(2) 4M^3 - N^2 > 0,\ |(a_{1111}a_{2122} - a_{1121}a_{1122})\sqrt{a_{2222}a_{1212} - a_{1222}^2} - \\ &(a_{1112}a_{2222} - a_{1222}a_{1122})\sqrt{a_{1111}a_{2121} - a_{1121}^2}| \leq \\ &2[(a_{1111}a_{2121} - a_{1121}^2)(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})(a_{2222}a_{1212} - a_{1222}^2) \end{aligned}$$

$$+ 2(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2)\sqrt{(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2)}]^{\frac{1}{2}},$$

and either

$$(i) - 2\sqrt{(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2)} \le a_{1111}a_{2222} + 2a_{1112}a_{2122}$$
$$- a_{1122}^2 - 2a_{1122}a_{1221} \le 6\sqrt{(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2)}, \quad \textit{or}$$
$$(ii) a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221}$$
$$> 6\sqrt{(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2)} \quad \textit{and}$$
$$|(a_{1111}a_{2122} - a_{1121}a_{1122})\sqrt{a_{2222}a_{1212} - a_{1222}^2} +$$
$$(a_{1112}a_{2222} - a_{1222}a_{1122})\sqrt{a_{1111}a_{2121} - a_{1121}^2}| \le 2[(a_{1111}a_{2121} - a_{1121}^2)$$
$$(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})(a_{2222}a_{1212} - a_{1222}^2) -$$
$$2(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2)\sqrt{(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2)}]^{\frac{1}{2}}.$$

Proof. (2.1) can be rewritten as

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) =& \mathcal{A}\mathbf{xyxy} \\ =& x_1^2(a_{1111}y_1^2 + 2a_{1112}y_1y_2 + a_{1212}y_2^2) \\ &+ 2x_1x_2(a_{1121}y_1^2 + (a_{1122} + a_{1221})y_1y_2 + a_{1222}y_2^2) + x_2^2(a_{2121}y_1^2 + 2a_{2122}y_1y_2 + a_{2222}y_2^2). \end{aligned}$$

Without loss of generality, we may assume $x_2 \neq 0$, $y_2 \neq 0$. Then for two vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x} = (x_1, x_2)^\top \neq 0$, $\mathbf{y} = (y_1, y_2)^\top \neq 0$, we have

$$\begin{aligned} \frac{\mathcal{A}\mathbf{xyxy}}{x_2^2} =& (a_{1111}y_1^2 + 2a_{1112}y_1y_2 + a_{1212}y_2^2)(\frac{x_1}{x_2})^2 \\ &+ 2(a_{1121}y_1^2 + (a_{1122} + a_{1221})y_1y_2 + a_{1222}y_2^2)(\frac{x_1}{x_2}) + a_{2121}y_1^2 + 2a_{2122}y_1y_2 + a_{2222}y_2^2. \end{aligned}$$

Let $\frac{x_1}{x_2} = \tau$ and $f(\tau) = \frac{\mathcal{A}\mathbf{xyxy}}{x_2^2}$, i.e.,

$$\begin{aligned} f(\tau) =& (a_{1111}y_1^2 + 2a_{1112}y_1y_2 + a_{1212}y_2^2)\tau^2 + 2(a_{1121}y_1^2 + (a_{1122} + a_{1221})y_1y_2 + a_{1222}y_2^2)\tau \\ &+ a_{2121}y_1^2 + 2a_{2122}y_1y_2 + a_{2222}y_2^2. \end{aligned}$$

Clearly, $f(\tau) > 0$ if and only if $\mathcal{A}\mathbf{xyxy} > 0$. It follows from Lemma 1.1, $f(\tau) > 0$ if and only if

$$a_{1111}y_1^2 + 2a_{1112}y_1y_2 + a_{1212}y_2^2 > 0, \tag{2.2}$$

$$\begin{aligned} &4(a_{1111}y_1^2 + 2a_{1112}y_1y_2 + a_{1212}y_2^2)(a_{2121}y_1^2 + 2a_{2122}y_1y_2 + a_{2222}y_2^2) \\ &- 4(a_{1121}y_1^2 + (a_{1122} + a_{1221})y_1y_2 + a_{1222}y_2^2)^2 > 0. \end{aligned} \tag{2.3}$$

Similar to the process discussed above, by Lemma 1.1, (2.2) holds if and only if

$$\begin{gathered} a_{1111} > 0, a_{1111}a_{1212} - a_{1112}^2 > 0 \\ \textit{or} \\ a_{1212} > 0, a_{1111}a_{1212} - a_{1112}^2 > 0. \end{gathered} \tag{2.4}$$

(2.3) can be rewritten as

$$\begin{aligned} &(a_{1111}a_{2121} - a_{1121}^2)y_1^4 + (2a_{1111}a_{2122} + 2a_{1112}a_{2121} - 2a_{1121}(a_{1122} + a_{1221}))y_1^3y_2 \\ &+ (a_{1111}a_{2222} + 4a_{1112}a_{2122} + a_{1212}^2 - 2a_{1121}a_{1222} - (a_{1122} + a_{1221})^2)y_1^2y_2^2 \\ &+ (2a_{1112}a_{2222} + 2a_{1212}a_{2122} - 2a_{1222}(a_{1122} + a_{1221}))y_1y_2^3 + (a_{2222}a_{1212} - a_{1222}^2)y_2^4 > 0. \end{aligned} \tag{2.5}$$

(2.5) holds if and only if

$$g(u) = au^4 + bu^3 + cu^2 + du + e > 0, \tag{2.6}$$

where $u = \frac{y_1}{y_2}$, $a = a_{1111}a_{2121} - a_{1121}^2$, $b = 2a_{1111}a_{2122} + 2a_{1112}a_{2121} - 2a_{1121}(a_{1122} + a_{1221}) = 2a_{1111}a_{2122} - 2a_{1121}a_{1122}$, $c = a_{1111}a_{2222} + 4a_{1112}a_{2122} + a_{1212}^2 - 2a_{1121}a_{1222} - (a_{1122} + a_{1221})^2 = a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221}$, $d = 2a_{1112}a_{2222} + 2a_{1212}a_{2122} - 2a_{1222}(a_{1122} + a_{1221}) = 2a_{1112}a_{2222} - 2a_{1222}a_{1122}$, $e = a_{2222}a_{1212} - a_{1222}^2$.

Thus, $f(\tau) > 0$ if and only if (2.4) and (2.6) are true.

Let

$$\begin{aligned}
\Delta =& 4(12ae - 3bd + c^2)^3 - (72ace + 9bcd - 2c^3 - 27ad^2 - 27b^2e)^2\\
=& 4[12(a_{1111}a_{2121} - a_{1121}^2)(a_{2222}a_{1212} - a_{1222}^2) - 3(2a_{1111}a_{2122} - 2a_{1121}a_{1122})(2a_{1112}a_{2222} -\\
& 2a_{1222}a_{1122}) + (a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})^2]^3 -\\
& [72(a_{1111}a_{2121} - a_{1121}^2)(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})(a_{2222}a_{1212} - a_{1222}^2)\\
& + 9(2a_{1111}a_{2122} - 2a_{1121}a_{1122})(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})\\
& (2a_{1112}a_{2222} - 2a_{1222}a_{1122}) - 2(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})^3\\
& - 27(a_{1111}a_{2121} - a_{1121}^2)(2a_{1112}a_{2222} - 2a_{1222}a_{1122})^2\\
& - 27(2a_{1111}a_{2122} - 2a_{1121}a_{1122})^2(a_{2222}a_{1212} - a_{1222}^2)]^2\\
=& 4M^3 - N^2.
\end{aligned}$$

Combined with the above discussion and Lemma 1.3, the desired conclusion can be obtained through a simple calculation. □

Remark 2.1. (1) *It follows from the proof of Theorem* 2.1 *that* $\mathcal{A}\boldsymbol{xyxy}$ *may be divided by* x_1^2 ($x_1 \neq 0$)*, then,*

$$\begin{aligned}
f(\zeta) =& (a_{2121}y_1^2 + 2a_{2122}y_1y_2 + a_{2222}y_2^2)\zeta^2 + 2(a_{1121}y_1^2 + (a_{1122} + a_{1221})y_1y_2 + a_{1222}y_2^2)\zeta\\
& + a_{1111}y_1^2 + 2a_{1112}y_1y_2 + a_{1212}y_2^2,
\end{aligned}$$

where $\zeta = \frac{x_2}{x_1}$*. Then* (2.4) *is replaced by*

$$a_{2121} > 0, a_{2121}a_{2222} - a_{2122}^2 > 0, \;\; or \;\; a_{2222} > 0, a_{2121}a_{2222} - a_{2122}^2 > 0.$$

The conclusion still holds.

(2) *In* (2.6)*, if we let* $u = \frac{y_2}{y_1}$*, then,* $a = a_{2222}a_{1212} - a_{1222}^2$*,* $e = a_{1111}a_{2121} - a_{1121}^2$*,* $b = 2a_{1112}a_{2222} - 2a_{1222}a_{1122}$*,* $d = 2a_{1111}a_{2122} - 2a_{1121}a_{1122}$*, and the value of* c *does not change. Obviously, the conclusion still holds.*

Using a similar argumentation technique, the following result can be obtained easily according to Lemma 1.2 and Lemma 1.3.

Theorem 2.2. *Let* $\mathcal{A}$ *be a fourth-order two dimensional partially symmetric tensor with* $a_{1111} > 0$*,* $a_{2222} > 0$*,* $a_{1212} > 0$*. Then,* $\mathcal{A}$ *is positive semidefinite if and only if* $a_{1111}a_{1212} - a_{1112}^2 \geq 0$*,* $a_{2222}a_{2121} - a_{2122}^2 \geq 0$ *and*

$$\begin{aligned}
& 4M^3 - N^2 \geq 0, \; |(a_{1111}a_{2122} - a_{1121}a_{1122})\sqrt{a_{2222}a_{1212} - a_{1222}^2} -\\
& (a_{1112}a_{2222} - a_{1222}a_{1122})\sqrt{a_{1111}a_{2121} - a_{1121}^2}| \leq\\
& 2[(a_{1111}a_{2121} - a_{1121}^2)(a_{1111}a_{2222} + 2a_{1112}a_{2122} - a_{1122}^2 - 2a_{1122}a_{1221})(a_{2222}a_{1212} - a_{1222}^2)
\end{aligned}$$

$$+2(a_{1111}a_{2121}-a_{1121}^2)(a_{2222}a_{1212}-a_{1222}^2)\sqrt{(a_{1111}a_{2121}-a_{1121}^2)(a_{2222}a_{1212}-a_{1222}^2)}]^{\frac{1}{2}},$$

and either

$$(i) -2\sqrt{(a_{1111}a_{2121}-a_{1121}^2)(a_{2222}a_{1212}-a_{1222}^2)} \le a_{1111}a_{2222}+2a_{1112}a_{2122} - a_{1122}^2-2a_{1122}a_{1221} \le 6\sqrt{(a_{1111}a_{2121}-a_{1121}^2)(a_{2222}a_{1212}-a_{1222}^2)};\ \textit{or}$$

$$(ii) a_{1111}a_{2222}+2a_{1112}a_{2122}-a_{1122}^2-2a_{1122}a_{1221} > 6\sqrt{(a_{1111}a_{2121}-a_{1121}^2)(a_{2222}a_{1212}-a_{1222}^2)}\ \textit{and}$$

$$|(a_{1111}a_{2122}-a_{1121}a_{1122})\sqrt{a_{2222}a_{1212}-a_{1222}^2}+ (a_{1112}a_{2222}-a_{1222}a_{1122})\sqrt{a_{1111}a_{2121}-a_{1121}^2}| \le 2[(a_{1111}a_{2121}-a_{1121}^2) (a_{1111}a_{2222}+2a_{1112}a_{2122}-a_{1122}^2-2a_{1122}a_{1221})(a_{2222}a_{1212}-a_{1222}^2)- 2(a_{1111}a_{2121}-a_{1121}^2)(a_{2222}a_{1212}-a_{1222}^2)\sqrt{(a_{1111}a_{2121}-a_{1121}^2)(a_{2222}a_{1212}-a_{1222}^2)}]^{\frac{1}{2}}.$$

§3 Rank-one positive definiteness of fourth-order three dimensional partially symmetric tensor

For a fourth-order three dimensional partially symmetric tensor, $\mathcal{A}\mathbf{xyxy}$ is a quartic polynomial with six variables. Therefore, it is not easy to give an analytical expression of the positive definiteness. However, we can give some sufficient conditions for the rank-one positive definiteness of the fourth-order three dimensional partially symmetric tensors.

Let $\mathcal{A}$ be a fourth-order three dimensional partially symmetric tensor defined in (1.1). For a vector $\mathbf{x}=(x_1,x_2,x_3)^\top \in \mathbb{R}^3$, we have

$$\begin{aligned}\mathcal{A}\mathbf{x}^4 = \mathcal{A}\mathbf{xxxx} = &\sum_{i,j,k,l=1}^{3} a_{ijkl}x_ix_jx_kx_l\\ &= a_{1111}x_1^4+a_{2222}x_2^4+a_{3333}x_3^4\\ &+4a_{1112}x_1^3x_2+4a_{1113}x_3x_1^3+4a_{1222}x_1x_2^3+4a_{2223}x_3x_2^3+4a_{1333}x_1x_3^3+4a_{2333}x_2x_3^3\\ &+(2a_{1122}+4a_{1212})x_1^2x_2^2+(2a_{1133}+4a_{1313})x_1^2x_3^2+(2a_{2233}+4a_{2323})x_2^2x_3^2\\ &+(4a_{1123}+8a_{1213})x_1^2x_2x_3+(4a_{2213}+8a_{2123})x_1x_2^2x_3+(4a_{3312}+8a_{3132})x_1x_2x_3^2.\end{aligned} \tag{3.1}$$

For simplicity, we abbreviate the above formula as follows

$$\begin{aligned}\mathcal{A}\mathbf{x}^4 = &\sum_{i,j,k,l=1}^{3} a_{ijkl}x_ix_jx_kx_l\\ =&a_1x_1^4+a_2x_2^4+a_3x_3^4+a_4x_1^3x_2+a_5x_3x_1^3+a_6x_1x_2^3+a_7x_3x_2^3+a_8x_1x_3^3+a_9x_2x_3^3\\ &+a_{10}x_1^2x_2^2+a_{11}x_1^2x_3^2+a_{12}x_2^2x_3^2+a_{13}x_1^2x_2x_3+a_{14}x_1x_2^2x_3+a_{15}x_1x_2x_3^2,\end{aligned} \tag{3.2}$$

where a_1, a_2, a_3, $\cdots$, a_{15} correspond to the coefficients in equation (3.1), respectively.

Theorem 3.1. *Let $\mathcal{A}$ be a fourth-order three dimensional partially symmetric tensor defined in (1.1). Suppose that*

$$a_1>0, a_2>0, a_3>0, a_{10}>0, a_{11}\ge 0, a_{12}\ge 0, 2a_1a_{10}-a_4^2>0, 2a_{10}a_2-a_6^2>0,$$

$$a_{11}a_{12}-a_{15}^2\geq 0, 2a_3a_{12}-a_9^2\geq 0, 2a_3a_{11}-a_8^2\geq 0,$$
$$2a_1a_{10}a_{11}+2a_4a_5a_{13}-a_5^2a_{10}-a_4^2a_{11}-2a_1a_{13}^2>0,$$
$$2a_{10}a_2a_{12}+2a_6a_7a_{14}-2a_{14}^2a_2-a_6^2a_{12}-a_7^2a_{10}>0,$$
$$2a_3a_{11}a_{12}+2a_8a_9a_{15}-a_{12}a_8^2-2a_3a_{15}^2-a_{11}a_9^2\geq 0.$$

Then, $\mathcal{A}$ is rank-one positive definite.

Proof. Let $\mathbf{x}=(x_1,x_2,x_3)^\top\in\mathbb{R}^3$ be an nonzero vector. Redistribute and combine the terms of polynomial (3.2), we can obtain

$$\begin{aligned}\mathcal{A}\mathbf{x}^4 =& x_1^2[a_1x_1^2+a_4x_1x_2+\frac{1}{2}a_{10}x_2^2+a_5x_1x_3+a_{13}x_2x_3+\frac{1}{2}a_{11}x_3^2]\\ &+x_2^2[a_2x_2^2+a_6x_1x_2+\frac{1}{2}a_{10}x_1^2+a_7x_2x_3+a_{14}x_1x_3+\frac{1}{2}a_{12}x_3^2]\\ &+x_3^2[a_3x_3^2+a_8x_1x_3+\frac{1}{2}a_{11}x_1^2+a_9x_2x_3+a_{15}x_1x_2+\frac{1}{2}a_{12}x_2^2]\\ =&\begin{pmatrix}x_1^2 & x_2^2 & x_3^2\end{pmatrix}\begin{pmatrix}\mathbf{x}^\top R\mathbf{x}\\ \mathbf{x}^\top S\mathbf{x}\\ \mathbf{x}^\top T\mathbf{x}\end{pmatrix}\end{aligned}$$

where R, S, and T are three symmetric matrices,

$$R=\begin{pmatrix}a_1 & \frac{1}{2}a_4 & \frac{1}{2}a_5\\ \frac{1}{2}a_4 & \frac{1}{2}a_{10} & \frac{1}{2}a_{13}\\ \frac{1}{2}a_5 & \frac{1}{2}a_{13} & \frac{1}{2}a_{11}\end{pmatrix},\ S=\begin{pmatrix}\frac{1}{2}a_{10} & \frac{1}{2}a_6 & \frac{1}{2}a_{14}\\ \frac{1}{2}a_6 & a_2 & \frac{1}{2}a_7\\ \frac{1}{2}a_{14} & \frac{1}{2}a_7 & \frac{1}{2}a_{12}\end{pmatrix},\ T=\begin{pmatrix}\frac{1}{2}a_{11} & \frac{1}{2}a_{15} & \frac{1}{2}a_8\\ \frac{1}{2}a_{15} & \frac{1}{2}a_{12} & \frac{1}{2}a_9\\ \frac{1}{2}a_8 & \frac{1}{2}a_9 & a_3\end{pmatrix}.$$

From Lemma 1.2, it follows that R is positive definite if and only if

$$a_1>0, 2a_1a_{10}-a_4^2>0, det(R)=\frac{1}{4}a_1a_{10}a_{11}+\frac{1}{4}a_4a_5a_{13}-\frac{1}{8}a_5^2a_{10}-\frac{1}{8}a_4^2a_{11}-\frac{1}{4}a_1a_{13}^2>0.$$

S is positive definite if and only if

$$a_{10}>0, 2a_{10}a_2-a_6^2>0, det(S)=\frac{1}{4}a_{10}a_2a_{12}+\frac{1}{4}a_6a_7a_{14}-\frac{1}{4}a_{14}^2a_2-\frac{1}{8}a_6^2a_{12}-\frac{1}{8}a_7^2a_{10}>0.$$

T is positive semidefinite if and only if

$$a_{11}\geq 0, a_{12}\geq 0, a_3\geq 0, a_{11}a_{12}-a_{15}^2\geq 0, 2a_3a_{12}-a_9^2\geq 0, 2a_3a_{11}-a_8^2\geq 0,$$
$$det(T)=\frac{1}{4}a_3a_{11}a_{12}+\frac{1}{4}a_8a_9a_{15}-\frac{1}{8}a_{12}a_8^2-\frac{1}{4}a_3a_{15}^2-\frac{1}{8}a_{11}a_9^2\geq 0.$$

Thus, the assumptions imply that both R and S are positive definite and T is positive semidefinite, i.e.,

$$\mathbf{x}^\top R\mathbf{x}>0,\ \mathbf{x}^\top S\mathbf{x}>0\ \ and\ \ \mathbf{x}^\top T\mathbf{x}\geq 0\ \ for\ all\ \ \mathbf{x}\in\mathbb{R}^3,\ \mathbf{x}\neq 0.$$

On the other hand, for each $\mathbf{x}=(x_1,x_2,x_3)^\top$, it is obvious that the vector $(x_1^2,x_2^2,x_3^2)^\top\geq 0$, and $\mathcal{A}\mathbf{x}^4=a_3x_3^4>0$ for $\mathbf{x}=(0,0,x_3)^\top\neq 0$, $\mathcal{A}\mathbf{x}^4=a_2x_2^4>0$ for $\mathbf{x}=(0,x_2,0)^\top\neq 0$. $\mathcal{A}\mathbf{x}^4=a_1x_1^4>0$ for $\mathbf{x}=(x_1,0,0)^\top\neq 0$. Then for all $\mathbf{x}=(x_1,x_2,x_3)^\top\neq 0$,

$$\mathcal{A}\mathbf{x}^4=x_1^2(\mathbf{x}^\top R\mathbf{x})+x_2^2(\mathbf{x}^\top S\mathbf{x})+x_3^2(\mathbf{x}^\top T\mathbf{x})>0.$$

That is, $\mathcal{A}$ is rank-one positive definite, as expected. □

From the proof of Theorem 3.1, the following conclusion can be obtained easily when S is positive semidefinite and both R and T are positive definite.

Theorem 3.2. *Let $\mathcal{A}$ be a fourth-order three dimensional partially symmetric tensor. Suppose that*

$$a_1 > 0, a_2 > 0, a_3 > 0, a_{10} \geq 0, a_{11} > 0, a_{12} \geq 0,$$
$$2a_1a_{10} - a_4^2 > 0, 2a_{10}a_2 - a_6^2 \geq 0, a_{10}a_{12} - a_{14}^2 \geq 0, 2a_2a_{12} - a_7^2 \geq 0,$$
$$a_{11}a_{12} - a_{15}^2 > 0, 2a_1a_{10}a_{11} + 2a_4a_5a_{13} - a_5^2a_{10} - a_4^2a_{11} - 2a_1a_{13}^2 > 0,$$
$$2a_{10}a_2a_{12} + 2a_6a_7a_{14} - 2a_{14}^2a_2 - a_6^2a_{12} - a_7^2a_{10} \geq 0,$$
$$2a_3a_{11}a_{12} + 2a_8a_9a_{15} - a_{12}a_8^2 - 2a_3a_{15}^2 - a_{11}a_9^2 > 0.$$

Then, $\mathcal{A}$ is rank-one positive definite.

When R is positive semidefinite and both S and T are positive definite, it is easy to obtain the following theorem.

Theorem 3.3. *Let $\mathcal{A}$ be a fourth-order three dimensional partially symmetric tensor. Suppose that*

$$a_1 > 0, a_2 > 0, a_3 > 0, a_{10} > 0, a_{11} > 0, a_{12} > 0,$$
$$2a_1a_{10} - a_4^2 \geq 0, 2a_1a_{11} - a_5^2 \geq 0, a_{10}a_{11} - a_{13}^2 \geq 0, 2a_{10}a_2 - a_6^2 > 0, a_{11}a_{12} - a_{15}^2 > 0,$$
$$2a_1a_{10}a_{11} + 2a_4a_5a_{13} - a_5^2a_{10} - a_4^2a_{11} - 2a_1a_{13}^2 \geq 0,$$
$$2a_{10}a_2a_{12} + 2a_6a_7a_{14} - 2a_{14}^2a_2 - a_6^2a_{12} - a_7^2a_{10} > 0,$$
$$2a_3a_{11}a_{12} + 2a_8a_9a_{15} - a_{12}a_8^2 - 2a_3a_{15}^2 - a_{11}a_9^2 > 0.$$

Then, $\mathcal{A}$ is rank-one positive definite.

When R, S, and T are positive definite, it is easy to obtain the following corollary.

Corollary 3.4. *Let $\mathcal{A}$ be a fourth-order three dimensional partially symmetric tensor. Suppose that*

$$a_1 > 0, a_2 > 0, a_3 > 0, a_{10} > 0, a_{11} > 0, 2a_1a_{10} - a_4^2 > 0, 2a_{10}a_2 - a_6^2 > 0, a_{11}a_{12} - a_{15}^2 > 0,$$
$$2a_1a_{10}a_{11} + 2a_4a_5a_{13} - a_5^2a_{10} - a_4^2a_{11} - 2a_1a_{13}^2 > 0,$$
$$2a_{10}a_2a_{12} + 2a_6a_7a_{14} - 2a_{14}^2a_2 - a_6^2a_{12} - a_7^2a_{10} > 0,$$
$$2a_3a_{11}a_{12} + 2a_8a_9a_{15} - a_{12}a_8^2 - 2a_3a_{15}^2 - a_{11}a_9^2 > 0.$$

Then, $\mathcal{A}$ is rank-one positive definite.

If we redistribute and combine the terms of polynomial (3.2) to yield

$$\begin{aligned}
\mathcal{A}\mathbf{x}^4 =& x_1^2[\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2}a_1 & \frac{1}{2}a_4 \\ \frac{1}{2}a_4 & \frac{1}{4}a_{10} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 & x_3 \end{pmatrix} \begin{pmatrix} \frac{1}{2}a_1 & \frac{1}{2}a_5 \\ \frac{1}{2}a_5 & \frac{1}{4}a_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \\
&+ \begin{pmatrix} x_2 & x_3 \end{pmatrix} \begin{pmatrix} \frac{1}{4}a_{10} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{13} & \frac{1}{4}a_{11} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}] + x_2^2[\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \frac{1}{4}a_{10} & \frac{1}{2}a_6 \\ \frac{1}{2}a_6 & \frac{1}{2}a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&+ \begin{pmatrix} x_1 & x_3 \end{pmatrix} \begin{pmatrix} \frac{1}{4}a_{10} & \frac{1}{2}a_{14} \\ \frac{1}{2}a_{14} & \frac{1}{4}a_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_2 & x_3 \end{pmatrix} \begin{pmatrix} \frac{1}{2}a_2 & \frac{1}{2}a_7 \\ \frac{1}{2}a_7 & \frac{1}{4}a_{12} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}] + \\
&x_3^2[\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \frac{1}{4}a_{11} & \frac{1}{2}a_{15} \\ \frac{1}{2}a_{15} & \frac{1}{4}a_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 & x_3 \end{pmatrix} \begin{pmatrix} \frac{1}{4}a_{11} & \frac{1}{2}a_8 \\ \frac{1}{2}a_8 & \frac{1}{2}a_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + \\
&\begin{pmatrix} x_2 & x_3 \end{pmatrix} \begin{pmatrix} \frac{1}{4}a_{12} & \frac{1}{2}a_9 \\ \frac{1}{2}a_9 & \frac{1}{2}a_3 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}].
\end{aligned}$$

Let

$$A=\begin{pmatrix}\frac{1}{2}a_1 & \frac{1}{2}a_4\\ \frac{1}{2}a_4 & \frac{1}{4}a_{10}\end{pmatrix},\ B=\begin{pmatrix}\frac{1}{2}a_1 & \frac{1}{2}a_5\\ \frac{1}{2}a_5 & \frac{1}{4}a_{11}\end{pmatrix},\ C=\begin{pmatrix}\frac{1}{4}a_{10} & \frac{1}{2}a_{13}\\ \frac{1}{2}a_{13} & \frac{1}{4}a_{11}\end{pmatrix},$$
$$D=\begin{pmatrix}\frac{1}{4}a_{10} & \frac{1}{2}a_6\\ \frac{1}{2}a_6 & \frac{1}{2}a_2\end{pmatrix},\ E=\begin{pmatrix}\frac{1}{4}a_{10} & \frac{1}{2}a_{14}\\ \frac{1}{2}a_{14} & \frac{1}{4}a_{12}\end{pmatrix},\ F=\begin{pmatrix}\frac{1}{2}a_2 & \frac{1}{2}a_7\\ \frac{1}{2}a_7 & \frac{1}{4}a_{12}\end{pmatrix},$$
$$G=\begin{pmatrix}\frac{1}{4}a_{11} & \frac{1}{2}a_{15}\\ \frac{1}{2}a_{15} & \frac{1}{4}a_{12}\end{pmatrix},\ H=\begin{pmatrix}\frac{1}{4}a_{11} & \frac{1}{2}a_8\\ \frac{1}{2}a_8 & \frac{1}{2}a_3\end{pmatrix},\ K=\begin{pmatrix}\frac{1}{4}a_{12} & \frac{1}{2}a_9\\ \frac{1}{2}a_9 & \frac{1}{2}a_3\end{pmatrix},$$

and $\mathbf{x}_1=(x_1,x_2)$, $\mathbf{x}_2=(x_1,x_3)$, $\mathbf{x}_3=(x_2,x_3)$. Then,

$$\begin{aligned}\mathcal{A}\mathbf{x}^4 =& x_1^2(\mathbf{x}_1A\mathbf{x}_1^\top+\mathbf{x}_2B\mathbf{x}_2^\top+\mathbf{x}_3C\mathbf{x}_3^\top)+x_2^2(\mathbf{x}_1D\mathbf{x}_1^\top+\mathbf{x}_2E\mathbf{x}_2^\top+\mathbf{x}_3F\mathbf{x}_3^\top)\\ &+x_3^2(\mathbf{x}_1G\mathbf{x}_1^\top+\mathbf{x}_2H\mathbf{x}_2^\top+\mathbf{x}_3K\mathbf{x}_3^\top).\end{aligned}$$

Obviously, we only need the matrices A, B, C, D, E, F, G, H, K to be positive definite, then $\mathcal{A}\mathbf{x}^4>0$ for all $\mathbf{x}\neq 0$. Thus, the following theorem can be obtained immediately.

Theorem 3.5. *Let $\mathcal{A}$ be a fourth-order three dimensional partially symmetric tensor. If*

$$a_1>0, a_2>0, a_3>0, a_{10}>0, a_{11}>0, a_{12}>0, a_1a_{10}-2a_4^2>0, a_1a_{11}-2a_5^2>0,$$
$$a_{10}a_{11}-4a_{13}^2>0, a_2a_{10}-2a_6^2>0, a_{10}a_{12}-4a_{14}^2>0, a_2a_{12}-2a_7^2>0,$$
$$a_{11}a_{12}-4a_{15}^2>0, a_3a_{11}-2a_8^2>0, a_3a_{12}-2a_9^2>0.$$

Then, $\mathcal{A}$ is rank-one positive definite.

Declarations

Conflict of interest The authors declare no conflict of interest.

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