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Improved population mean estimator with exponential function under non-response

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Abstract. In this article, we consider a new family of exponential type estimators for estimating the unknown population mean of the study variable. We propose estimators taking advantage of the auxiliary variable information under the first and second non-response cases separately. The required theoretical comparisons are obtained and the numerical studies are conducted. In conclusion, the results show that the proposed family of estimators is the most efficient estimator with respect to the estimators in literature under the obtained conditions for both cases.

§1 Introduction

In sample surveys, efficient estimators can be used to obtain the unknown population parameters such as mean, proportion, variance, and sum as well. The common and main method is the use of the information of the auxiliary variable *(x)* for obtaining efficient estimators. For this reason, ratio, product, logarithmic, regression, and exponential type estimators can be seen in literature in order to obtain the most efficient estimator for the estimation of the unknown parameters.

In sample surveys, it is possible that all information on different variables will not be available at all times. Hansen and Hurwitz (1946) introduced a new sub-sampling technique to deal with this situation that is still significant in the sampling theory. In this technique, the non-response units are considered in order to estimate the unknown population parameters. Thus, the effect of non-response units can be reduced with this method.

In the sub-sampling technique, we consider a population *(N)* that is composed of two groups as N_1 and N_2. Here, the response units are available on N_1 while the non-response units are available only on N_2 $(N_2 = N - N_1)$ units in the population. The sample size is determined as n units by the Simple Random Sampling Without Replacement (SRSWOR). Here, the response units are symbolized as n_1 while the non-response units are symbolized as n_2 $(n_2 = n - n_1)$ in the sample. In this technique, r units $(r = {}^{n_2}/_{z}\ ,\ z > 1)$ are obtained from non-response units in

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the sample with extra effort. Using $(n_1 + r)$ units, the unknown population parameters can be estimated with this sub-sampling method. The unbiased population mean estimator proposed by Hansen and Hurwitz (1946) to deal with the non-response situation using the sub-sampling method is as follows:

$$t_{Nr} = w_1 \bar{y}_1 + w_2 \bar{y}_{2(r)} \tag{1}$$

whose variance is

$$V(t_{Nr}) = \bar{Y}^2 \left(\lambda C_y^2 + \frac{W_2 (z-1)}{n} C_{y(2)}^2 \right) \tag{2}$$

where $f = \frac{n}{N}$, $\lambda = \frac{1-f}{n}$, $C_y^2 = \frac{S_y^2}{\bar{Y}^2}$, $C_{y(2)}^2 = S_{y(2)}^2 \big/ \bar{Y}^2$ and $W_2 = N_2/N$ is the weight of non-response units for the population, and the population mean of y is symbolized as $\bar{Y}$.In the t_{Nr} estimator, $w_2 = \frac{n_2}{n}$ symbolizes the weight of the non-response units for the sample while $w_1 = \frac{n_1}{n}$ is the weight of response. The $\bar{y}_1$ is the sample mean of y according to n_1 units while $\bar{y}_{2(r)}$ is the sample mean of y according to r units.

The non-response case is examined under the first and second cases. In the first case, the non-response units are available only on y while x has all the information. The population mean of x is known.

Rao (1986) proposed the classical ratio and classical regression estimators under the first case, respectively, as

$$t_{R1} = \bar{y}^* \frac{\bar{X}}{\bar{x}} \tag{3}$$

$$t_{reg1} = \bar{y}^* + b^* \left(\bar{X} - \bar{x} \right) \tag{4}$$

where $b^* = S_{xy}^* \big/ S_x^{*2}$. In these estimators, the sample mean of y is symbolized as $\bar{y}^*$ under the non-response scheme. In addition, the sample and the population mean of x are denoted as $\bar{x}$ and $\bar{X}$,respectively, while $\bar{Y}$ is the population mean of y.

The MSE equations of the t_{R1} and t_{reg1} estimators are, respectively, given by

$$MSE(t_{R1}) = \bar{Y}^2 \left(\lambda \left(C_x^2 - 2C_{yx} + C_y^2 \right) + \frac{W_2 (z-1)}{n} C_{y(2)}^2 \right) \tag{5}$$

$$MSE(t_{reg1}) = \bar{Y}^2 \left(\lambda C_y^2 \left(1 - \rho_{xy}^2 \right) + \frac{W_2 (z-1)}{n} C_{y(2)}^2 \right) \tag{6}$$

where $C_x^2 = \frac{S_x^2}{\bar{X}^2}$, $C_{yx} = \rho_{xy} C_y C_x$ and the population correlation coefficient between y and x is symbolized as ρ_{xy}. In the second non-response case, the non-response units are available on both x and also y. Similar with the first case, $\bar{X}$ is also known in the second case.

Cochran (1977) proposed the classical ratio and classical regression estimators under the second case, respectively, as

$$t_{R2} = \bar{y}^* \frac{\bar{X}}{\bar{x}^*} \tag{7}$$

$$t_{reg2} = \bar{y}^* + b^* \left(\bar{X} - \bar{x}^* \right) \tag{8}$$

where the sample mean of x in the case of non-response is symbolized as $\bar{x}^*$.

The MSE equations of t_{R2} and t_{reg2} estimators are, respectively, given by

$$MSE(t_{R2}) = \bar{Y}^2 \left(\lambda \left(C_x^2 - 2C_{yx} + C_y^2 \right) + \frac{W_2(z-1)}{n} \left(C_{y(2)}^2 + C_{x(2)}^2 - 2C_{yx(2)} \right) \right) \tag{9}$$

$$MSE\left(t_{reg2}\right)=\bar{Y}^{2}\left(\lambda C_{y}^{2}\left(1-\rho_{xy}^{2}\right)+\frac{W_{2}(z-1)}{n}\left(C_{y(2)}^{2}+\rho_{xy}^{2}\frac{C_{y}^{2}}{C_{x}^{2}}C_{x(2)}^{2}-2\rho_{xy}\frac{C_{y}}{C_{x}}C_{yx(2)}\right)\right) \quad (10)$$

where $C_{x(2)}^{2}=S_{x(2)}^{2}\Big/\bar{X}^{2}$, $C_{yx(2)}=\rho_{xy(2)}C_{y(2)}C_{x(2)}$ and $\rho_{yx(2)}$ symbolizes the coefficient of the population correlation between x and y in the presence of non-response group.

In literature, Kumar (2013) proposed a population mean estimator, and this estimator is supported by numerical study about the physical growth of the upper socio-economic (Khare and Sinha, 2007). Using a similar numerical example, Dahiru *et al.* (2021) also contributed to the literature with a product-log-type estimator as well as Zubair and Ali (2019) introduced a generalized regression-product type estimator for both Cases I and II. Khare and Rehman (2014) proposed a Chain Ratio-Cum-Regression Type estimator for only Case I. For another numerical example, Unal and Kadilar (2020), Singh and Usman (2021), and Sinha and Kumar (2017) proposed various estimators for the population mean using with related to agriculture (Khare and Sinha, (2009), Khare and Srivastava, (1993), Khare and Kumar, (2009), Sinha and Kumar, (2015)). Satici and Kadilar (2011) introduced an application about education in the presence of non-response for estimating the population means and Yaqub et al. (2017), Pal and Singh (2021), and Unal and Kadilar (2021) used this numerical example. In addition, the estimators, proposed by Pal and Singh (2020, 2021), Pal et al. (2020), Kadilar and Cekim (2015, 2017), and Cekim and Cingi (2016), are also important estimators in the literature.

In this study, the exponential type estimators in literature under the first and second cases are given in the next section. After that, the proposed family of estimators is extensively examined for both cases in Section 3. In Section 4, the theoretical comparisons are obtained and then the numerical studies are performed in Section 5. In conclusion, the results are discussed.

§2 Existing Exponential Type Estimators in Literature

Following the sub-sampling method, we can see many types of estimators for estimating the population mean. According to both first and second cases, the exponential type estimators and their MSE equations, up to the first order of approximation, are given in Table 1 and Table 2, respectively.

Table 1. List of estimators for the first case.

Authors	Estimators	MSE of the Estimators
Singh *et al.* (2009)	$t_{\exp 1} = \bar{y}^* exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	$MSE\left(t_{\exp 1}\right) = \bar{Y}^2\left(\lambda\left(C_y^2 + \frac{C_x^2}{4} - C_{yx}\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)$
Olufadi and Kumar (2014)	$t_{YK1} = \bar{y}^*\left\{\alpha \exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) + (1-\alpha)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right\}$	$MSE\left(t_{YK1}\right) = \bar{Y}^2\left(\lambda C_y^2\left(1-\rho_{xy}^2\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)$
Dansawad (2019)	$t_D = \bar{y}^* exp\left(\frac{\left(a\bar{X}+b\right)-\left(a\bar{x}+b\right)}{\left(a\bar{X}+b\right)+\left(a\bar{x}+b\right)}\right)$	$MSE\left(t_D\right) = \bar{Y}^2\left(\lambda\left(C_y^2 + \theta_i C_x^2\left(\theta_i - 2\rho_{xy}\frac{C_y}{C_x}\right)\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)$
Pal and Singh (2017)	$t_{PS1} = a^*\bar{y}^*\left(\frac{\bar{X}}{\bar{x}}\right) + (1-a^*)\,\bar{y}^* exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	$MSE\left(t_{PS1}\right) = \bar{Y}^2\left(\lambda C_y^2\left(1-\rho_{xy}^2\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)$
Singh and Usman (2019)	$t_{US1} = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}}\right)^{\theta} exp\left\{a\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right\}$	$MSE\left(t_{US1}\right) = \bar{Y}^2\left(\lambda C_y^2\left(1-\rho_{xy}^2\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)$
Yadav *et al.* (2016)	$t_{Y1} = \bar{y}^* exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+(a-1)\bar{x}}\right)$	$MSE_{\min}\left(t_{Y1}\right) = \bar{Y}^2\left(\lambda C_y^2\left(1-\rho_{xy}^2\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)$
Pal and Singh (2016)	$t_{(\alpha_1,\delta_1)} = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha_1} exp\left\{\left(\frac{\delta_1\left(\bar{x}-\bar{X}\right)}{\bar{x}+\bar{X}}\right)\right\}$	$MSE\left(t_{(\alpha_1,\delta_1)}\right) = \bar{Y}^2\left(\lambda C_y^2\left(1-\rho_{xy}^2\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)$
Pal and Singh (2018)	$t_{(\eta,\delta)} = \bar{y}^*\left[\eta\left\{\frac{(1-\delta)\bar{x}+\delta\bar{X}}{\delta\bar{x}+(1-\delta)\bar{X}}\right\} + (1-\delta)\left\{\frac{\delta\bar{x}+(1-\delta)\bar{X}}{(1-\delta)\bar{x}+\delta\bar{X}}\right\}\right\}$	$MSE\left(t_{(\eta,\delta)}\right) = \bar{Y}^2\left(\lambda C_y^2\left(1-\rho_{xy}^2\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)$
Unal and Kadilar(2021)	$t_{1,i} = k\bar{y}^*\left(\frac{a_i\bar{X}+b_i}{a_i\bar{x}+b_i}\right)^{\alpha}\exp\left(\frac{a_i\left(\bar{X}-\bar{x}\right)}{a_i\left(\bar{X}+\bar{x}\right)+2b_i}\right)$	$MSE_{\min}\left(t_{1,i}\right) = \bar{Y}^2\left(1-\frac{E_1^2}{2E_2}\right),\ i=1,2,...,10$ $\theta_i = \frac{a\bar{X}}{a\bar{X}+b}, i=1,2,...,10$ $E_1 = \lambda\left(C_x^2\theta_i^2\left(\alpha^2+\frac{3}{4}\right) - C_{yx}\theta_i\left(1+2\alpha\right)\right)+2$ $E_2 = \left(\lambda\left(2\theta_i^2C_x^2 + 4\alpha^2\theta_i^2C_x^2 + 2\alpha\theta_i^2C_x^2 + 8\alpha\theta_iC_{yx}\right) + 2C_y^2 - 4\theta_iC_{yx} + 2\frac{W_2(z-1)}{n}C_{y(2)}^2 + 2\right)$

Table 2. List of estimators for the second case.

Authors	Estimators	MSE of the Estimators
Singh *et al.* (2009)	$t_{\exp 2} = \bar{y}^* exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right)$	$MSE\left(t_{\exp 2}\right) = \bar{Y}^2\left(\lambda C_y^2 + \lambda\frac{C_x^2}{4} - \lambda C_{yx} + \frac{W_2(z-1)}{n}\left(C_{y(2)}^2 + \frac{C_{x(2)}^2}{4} - C_{yx(2)}\right)\right)$
Kumar and Bhougal (2011)	$t_{KB} = \bar{y}^*\left\{\alpha \exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right) + (1-\alpha)\exp\left(\frac{\bar{x}^*-\bar{X}}{\bar{x}^*+\bar{X}}\right)\right\}$	$MSE_{\min}\left(t_{KB}\right) = \bar{Y}^2\left(\lambda C_y^2\left(1-\rho_{xy}^2\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\left(1-\rho_{xy(2)}^2\right)\right)$
Kumar (2013)	$t_K = \bar{y}^* exp\left(\frac{\left(a\bar{X}+b\right)-\left(a\bar{x}^*+b\right)}{\left(a\bar{X}+b\right)+\left(a\bar{x}^*+b\right)}\right)$	$MSE\left(t_K\right) = \bar{Y}^2\left(\lambda\left(C_y^2 + C_x^2 - 2\phi C_{yx}\right) + \frac{W_2(z-1)}{n}\left(C_{y(2)}^2 + C_{x(2)}^2 - 2\phi C_{yx(2)}\right)\right)$ $\phi = \frac{a\bar{X}}{2\left(a\bar{X}+b\right)}$
Singh and Usman (2021)	$t_{SU} = \left[d_1^*\bar{y}^* + d_2^*\left(\bar{X}-\bar{x}^*\right)\right] exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right)$	$MSE_{\min}\left(t_{SU}\right) = \bar{Y}^2\left[\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right) - \frac{\left(\lambda C_{xy}+\frac{W_2(z-1)}{n}C_{yx(2)}\right)^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)}\right]$
Unal and Kadilar (2020)	$t_{UK} = \bar{y}^*\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha}\exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right)$	$MSE_{\min}\left(t_{UK}\right) = \bar{Y}^2\left[\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right) - \frac{\left(\lambda C_{xy}+\frac{W_2(z-1)}{n}C_{yx(2)}\right)^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)}\right]$
Pal and Singh (2018)	$t_{(\alpha,\beta)} = \bar{y}^*\left[\alpha\left\{\frac{(1-\beta)\bar{x}^*+\beta\bar{X}}{\beta\bar{x}^*+(1-\beta)\bar{X}}\right\} + (1-\alpha)\left\{\frac{\beta\bar{x}^*+(1-\beta)\bar{X}}{(1-\beta)\bar{x}^*+\beta\bar{X}}\right\}\right]$	$MSE_{\min}\left(t_{(\alpha,\beta)}\right) = \bar{Y}^2\left[\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right) - \frac{\left(\lambda C_{xy}+\frac{W_2(z-1)}{n}C_{yx(2)}\right)^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)}\right]$
Pal and Singh (2016)	$t_{(\alpha 2,\delta 2)} = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}^*}\right)^{\alpha_1} exp\left\{\left(\frac{\delta\left(\bar{X}-\bar{x}^*\right)}{\bar{X}+\bar{x}^*}\right)\right\}$	$MSE_{\min}\left(t_{(\alpha 2,\delta 2)}\right) = \bar{Y}^2\left[\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right) - \frac{\left(\lambda C_{xy}+\frac{W_2(z-1)}{n}C_{yx(2)}\right)^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)}\right]$

Table 2. Continued.

Authors	Estimators	MSE of the Estimators
Pal and Singh (2017)	$t_{PS2} = a^* \bar{y}^* \left(\frac{\bar{X}}{\bar{x}^*}\right) + (1-a^*)\,\bar{y}^* exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right)$	$MSE_{\min}(t_{PS2}) = \bar{Y}^2 \left[\left(\lambda C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2\right) - \frac{\left(\lambda C_{xy} + \frac{W_2(z-1)}{n} C_{yx(2)}\right)^2}{\left(\lambda C_x^2 + \frac{W_2(z-1)}{n} C_{x(2)}^2\right)}\right]$
Singh and Usman (2019)	$t_{US2} = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}^*}\right)^{\theta} exp\left\{a \frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right\}$	$MSE_{\min}(t_{US2}) = \bar{Y}^2 \left[\left(\lambda C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2\right) - \frac{\left(\lambda C_{xy} + \frac{W_2(z-1)}{n} C_{yx(2)}\right)^2}{\left(\lambda C_x^2 + \frac{W_2(z-1)}{n} C_{x(2)}^2\right)}\right]$
Unal and Kadilar(2021)	$t_{2,i} = k\bar{y}^* \left(\frac{a_i \bar{X}+b_i}{a_i \bar{x}^*+b_i}\right)^{\alpha} \exp\left(\frac{a_i(\bar{X}-\bar{x}^*)}{a_i(\bar{X}+\bar{x}^*)+2b_i}\right)$	$MSE_{\min}(t_{1,i}) = \bar{Y}^2 \left(1 - \frac{E_3^2}{2E_4}\right),\ i = 1,2,...,10$ $E_3 = \left[\theta_i^2 \left(\alpha^2 + \frac{3}{4}\right)\left(\lambda C_x^2 + \frac{W_2(z-1)}{n} C_{x(2)}^2\right)\right.$ $E_4 = 2\left(\lambda C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2\right)$ $+2\theta_i^2 \left(2\alpha^2 + \alpha + 1\right)\left(\lambda C_x^2 + \frac{W_2(z-1)}{n} C_{x(2)}^2\right)$ $-4\theta_i \left(1+2\alpha\right)\left(\lambda C_{yx} + \frac{W_2(z-1)}{n} C_{yx(2)}\right)$
Riaz *et al.* (2020)	$t_{RNAQ} = \bar{y}^* \left[w_1 + w_2\left(a\bar{X} - a\bar{x}^*\right)\right] \exp\left(\frac{\delta\left(a\bar{X}-a\bar{x}^*\right)}{\left(a\bar{X}-a\bar{x}^*\right)+2b}\right)$	$MSE_{\min}(t_{RNAQ}) = \bar{Y}^2 \left(1 - \frac{A_1 A_4^2 + A_2 A_3^2 - 2A_3 A_4 A_5}{A_1 A_2 - A_5^2}\right)$ $A_1 = 1 + \lambda C_y^2 + \lambda\left(k_2^2 + 2k_3\right) C_x^2 - 4\lambda k_2 C_{yx},$ $A_2 = \lambda k_1^2 C_x^2,$ $A_3 = 1 + \lambda k_3 C_x^2 - \lambda k_2 C_{yx},$ $A_4 = \lambda k_1 k_2 C_x^2 - \lambda k_1 C_{yx},$ $A_5 = 2\lambda k_1 k_2 C_x^2 - 2\lambda k_1 C_{yx},$ $k_1 = a\bar{X},\ k_2 = \frac{\theta\delta}{2}, k_3 = \frac{\theta^2\delta}{4} + \frac{\theta^2\delta^2}{8}$

§3 The Proposed Family of Estimators

Singh *et al.* (2011) introduced general class of estimators to estimate the population mean as follows:

$$t = \left[\varepsilon_1 \bar{y} + \varepsilon_2 \left(\bar{X} - \bar{x}\right)\right] \left[\xi \exp\left(\frac{a\bar{X}+b}{a\bar{x}+b}\right) + (1-\xi)\exp\left\{\frac{a\left(\bar{X}-\bar{x}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\}\right] \tag{11}$$

where values of ε_1 and ε_2 are unknown constants for obtaining the minimum MSE of the t estimator that will be computed later and ξ is a constant to generate the different types estimator. Besides, a and b are either real numbers or the functions of the known parameters of the auxiliary variable.

The main aim of this study is to propose a family of estimators by adapting the t estimator introduced by Singh et al. (2011) to the non-response cases. In this section, the situation of non-response is examined under two different cases as first case and second case.

3.1 First Case

Along the lines of Singh *et al.* (2011), we propose t^* family of estimators for the first case as follows:

$$t^* = \left[\varepsilon_1 \bar{y}^* + \varepsilon_2 \left(\bar{X} - \bar{x}\right)\right] \left[\xi \exp\left(\frac{a\bar{X}+b}{a\bar{x}+b}\right) + (1-\xi)\exp\left\{\frac{a\left(\bar{X}-\bar{x}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\}\right] \tag{12}$$

Some notations are used for the first case to obtain the $Bias\left(t^*\right)$ and $MSE\left(t^*\right)$ and the minimum $MSE\left(t^*\right)$ is as follows:
$\bar{x} = \bar{X}\left(e_x+1\right),\ \bar{y}^* = \bar{Y}\left(e_y^*+1\right),\ E\left(e_y^*\right) = E\left(e_x\right) = 0,\ E\left({e_y^*}^2\right) = \left(\lambda C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2\right),$
$E\left(e_x^2\right) = \lambda C_x^2,\ E\left(e_y^* e_x\right) = \lambda C_{yx}$.

Using these notations, we re-write the t^* estimator in Eq. (12) as

$$\begin{aligned} t^* = \bar{Y}&\left\{\varepsilon_1\left(1 - \frac{\theta e_x}{2} + \frac{3\theta^2 e_x^2}{8} - \frac{\xi\theta e_x}{2} + \frac{5\xi\theta^2 e_x^2}{8} + e_y^* - \frac{\theta e_y^* e_x}{2} - \frac{\xi\theta e_y^* e_x}{2}\right)\right\} \\ &-\bar{X}\left\{\varepsilon_2\left(e_x - \frac{\theta e_x^2}{2} - \frac{\xi\theta e_x^2}{2}\right)\right\} \end{aligned} \tag{13}$$

where θ was defined in Section 2. Here, we can derive the members of θ values using different α and b values. Some members of the θ are given in Table 3.
We neglect two and higher powers of e_y^* and e_x and obtain

$$\begin{aligned} \left(t^* - \bar{Y}\right) = \bar{Y}&\left\{\varepsilon_1\left(1 - \frac{\theta e_x}{2} + \frac{3\theta^2 e_x^2}{8} - \frac{\xi\theta e_x}{2} + \frac{5\xi\theta^2 e_x^2}{8} + e_y^*\right.\right. \\ &\left.\left. - \frac{\theta e_y^* e_x}{2} - \frac{\xi\theta e_y^* e_x}{2}\right) - 1\right\} - \bar{X}\left\{\varepsilon_2\left(e_x - \frac{\theta e_x^2}{2} - \frac{\xi\theta e_x^2}{2}\right)\right\} \end{aligned} \tag{14}$$

Using Eq. (14), the $Bias\left(t^*\right)$ is obtained as

$$Bias\left(t^*\right) = \bar{Y}\left\{\varepsilon_1\left(1 + \frac{\theta^2\lambda C_x^2}{8}(3+5\xi) - \frac{\theta\lambda C_{yx}}{2}(1+\xi)\right) - 1\right\} + \bar{X}\frac{\varepsilon_2\theta\lambda C_x^2}{2}(1+\xi) \tag{15}$$

Taking the square of the both sides of Eq.(14) and then expectation, we obtain the $MSE\left(t^*\right)$ for the first case as follows:

$$\begin{aligned} MSE\left(t^*\right) = \Big\{\bar{Y}^2&\left(\left(\varepsilon_1 - 1\right)^2 + \frac{\varepsilon_1^2\theta^2}{4}\lambda C_x^2(\xi+1)^2 - \varepsilon_1^2\theta\lambda C_{yx}(\xi+1)\right. \\ &\left. + \varepsilon_1^2\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right) + \bar{X}^2\varepsilon_2^2\lambda C_x^2 \\ &+ \bar{X}\bar{Y}\varepsilon_1\varepsilon_2\left(\theta\lambda C_x^2(1+\xi) - 2\lambda C_{yx}\right)\Big\} \end{aligned} \tag{16}$$

Table 4. Some members of the θ values.

	α	b	θ
1	1	1	$\theta_1 = \frac{\bar{X}}{(\bar{X}+1)}$
2	1	$\beta_2(x)$	$\theta_2 = \frac{\bar{X}}{(\bar{X}+\beta_2(x))}$
3	1	C_x	$\theta_3 = \frac{\bar{X}}{(\bar{X}+C_x)}$
4	1	ρ	$\theta_4 = \frac{\bar{X}}{(\bar{X}+\rho)}$
5	$\beta_2(x)$	C_x	$\theta_5 = \frac{\beta_2(x)\bar{X}}{(\beta_2(x)\bar{X}+C_x)}$
6	C_x	$\beta_2(x)$	$\theta_6 = \frac{C_x\bar{X}}{(C_x\bar{X}+\beta_2(x))}$
7	C_x	ρ	$\theta_7 = \frac{C_x\bar{X}}{(C_x\bar{X}+\rho)}$
8	ρ	C_x	$\theta_8 = \frac{\rho\bar{X}}{(\rho\bar{X}+C_x)}$
9	$\beta_2(x)$	ρ	$\theta_9 = \frac{\beta_2(x)\bar{X}}{(\beta_2(x)\bar{X}+\rho)}$
10	ρ	$\beta_2(x)$	$\theta_{10} = \frac{\rho\bar{X}}{(\rho\bar{X}+\beta_2(x))}$

Taking the derivation of the $MSE(t^*)$, the optimum values of ϵ_1 and ϵ_2 , ϵ_1^* and ϵ_2^* are obtained as

$$\begin{aligned} \varepsilon_1^* &= \frac{\lambda C_x^2}{\lambda C_x^2\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)+(\lambda C_{yx})^2}, \\ \varepsilon_2^* &= \frac{\bar{Y}\left(2\lambda C_{yx}-\theta\lambda C_x^2-\xi\theta\lambda C_x^2\right)}{2\bar{X}\left(\lambda C_x^2\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)-(\lambda C_{yx})^2\right)} \end{aligned} \tag{17}$$

Substituting ϵ_1^* and ϵ_2^* instead of ϵ_1 and ϵ_2 in Eq.(16), we obtain the minimum $MSE(t^*)$ for the first case as follows:

$$MSE_{\min}(t^*) = \bar{Y}^2\left[\frac{(\lambda C_x^2)\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)-(\lambda C_{yx})^2}{(\lambda C_x^2)\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)-(\lambda C_{yx})^2}\right] \tag{18}$$

3.2 Second Case

Along the lines of Singh et al. (2011), we propose t^{**} estimator for the second case as follows:

$$t^{**} = \left[\varepsilon_1\bar{y}^* + \varepsilon_2\left(\bar{X}-\bar{x}^*\right)\right]\left[\xi\exp\left(\frac{a\bar{X}+b}{a\bar{x}^*+b}\right)+(1-\xi)\exp\left\{\frac{a\left(\bar{X}-\bar{x}^*\right)}{a\left(\bar{X}+\bar{x}^*\right)+2b}\right\}\right] \tag{19}$$

To obtain the $Bias(t^{**})$, $MSE(t^{**})$ and the minimum $MSE(t^{**})$, under the second case, we use some notations as

$\bar{y}^* = \bar{Y}\left(e_y^*+1\right), \bar{x}^* = \bar{X}\left(e_x^*+1\right), E\left(e_x^*\right)=0, E\left(e_y^*\right)=0, E\left(e_x^{*2}\right) = \left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right),$
$E\left(e_y^*e_x^*\right) = \left(\lambda C_{yx}+\frac{W_2(z-1)}{n}C_{yx(2)}\right), E\left(e_y^{*2}\right) = \left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right).$

Using these notations, we can write the t^{**} estimator as

$$t^{**} = \bar{Y}\left\{\varepsilon_1\left(1 + e_y^* - \frac{\theta e_x^*}{2} - \frac{\xi\theta e_x^*}{2} + \frac{3\theta^2 {e_x^*}^2}{8} + \frac{5\xi\theta^2 {e_x^*}^2}{8} - \frac{\theta e_y^* e_x^*}{2} - \frac{\xi\theta e_y^* e_x^*}{2}\right)\right\} - \bar{X}\left\{\varepsilon_2\left(e_x^* - \frac{\theta {e_x^*}^2}{2} - \frac{\xi\theta {e_x^*}^2}{2}\right)\right\} \tag{20}$$

We can also write some members of the family of t^{**} estimators under the second case as in Table 3. Expressions of the $Bias\,(t^{**})$ and $MSE\,(t^{**})$ are computed for the second case, respectively, as follows:

$$E\left(t^{**} - \bar{Y}\right) = \bar{Y}\left\{\varepsilon_1\left(1 + \frac{\theta^2 E\left({e_x^*}^2\right)}{8}(3+5\xi) - \frac{\theta E\left(e_y^* e_x^*\right)}{2}(1+\xi)\right) - 1\right\} + \bar{X}\frac{\varepsilon_2 \theta E\left({e_x^*}^2\right)}{2}(1+\xi) \tag{21}$$

$$\begin{aligned} Bias\,(t^{**}) = \Big\{\bar{Y}\Big\{\varepsilon_1\Big(1 + \tfrac{\theta^2}{8}\left(\lambda C_x^2 + \tfrac{W_2(z-1)}{n}C_{x(2)}^2\right)(3+5\xi) \\ - \tfrac{\theta}{2}\left(\lambda C_{yx} + \tfrac{W_2(z-1)}{n}C_{yx(2)}\right)(1+\xi)\Big) - 1\Big\} \\ + \bar{X}\tfrac{\varepsilon_2\theta}{2}\left(\lambda C_x^2 + \tfrac{W_2(z-1)}{n}C_{x(2)}^2\right)(1+\xi)\Big\} \end{aligned} \tag{22}$$

$$\begin{aligned} MSE\,(t^{**}) = \Big\{\bar{Y}^2 + \Big(\varepsilon_1^2\left(\lambda C_y^2 + \tfrac{W_2(z-1)}{n}C_{y(2)}^2\right) \\ + \tfrac{\varepsilon_1^2\theta^2}{4}\left(\lambda C_x^2 + \tfrac{W_2(z-1)}{n}C_{x(2)}^2\right)(\xi+1)^2\Big) \\ - \varepsilon_1^2\theta\left(\lambda C_{yx} + \tfrac{W_2(z-1)}{n}C_{yx(2)}\right)(\xi+1)\Big) \\ + \bar{X}^2\varepsilon_2^2\left(\lambda C_x^2 + \tfrac{W_2(z-1)}{n}C_{x(2)}^2\right) \\ + \Big(\bar{X}\bar{Y}\varepsilon_1\varepsilon_2\Big(\theta\left(\lambda C_x^2 + \tfrac{W_2(z-1)}{n}C_{x(2)}^2\right)(1+\xi) \\ - 2\left(\lambda C_{yx} + \tfrac{W_2(z-1)}{n}C_{yx(2)}\right)\Big)\Big\} \end{aligned} \tag{23}$$

The optimal values, ϵ_1^{**} and ϵ_2^{**} are obtained as follows:

$$\begin{aligned} \varepsilon_1^{**} &= \frac{\left(\lambda C_x^2 + \frac{W_2(z-1)}{n}C_{x(2)}^2\right)}{\left(\lambda C_x^2 + \frac{W_2(z-1)}{n}C_{x(2)}^2\right)\left(1 + \left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right) + \left(\lambda C_{yx} + \frac{W_2(z-1)}{n}C_{yx(2)}\right)^2}, \\ \varepsilon_2^{**} &= \frac{\bar{Y}\left\{2\left(\lambda C_{yx} + \frac{W_2(z-1)}{n}C_{yx(2)}\right) - (\theta+\xi\theta)\left(\lambda C_x^2 + \frac{W_2(z-1)}{n}C_{x(2)}^2\right)\right\}}{2\bar{X}\left\{\left(\lambda C_x^2 + \frac{W_2(z-1)}{n}C_{x(2)}^2\right)\left(1 + \left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right) - \left(\lambda C_{yx} + \frac{W_2(z-1)}{n}C_{yx(2)}\right)^2\right\}}. \end{aligned} \tag{24}$$

The optimum values are substituted in $MSE\,(t^{**})$ and then the minimum MSE for the second case is obtained. In order to simplify the mathematical notations, we can write the $MSE\,(t^{**})$ for the Case II as

$$MSE_{\min}\,(t^{**}) = \bar{Y}^2\left[\frac{k_1 - k_2^2}{\left(\lambda C_x^2 + \frac{W_2(z-1)}{n}C_{x(2)}^2\right) + k_1 - k_2^2}\right] \tag{25}$$

where $k_1 = \left(\lambda C_x^2 + \frac{W_2(z-1)}{n}C_{x(2)}^2\right)\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right)$ and $k_2 = \left(\lambda C_{yx} + \frac{W_2(z-1)}{n}C_{yx(2)}\right)$.

§4 Efficiency Comparisons

To show the appropriateness of t^* and t^{**} estimators theoretically, we compare the efficiency of the proposed estimators with the unbiased, ratio, regression and exponential type estimators for the first and the second non-response cases, respectively.

4.1 Efficiency Comparisons for the First Case

We compare the MSE of the t^* estimator with the MSE equations of the unbiased, ratio, regression and the exponential estimators given in Table 1. We obtain the conditions for the first case as follows:

$$MSE_{\min}(t^*) < V(t_{Nr})$$

$$\left[\frac{(\lambda C_x^2)\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)-(\lambda C_{yx})^2}{(\lambda C_x^2)\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)-(\lambda C_{yx})^2}\right]-\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)<0 \tag{26}$$

$$MSE_{\min}(t^*) < MSE(t_{R1})$$

$$\left[\frac{(\lambda C_x^2)\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)-(\lambda C_{yx})^2}{(\lambda C_x^2)\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)-(\lambda C_{yx})^2}\right]-\left(\lambda\left(C_x^2-2C_{yx}+C_y^2\right)+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)<0 \tag{27}$$

$$MSE_{\min}(t^*) < MSE(t_{\exp 1})$$

$$\left[\frac{(\lambda C_x^2)\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)-(\lambda C_{yx})^2}{(\lambda C_x^2)\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)-(\lambda C_{yx})^2}\right]-\left(\lambda\left(C_y^2+\frac{C_x^2}{4}-C_{yx}\right)+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)<0 \tag{28}$$

$$MSE_{\min}(t^*) < MSE(t_{reg1})$$

$$\left[\frac{(\lambda C_x^2)\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)-(\lambda C_{yx})^2}{(\lambda C_x^2)\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)-(\lambda C_{yx})^2}\right]-\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)<0 \tag{29}$$

$$MSE_{\min}(t^*) < MSE(t_D)$$

$$\begin{aligned}&\left[\frac{(\lambda C_x^2)\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)-(\lambda C_{yx})^2}{(\lambda C_x^2)\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)-(\lambda C_{yx})^2}\right]\\&\qquad-\left(\lambda\left(C_y^2+\theta_i C_x^2\left(\theta_i-2\rho_{xy}\frac{C_y}{C_x}\right)\right)+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)<0\end{aligned} \tag{30}$$

$$MSE_{\min}(t^*) < MSE_{\min}(t_{YK1})$$

$$\begin{aligned}&\left[\frac{(\lambda C_x^2)\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)-(\lambda C_{yx})^2}{(\lambda C_x^2)\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)-(\lambda C_{yx})^2}\right]\\&-\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)+\frac{\left[\lambda C_x^2+8\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right]^2}{64\left[1+\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right]}<0\end{aligned} \tag{31}$$

$$MSE_{\min}(t^*) < MSE_{\min}(t_{1,i})$$

$$\left[\frac{\left(\lambda C_x^2\right)\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)-\left(\lambda C_{yx}\right)^2}{\left(\lambda C_x^2\right)\left(1+\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)\right)-\left(\lambda C_{yx}\right)^2}\right]-\left(1-\frac{E_1^2}{2E_2}\right)<0 \tag{32}$$

The efficiency conditions of $t_{PS1}, t_{US1},\ t_{Y1},\ t_{(\alpha_1,\delta_1)}$ and $t_{(\eta,\delta)}$ estimators are obtained as in the condition (29). We conclude that the t^* estimator is more effective than others in literature based on the conditions from (26) to (32) for the first case.

4.2 Efficiency comparisons for the second case

We compare the MSE of the t^{**} estimator with the MSE equations of the unbiased, ratio, regression and the exponential estimators given in Table 2. We obtain the conditions for the second case as follows:

$$MSE_{\min}\left(t^{**}\right)<V\left(t_{Nr}\right)$$

$$\left[\frac{k_1-k_2^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)+k_1-k_2^2}\right]-\left(\lambda C_y^2+\frac{W_2\left(z-1\right)}{n}C_{y(2)}^2\right)<0 \tag{33}$$

$$MSE_{\min}\left(t^{**}\right)<MSE\left(t_{R2}\right)$$

$$\begin{aligned}\left[\frac{k_1-k_2^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)+k_1-k_2^2}\right]&-\left(\lambda\left(C_x^2-2C_{yx}+C_y^2\right)\right.\\&\left.+\frac{W_2(z-1)}{n}\left(C_{y(2)}^2+C_{x(2)}^2-2C_{yx(2)}\right)\right)<0\end{aligned} \tag{34}$$

$$MSE_{\min}\left(t^{**}\right)<MSE\left(t_{reg2}\right)$$

$$\begin{aligned}\left[\frac{k_1-k_2^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)+k_1-k_2^2}\right]&-\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)\right.\\&\left.+\frac{W_2(z-1)}{n}\left(C_{y(2)}^2+\rho_{xy}^2\frac{C_y^2}{C_x^2}C_{x(2)}^2-2\rho_{xy}\frac{C_y}{C_x}C_{yx(2)}\right)\right)<0\end{aligned} \tag{35}$$

$$MSE_{\min}\left(t^{**}\right)<MSE\left(t_{\exp 2}\right)$$

$$\begin{aligned}\left[\frac{k_1-k_2^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)+k_1-k_2^2}\right]&-\left(\lambda C_y^2+\lambda\frac{C_x^2}{4}-\lambda C_{yx}\right.\\&\left.+\frac{W_2(z-1)}{n}\left(C_{y(2)}^2+\frac{C_{x(2)}^2}{4}-C_{yx(2)}\right)\right)<0\end{aligned} \tag{36}$$

$$MSE_{\min}\left(t^{**}\right)<MSE_{\min}\left(t_{KB}\right)$$

$$\left[\frac{k_1-k_2^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)+k_1-k_2^2}\right]-\left(\lambda C_y^2\left(1-\rho_{xy}^2\right)+\frac{W_2(z-1)}{n}C_{y(2)}^2\left(1-\rho_{xy(2)}^2\right)\right)<0 \tag{37}$$

$$MSE_{\min}\left(t^{**}\right)<MSE\left(t_K\right)$$

$$\left[\frac{k_1-k_2^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)+k_1-k_2^2}\right]-\left(\lambda\left(C_y^2+C_x^2-2\phi C_{yx}\right)\right.$$
$$\left.+\frac{W_2(z-1)}{n}\left(C_{y(2)}^2+C_{x(2)}^2-2\phi C_{yx(2)}\right)\right)<0 \tag{38}$$

$$MSE_{\min}\left(t^{**}\right)<MSE_{\min}\left(t_{SU}\right)$$

$$\left[\frac{k_1-k_2^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)+k_1-k_2^2}\right]-\left[\left(\lambda C_y^2+\frac{W_2(z-1)}{n}C_{y(2)}^2\right)-\frac{\left(\lambda C_{xy}+\frac{W_2(z-1)}{n}C_{yx(2)}\right)^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)}\right]<0 \tag{39}$$

$$MSE_{\min}\left(t^{**}\right)<MSE_{\min}\left(t_{2,i}\right)$$

$$\left[\frac{k_1-k_2^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)+k_1-k_2^2}\right]-\left(1-\frac{E_3^2}{2E_4}\right)<0 \tag{40}$$

$$MSE_{\min}\left(t^{**}\right)<MSE_{\min}\left(t_{RNAQ}\right)$$

$$\left[\frac{k_1-k_2^2}{\left(\lambda C_x^2+\frac{W_2(z-1)}{n}C_{x(2)}^2\right)+k_1-k_2^2}\right]-\left(1-\frac{A_1A_4^2+A_2A_3^2-2A_3A_4A_5}{A_1A_2-A_5^2}\right)<0 \tag{41}$$

The efficiency conditions of $t_{PS2},\ t_{UK},\ t_{(\alpha,\beta)},\ t_{(\alpha2,\delta2)}$ and t_{US2} estimators are obtained as in the condition (39). We conclude that the t^{**} estimator is more effective than other estimators in literature based on the conditions from (33) to (41) for the second case.

§5 Numerical Studies

After appropriateness of the theoretical inferences and comparisons, we calculated the MSE values of the proposed and existing estimators in literature with numerical studies using real data sets. In first data set, we use real Covid-19 data. The deaths and the new confirmed cases in a day are considered based on all cities (85 cities) in Russia (COVID-19, 2021). Here, the death number is taken as the study variable *(y)* while the number of new confirmed cases is taken as the auxiliary variable *(x)*. The missing data (non-response) rate is considered as the last %25, 21 units, of the population. The population statistics are summarized in Table 4.

Table 5. First data set parameters.

$N = 85$, n=30	$\bar{X} = 99.59$	$\rho_{yx(2)} = 0.816$	$\rho_{yx} = 0.902$	$C_{yx} = 5.635$
$\beta_2(x) = 68.088$	$\bar{Y} = 1337.02$	$C_y = 1.92$	$C_x = 3.254$	$C_{yx(2)} = 0.0115$
$W_2 = 0.25$	$\lambda = 0.0216$	$C_{y(2)} = 0.1265$	$C_{x(2)} = 0.1113$	$f = 0.353$

We obtain the MSE values of the existing unbiased, ratio, regression, exponential type estimators, given in Tables 1- 2, using different values of z and the MSE of the proposed families of estimators, t^* and t^{**}, under the first and second cases based on the COVID-19 data set. The results for the first and second cases are given in Tables 5-6, respectively.

Table 6. MSE values of the existing and t^* estimators for the first case.

Estimators	z=2	z=3	z=4	z=5	z=6
t_{Nr}	142580.0398	142818.4230	143056.8062	143295.1894	143533.5726
t_{R1}	116266.140	116504.523	116742.906	116981.289	117219.672
t_{exp1}	27210.436	27448.819	27687.202	27925.585	28163.969
t_{reg1}	26770.299	27008.682	27247.065	27485.448	27723.831
t_{YK1}	26770.299	27008.682	27247.065	27485.448	27723.831
$t_{D,1}$	112472.471	112710.854	112949.237	113187.620	113426.004
$t_{D,2}$	28327.615	28565.998	28804.382	29042.765	29281.148
$t_{D,3}$	104541.456	104779.839	105018.222	105256.606	105494.989
$t_{D,4}$	112834.315	113072.698	113311.081	113549.465	113787.848
$t_{D,5}$	116051.307	116289.690	116528.074	116766.457	117004.840
$t_{D,6}$	62147.094	62385.477	62623.861	62862.244	63100.627
$t_{D,7}$	115176.127	115414.510	115652.893	115891.276	116129.660
$t_{D,8}$	103362.843	103601.226	103839.609	104077.992	104316.376
$t_{D,9}$	116183.814	116422.197	116660.580	116898.963	117137.347
$t_{D,10}$	27318.613	27556.997	27795.380	28033.763	28272.146
t_{PS1}	26770.299	27008.682	27247.065	27485.448	27723.831
t_{US1}	26770.299	27008.682	27247.065	27485.448	27723.831
t_{Y1}	26770.299	27008.682	27247.065	27485.448	27723.831
$t_{(\alpha 1,\delta 1)}$	26770.299	27008.682	27247.065	27485.448	27723.831
$t_{(\eta,\delta)}$	26770.299	27008.682	27247.065	27485.448	27723.831
$t_{1;1}$	956458.630	956508.609	956558.583	956608.550	956658.512
$t_{1;2}$	717926.604	718018.801	718110.981	718203.146	718295.295
$t_{1;3}$	947141.761	947193.328	947244.888	947296.442	947347.990
$t_{1;4}$	956864.133	956914.044	956963.948	957013.847	957063.739
$t_{1;5}$	960399.517	960448.826	960498.128	960547.425	960596.716
$t_{1;6}$	876040.957	876104.742	876168.518	876232.285	876296.043
$t_{1;7}$	959449.932	959499.403	959548.867	959598.325	959647.778
$t_{1;8}$	945682.687	945734.502	945786.311	945838.114	945889.910
$t_{1;9}$	960542.509	960591.794	960641.072	960690.344	960739.611
$t_{1;10}$	697608.150	697704.133	697800.099	697896.048	697991.980
t^*	26391.570	26622.936	26854.241	27085.486	27316.670

Table 7. Second data set parameters.

$N = 96,\ n = 40$	$\bar{X} = 144.87$	$\rho_{yx(2)} = 0.72$	$\rho_{yx} = 0.77$	$C_{yx} = 0.823284$
$\beta_2(x) = 1.19997$	$\bar{Y} = 137.92$	$C_y = 1.32$	$C_x = 0.81$	$C_{yx(2)} = 1.407744$
$W_2 = 0.25$	$\lambda = 0.014583$	$C_{y(2)} = 1.7424$	$C_{x(2)} = 0.94$	$f = 0.4167$

Table 8. MSE values of the existing and t^{**} estimators for the second case.

Estimators	z=2	z=3	z=4	z=5	z=6
t_{Nr}	142580.0398	142818.4230	143056.8062	143295.1894	143533.5726
t_{R2}	116108.050	116188.343	116268.636	116348.929	116429.222
t_{exp2}	27085.256	27198.460	27311.664	27424.868	27538.072
t_{reg2}	26640.217	26748.519	26856.821	26965.123	27073.425
t_{KB}	26611.570	26691.224	26770.878	26850.533	26930.187
$t_{K,1}$	336026.385	336279.695	336533.005	336786.315	337039.625
$t_{K,2}$	422283.508	422604.680	422925.851	423247.022	423568.194
$t_{K,3}$	340751.378	341008.405	341265.433	341522.460	341779.487
$t_{K,4}$	335816.142	336069.287	336322.432	336575.576	336828.721
$t_{K,5}$	333966.066	334217.756	334469.445	334721.134	334972.823
$t_{K,6}$	371669.244	371950.595	372231.947	372513.298	372794.650
$t_{K,7}$	334466.031	334718.113	334970.196	335222.278	335474.361
$t_{K,8}$	341473.765	341731.360	341988.956	342246.552	342504.147
$t_{K,9}$	333890.583	334142.213	334393.843	334645.473	334897.102
$t_{K,10}$	427748.248	428073.718	428399.189	428724.660	429050.130
t_{SU}	26656.952	26765.220	26873.463	26981.679	27089.869
t_{PS2}	26656.952	26765.220	26873.463	26981.679	27089.869
$t_{2;1}$	317171.731	317290.924	317410.114	317529.302	317648.488
$t_{2;2}$	79163.896	79243.709	79323.521	79403.333	79483.144
$t_{2;3}$	303015.415	303131.765	303248.113	303364.459	303480.804
$t_{2;4}$	317798.966	317918.283	318037.598	318156.910	318276.221
$t_{2;5}$	323307.451	323427.856	323548.259	323668.659	323789.056
$t_{2;6}$	209723.039	209820.107	209917.175	210014.243	210111.312
$t_{2;7}$	321820.835	321940.947	322061.057	322181.165	322301.270
$t_{2;8}$	300842.095	300958.004	301073.911	301189.816	301305.720
$t_{2;9}$	323531.763	323652.213	323772.659	323893.103	324013.545
$t_{2;10}$	68777.484	68857.851	68938.217	69018.583	69098.949
t_{RNAQ}	26656.952	26765.220	26873.463	26981.679	27089.869
t_{UK}	26656.952	26765.220	26873.463	26981.679	27089.869
$t_{(\alpha,\beta)}$	26656.952	26765.220	26873.463	26981.679	27089.869
$t_{(\alpha 2,\delta 2)}$	26656.952	26765.220	26873.463	26981.679	27089.869
t_{US2}	26656.952	26765.220	26873.463	26981.679	27089.869
t^{**}	26265.285	26370.389	26475.455	26580.483	26685.474

According to the results in Table 5, the proposed t^* family of estimators has the minimum MSE value in comparison with other estimators in the first case. We can see similar situation for the second case in Table 6 as well. The second proposed t^{**} family of estimators has the minimum MSE value among compared estimators. Therefore, using the proposed t^* and t^{**} families of estimators is appropriate while estimating the population mean under the first and second cases, respectively.

Secondly, we use Khare and Sinha (2009) data set. Here, the number of agriculture labors is taken as the study variable *(y)* while the area of the village is taken as the auxiliary variable *(x)*. The missing data (non-response) rate is considered as the last %25, 24 units, of the population. The descriptive statistics are summarized in Table 8.

We obtain the MSE values of the similar estimators, given in Tables 1-2, according to the determinated z values and the MSE of the proposed families of estimators, t^* and t^{**}, under the first and second cases based on the Khare and Sinha (2009) data set. The results for the first and second cases are given in Table 7, Table 9, respectively.

Table 9. MSE values of the existing and t^* estimators for the first case.

Estimators	z=2	z=3	z=4	z=5	z=6
t_{Nr}	997.70	1512.053	2026.41	2540.76	3055.112
t_{R1}	722.941	1237.294	1751.647	2266.00	2780.353
t_{exp1}	814.820	1329.172	1843.525	2357.878	2872.231
t_{reg1}	711.124	1225.476	1739.829	2254.182	2768.535
t_{YK1}	711.124	1225.476	1739.829	2254.182	2768.535
$t_{D,1}$	723.586	1237.938	1752.291	2266.644	2780.997
$t_{D,2}$	723.715	1238.068	1752.421	2266.774	2781.127
$t_{D,3}$	723.463	1237.815	1752.168	2266.521	2780.874
$t_{D,4}$	723.463	1237.790	1752.142	2266.495	2780.848
$t_{D,5}$	723.375	1237.728	1752.081	2266.434	2780.787
$t_{D,6}$	723.899	1238.252	1752.604	2266.957	2781.31
$t_{D,7}$	723.554	1237.906	1752.259	2266.612	2780.965
$t_{D,8}$	723.619	1237.972	1752.325	2266.678	2781.031
$t_{D,9}$	723.354	1237.707	1752.06	2266.412	2780.765
$t_{D,10}$	723.949	1238.302	1752.655	2267.008	2781.360
t_{PS1}	711.124	1225.476	1739.829	2254.182	2768.535
t_{US1}	711.124	1225.476	1739.829	2254.182	2768.535
t_{Y1}	711.124	1225.476	1739.829	2254.182	2768.535
$t_{(\alpha 1,\delta 1)}$	711.124	1225.476	1739.829	2254.182	2768.535
$t_{(\eta,\delta)}$	711.124	1225.476	1739.829	2254.182	2768.535
$t_{1;1}$	2274.526	2671.384	3049.869	3411.228	3756.598
$t_{1;2}$	2272.418	2669.374	3047.949	3409.393	3754.841
$t_{1;3}$	2276.535	2673.300	3051.699	3412.977	3758.271
$t_{1;4}$	2276.958	2673.704	3052.085	3413.346	3758.624
$t_{1;5}$	2277.965	2674.665	3053.002	3414.223	3759.463
$t_{1;6}$	2269.462	2666.554	3045.257	3406.819	3752.378
$t_{1;7}$	2275.048	2671.882	3050.344	3411.682	3757.032
$t_{1;8}$	2273.978	2670.861	3049.370	3410.751	3756.141
$t_{1;9}$	2278.319	2675.002	3053.324	3414.531	3759.758
$t_{1;10}$	2268.656	2665.785	3044.522	3406.117	3751.706
t^*	685.497	1151.304	1594.032	2015.354	2416.785

Table 10. MSE values of the existing and t^{**} estimators for the second case.

Estimators	z=2	z=3	z=4	z=5	z=6
t_{Nr}	997.70	1512.053	2026.41	2540.76	3055.112
t_{R2}	493.265	777.941	1062.618	1347.294	1631.970
t_{exp2}	673.719	1046.972	1420.224	1793.477	2166.729
t_{reg2}	456.511	716.251	975.991	1235.732	1495.472
t_{KB}	444.483	692.195	939.908	1187.620	1435.332
$t_{K,1}$	891.722	1344.908	1798.094	2251.281	2704.467
$t_{K,2}$	892.260	1345.674	1799.088	2252.501	2705.915
$t_{K,3}$	891.209	1344.179	1797.148	2250.118	2703.087
$t_{K,4}$	891.101	1344.025	1796.949	2249.872	2702.796
$t_{K,5}$	890.844	1343.659	1796.474	2249.290	2702.105
$t_{K,6}$	893.015	1346.748	1800.481	2254.214	2707.947
$t_{K,7}$	891.589	1344.719	1797.849	2250.979	2704.109
$t_{K,8}$	891.862	1345.107	1798.353	2251.598	2704.844
$t_{K,9}$	890.754	1343.531	1796.308	2249.085	2701.862
$t_{K,10}$	893.221	1347.041	1800.861	2254.681	2708.501
t_{SU}	452.109	703.361	953.118	1202.161	1450.809
t_{PS2}	452.109	703.361	953.118	1202.161	1450.809
$t_{2;1}$	454.961	702.265	948.686	1194.214	1438.839
$t_{2;2}$	454.832	702.178	948.638	1194.203	1438.862
$t_{2;3}$	455.086	702.351	948.735	1194.229	1438.823
$t_{2;4}$	455.112	702.369	948.746	1194.233	1438.82
$t_{2;5}$	455.176	702.414	948.772	1194.242	1438.814
$t_{2;6}$	454.655	702.061	948.577	1194.195	1438.903
$t_{2;7}$	454.993	702.287	948.698	1194.217	1438.835
$t_{2;8}$	454.927	702.242	948.673	1194.210	1438.845
$t_{2;9}$	455.199	702.429	948.782	1194.246	1438.812
$t_{2;10}$	454.608	702.030	948.562	1194.194	1438.916
t_{RNAQ}	452.109	703.361	953.118	1202.161	1450.809
t_{UK}	452.109	703.361	953.118	1202.161	1450.809
$t_{(\alpha,\beta)}$	452.109	703.361	953.118	1202.161	1450.809
$t_{(\alpha 2,\delta 2)}$	452.109	703.361	953.118	1202.161	1450.809
t_{US2}	452.109	703.361	953.118	1202.161	1450.809
t^{**}	441.613	678.281	907.639	1130.702	1347.997

According to the results in Table 8, the proposed t^* family of estimators has the minimum MSE value in comparison with other estimators in the first case.We can see similar situation for the second case in Table 9 as well. In conclusion, using the proposed t^* and t^{**} families of estimators is appropriate while estimating the population mean under the first and second cases, respectively, for both data sets.

§6 Conclusion

In this article, we consider a new family of exponential type estimators using the Hansen-Hurwitz method for the first and second non-response cases. Using this method, the proposed family of estimators is examined under the non-response schemes separately. The theoretical required comparisons are obtained and we show that the proposed estimator is more efficient than compared main unbiased, ratio, regression and exponential estimators in literature under the conditions for both cases. In literature, the real COVID-19 data set is used for the numerical study. In addition, we consider a common data set which is used by Khare and Sinha (2009). We calculate the MSE values of all mentioned and proposed estimators based on these data sets. The results show that the proposed estimators are the most appropriate estimators in order to estimate the unknown population mean. For this reason, we recommend the proposed families of estimators for both cases in the presence of non-response.

Declarations

Conflict of interest The authors declare no conflict of interest.

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