

The asymptotic relation between the first crossing point and the last exit time of Gaussian order statistics sequences

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Abstract. In this paper, we study the asymptotic relation between the first crossing point and the last exit time for Gaussian order statistics which are generated by stationary weakly and strongly dependent Gaussian sequences. It is shown that the first crossing point and the last exit time are asymptotically independent and dependent for weakly and strongly dependent cases, respectively. The asymptotic relations between the first crossing point and the last exit time for stationary weakly and strongly dependent Gaussian sequences are also obtained.

§1 Introduction

The first crossing point (also known as the first exit time) and the last exit time have been studied extensively in applied probability. In risk theory, they play a very important role, since the first exit time corresponds to the ruin time of a risk process and the last exit time can be regarded as the final recovery time after when there will be no more ruin, see e.g., [8]. In practice, when we consider the random noises problems, it usually adds the first zero crossing of a sine wave to the random noises. Instead of considering the zero crossing, we can transform it to a stationary random sequence and consider therefore the first crossing of the sequence, see e.g., [9].

Let $\{X_i, i \in \mathbb{Z}\}$ be a sequence of stationary random variables with $EX_1^+ < \infty$, where X^+ means $\max\{X, 0\}$. Define for some $\beta > 0$

$$\Upsilon(X) = \min\{i \in \mathbb{Z} : X_i > -\beta i\}$$

and

$$\tau(X) = \max\{i \geq 0 : X_i > \beta i\}$$

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the first crossing point and the last exit time of the sequence $\{X_i, i \in \mathbb{Z}\}$, respectively. Because of the existence of EX_1^+ , both $\Upsilon(X)$ and $\tau(X)$ are well-defined and also finite almost surely, see e.g., [9].

[9] first studied the limit distributions of the first crossing point for stationary Gaussian sequences and showed that with appropriately chosen normalization the limit distribution of the first crossing point and the maxima are the same. [10] discussed the almost sure limiting behaviour including the a.s. stability and a.s. relative stability and also derived an iterated logarithm law for the first crossing point for stationary Gaussian sequences. It is easy to see from the definition of the first crossing point and the last exit time that their limit properties of them are equal. The limit properties of the last exit time for independently and identically distributed random sequences and stationary random sequences can be found in [11],[12] and [13]. Recently, [16] considered a similar problem for continuous time weakly dependent Gaussian processes and derived the limiting distribution of the scaled last exit time over a slowly growing linear boundary, which was extended in [21] to strongly dependent cases. All of the above-mentioned results only consider the last exit time or the first crossing point for the linear boundary. [15] studied the last exit time over a moving nonlinear boundary for a Gaussian process. For some related results, see e.g., [2] and [18].

A natural question arises what is the joint limit distribution of the first and the last exit time? For certain of Gaussian processes, [14] showed that the conditional joint limit distribution of the first and the last exit time, conditioned on ruin occurring, is a difference of two standard normal distributions. [22] discussed the joint limit properties for the first crossing point and the last exit time for some dependent chi-sequences.

In this paper, we are interested in the asymptotic relation between the first crossing point and last exit time for Gaussian order statistics sequences. Let $\{X_i, i \in \mathbb{Z}\}$ be a sequence of stationary standard (mean 0, variance 1) Gaussian random variables with covariance function $r_i = E(X_1 X_{i+1})$ and $\{X_{ij}, i \in \mathbb{Z}\}, j = 1, 2, \dots, d$ with $d \geq 1$ be independent copies of $\{X_i, i \in \mathbb{Z}\}$. Define $\{O_{id}^{(r)}(X), i \in \mathbb{Z}\}$ a sequence of Gaussian r -th order statistics generated by X as follows

$$O_{id}^{(d)}(X) := \min_{j=1}^d X_{ij} \leq \dots \leq O_{id}^{(r)}(X) \leq \dots \leq O_{id}^{(1)}(X) := \max_{j=1}^d X_{ij}, \quad i \in \mathbb{Z},$$

where $r \in \{1, 2, \dots, d\}$. The Gaussian r -th order statistics random variables play a very important role in applied fields, for instance, in models concerned with the analysis of the surface roughness during all machinery processes, see e.g., [1]. The limit properties of extremes of Gaussian r -th order statistics sequences and processes are studied in [3,4,5,6,7,23,24,25].

It is worth mentioning the following result, since it is highly relative to the main result of this paper. Based on the results of [4], [24] derived the weak limit theorems for the maxima of Gaussian r -th order statistics variables.

Theorem 1.1. *Let $\{O_{id}^{(r)}(X), i \in \mathbb{Z}\}$ be defined as above. If the covariance function r_n satisfies*

$$r_n \ln n \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

then for any $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} P \left(a_n \left(\max_{i=1}^n O_{id}^{(r)}(X) - b_n \right) \leq x \right) = \exp(-e^{-x}), \quad (1)$$

where

$$a_n = (2r \ln n)^{1/2}, \quad b_n = \frac{1}{r}a_n + a_n^{-1} \ln\left(a_n^{-r} C_d^r (2\pi)^{-r/2}\right) \tag{2}$$

with $C_d^r = \frac{d!}{r!(d-r)!}$.

Theorem 1.1 obtained the limit distributions of maxima of Gaussian order statistics variables for weakly dependent cases. In the literature, assumption $r_n \ln n \rightarrow 0$ is referred to as the weak dependence or the Berman condition (see e.g., [17] and [20]), and, consequently, the stationary Gaussian sequences $\{X_i, i \in \mathbb{Z}\}$ is called weakly dependent sequence. We called also $\{O_{id}^{(r)}(X), i \in \mathbb{Z}\}$ weakly dependent Gaussian r -th order statistics sequence. In analogy, $\{O_{id}^{(r)}(X), i \in \mathbb{Z}\}$ with correlation function satisfying assumption $r_n \ln n \rightarrow \gamma > 0$ is called strongly dependent Gaussian r -th order statistics sequence.

In this paper, we will investigate the asymptotic relation between the first crossing point and the last exit time for Gaussian order statistics sequences. We showed that they are asymptotic independent if the Gaussian order statistics sequences are weakly dependent and are asymptotic dependent if the Gaussian order statistics sequences are strongly dependent. We also showed that the limit distribution functions of the first crossing point and the last exit time are very similar with that of the maxima. Section 2 presents our main results and their proofs are provided in Section 3.

§2 Main results

We define the first crossing point and the last exit time of Gaussian order statistics sequence $\{O_{id}^{(r)}(X), i \in \mathbb{Z}\}$ with function $i\beta$ as follows:

$$\Upsilon(O_X^{(r)}) = \min\{i \leq 0 : O_{id}^{(r)}(X) > -\beta i\}$$

and

$$\tau(O_X^{(r)}) = \max\{i \geq 0 : O_{id}^{(r)}(X) > \beta i\}$$

with $\beta > 0$. We consider the joint limit distribution of $\Upsilon(O_X^{(r)})$ and $\tau(O_X^{(r)})$ as $\beta \rightarrow 0$.

Now we state our main results.

Theorem 2.1. *Let $\{O_{id}^{(r)}(X), i \in \mathbb{Z}\}$ be defined as above. If the covariance function r_n satisfies*

$$r_n \ln n \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

then for any $x, y \in \mathbb{R}$

$$\lim_{\beta \rightarrow 0} P\left(\Upsilon(O_X^{(r)}) \geq -u_\beta(x), \tau(O_X^{(r)}) \leq u_\beta(y)\right) = \exp(-(e^{-x} + e^{-y})) \tag{3}$$

and

$$\lim_{\beta \rightarrow 0} P\left(\Upsilon(O_X^{(r)}) \geq -u_\beta(x)\right) = \lim_{\beta \rightarrow 0} P\left(\tau(O_X^{(r)}) \leq u_\beta(x)\right) = \exp(-e^{-x}), \tag{4}$$

where

$$u_\beta(x) = \left(a(\beta, r) + \frac{x}{ra(\beta, r)}\right) \beta^{-1} \tag{5}$$

with

$$a(\beta, r) = \sqrt{\frac{2}{r} \left(\ln \beta^{-1} - \frac{r+1}{2} \ln \left(\frac{2}{r} \ln \beta^{-1} \right) + \ln(C_d^r r^{-1} (2\pi)^{-r/2}) \right)}. \tag{6}$$

For the strongly dependent case, we can only deal with the cases $r = 1$ and $r = d$.

Theorem 2.2. *Let $\{O_{id}^{(r)}(X), i \in \mathbb{Z}\}$ be defined as above. Suppose that the covariance*

function r_n satisfies

$$r_n \ln n \rightarrow \gamma > 0 \text{ as } n \rightarrow \infty.$$

Define $\rho_\beta = \gamma/|\ln \beta|$ and $b(\beta, \rho_\beta, r) = \sqrt{1 - \rho_\beta}a(\beta/\sqrt{1 - \rho_\beta}, r)$ with $a(\cdot, \cdot)$ defined as in (6).

1.) For the case $r = 1$, we have for any $x, y \in \mathbb{R}$

$$\begin{aligned} & \lim_{\beta \rightarrow 0} P\left(\Upsilon(O_X^{(1)}) \geq -u_\beta(x), \tau(O_X^{(1)}) \leq u_\beta(y)\right) \\ &= \int_{\mathbb{R}^d} \exp\left(-\sum_{j=1}^d (e^{-\sqrt{2\gamma}(x-z_j)} + e^{-\sqrt{2\gamma}(y-z_j)})\right) d\Phi(z_1) \cdots d\Phi(z_d) \end{aligned} \quad (7)$$

and

$$\begin{aligned} \lim_{\beta \rightarrow 0} P\left(\Upsilon(O_X^{(1)}) \geq -u_\beta(x)\right) &= \lim_{\beta \rightarrow 0} P\left(\tau(O_X^{(1)}) \leq u_\beta(x)\right) \\ &= \int_{\mathbb{R}^d} \exp\left(-\sum_{j=1}^d e^{-\sqrt{2\gamma}(x-z_j)}\right) d\Phi(z_1) \cdots d\Phi(z_d), \end{aligned} \quad (8)$$

where

$$u_\beta(x) = (b(\beta, \rho_\beta, 1) + \sqrt{\rho_\beta}x) \beta^{-1} \quad (9)$$

and $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal variable.

2.) For the case $r = d$, we have for any $x, y \in \mathbb{R}$

$$\begin{aligned} & \lim_{\beta \rightarrow 0} P\left(\Upsilon(O_X^{(d)}) \geq -u_\beta(x), \tau(O_X^{(d)}) \leq u_\beta(y)\right) \\ &= \int_{\mathbb{R}^d} \exp\left(-\left(e^{-\sqrt{2d^{-1}\gamma}(x-\sum_{j=1}^d z_j)} + e^{-\sqrt{2d^{-1}\gamma}(y-\sum_{j=1}^d z_j)}\right)\right) d\Phi(z_1) \cdots d\Phi(z_d) \end{aligned} \quad (10)$$

and

$$\begin{aligned} \lim_{\beta \rightarrow 0} P\left(\Upsilon(O_X^{(d)}) \geq -u_\beta(x)\right) &= \lim_{\beta \rightarrow 0} P\left(\tau(O_X^{(d)}) \leq u_\beta(x)\right) \\ &= \int_{\mathbb{R}^d} \exp\left(-e^{-\sqrt{2d^{-1}\gamma}(x-\sum_{j=1}^d z_j)}\right) d\Phi(z_1) \cdots d\Phi(z_d), \end{aligned} \quad (11)$$

where

$$u_\beta(x) = (b(\beta, \rho_\beta, d) + \sqrt{\rho_\beta}d^{-1}x) \beta^{-1}. \quad (12)$$

Next, let us consider the special case $d = 1$, namely the stationary Gaussian case.

Corollary 2.1. Let $\{X_i, i \in \mathbb{Z}\}$ be a sequence of stationary standard Gaussian random variables with covariance function $r_i = E(X_1 X_{i+1})$.

1.) Under the conditions of Theorem 2.1, we have for any $x, y \in \mathbb{R}$

$$\lim_{\beta \rightarrow 0} P(\Upsilon(X) \geq -u_\beta(x), \tau(X) \leq u_\beta(y)) = \exp\left(-\left(e^{-x} + e^{-y}\right)\right),$$

where $u_\beta(x)$ is defined in (5) with $r = d = 1$.

2.) Under the conditions of Theorem 2.2, we have for any $x, y \in \mathbb{R}$

$$\begin{aligned} & \lim_{\beta \rightarrow 0} P(\Upsilon(X) \geq -u_\beta(x), \tau(X) \leq u_\beta(y)) \\ &= \int_{\mathbb{R}} \exp\left(-\left(e^{-\sqrt{2\gamma}(x-z)} + e^{-\sqrt{2\gamma}(y-z)}\right)\right) d\Phi(z), \end{aligned}$$

where $u_\beta(x)$ is defined in (12) with $d = 1$.

Corollary 2.1 extends Theorems 2.1 and 2.2 of [9] which derived only the limit distribution for $\Upsilon(X)$. But there is still a case which can not be derived from Theorems 2.1 and 2.2 and

will be discussed in next theorem.

Theorem 2.3. Let $\{X_i, i \in \mathbb{Z}\}$ be a sequence of stationary standard Gaussian random variables with covariance function $r_i = E(X_1 X_{i+1})$. Suppose that r_n is convex for all $n \geq 0$, $r_n = o(1)$, and $(r_n \ln n)^{-1}$ is monotone for large n and $o(1)$. Then for any $x, y \in \mathbb{R}$

$$\lim_{\beta \rightarrow 0} P(\Upsilon(X) \geq -u_\beta(x), \tau(X) \leq u_\beta(y)) = \Phi(\min\{x, y\}), \tag{13}$$

where $u_\beta(x)$ is defined in (12) with $d = 1$ and $\rho_\beta = r_{[\beta^{-1}]}$. Here $[x]$ means the integral parts of x .

Remark 2.1. i). It can be seen that with appropriately chosen normalization the limit distributions of $\max_{i=1}^n O_{id}^{(r)}(X)$ ($n \rightarrow \infty$), $\tau(O_X^{(r)})$ and $\Upsilon(O_X^{(r)})$ ($\beta \rightarrow 0$) are the same for the weakly dependent case.

ii). As in [9], the above results can be generalized for the first crossing point $\Upsilon(O_X^{(r)}) = \min\{i \leq 0 : O_{id}^{(r)}(X) > -\beta f(i)\}$ and the last exit time $\tau(O_X^{(r)}) = \max\{i \geq 0 : O_{id}^{(r)}(X) > \beta f(i)\}$ with $f(t) = t^\vartheta L(t)$ for $t \rightarrow \infty$, where $\vartheta > 0$ and $L(t)$ varies slowly.

§3 Proofs

In this section, we give the proofs. As usual, $a_n \ll b_n$ means $\limsup_{n \rightarrow \infty} |a_n/b_n| < +\infty$. K will denote a constant whose value will change from line to line.

The following lemma plays a crucial role in our proofs.

Lemma 3.1. Denote by $\mathcal{Y} = (Y_{il})_{n \times d}$ and $\mathcal{Z} = (Z_{il})_{n \times d}$ two random arrays with $N(0, 1)$ components, and let $(\sigma_{il,jk}^{(1)})_{dn \times dn}$ and $(\sigma_{il,jk}^{(0)})_{dn \times dn}$ be the covariance matrices of \mathcal{Y} and \mathcal{Z} , respectively, with $\sigma_{il,jk}^{(1)} := EY_{il}Y_{jk}$ and $\sigma_{il,jk}^{(0)} := EZ_{il}Z_{jk}$, $1 \leq i, j \leq n, 1 \leq l, k \leq d$. Suppose that the columns of both \mathcal{Y} and \mathcal{Z} are mutually independent, i.e.,

$$\sigma_{il,jk}^{(\kappa)} = \sigma_{ik,jk}^{(\kappa)} \mathbf{1}(k = l), 1 \leq i, j \leq n, 1 \leq k, l \leq d, \kappa = 0, 1.$$

Further, suppose that $\sigma_{i,j} = \max_{1 \leq k, l \leq d} \{|\sigma_{ik,jk}^{(0)}|, |\sigma_{ik,jk}^{(1)}|\} < 1$, where $\mathbf{1}(\cdot)$ will denote an indicator function. Define $(O_{1d}^{(r)}(Y), \dots, O_{nd}^{(r)}(Y))$ to be the r -th order statistics vector generated by \mathcal{X} as follows

$$\min_{1 \leq k \leq d} Y_{ik} = O_{id}^{(d)}(Y) \leq \dots \leq O_{id}^{(r)}(Y) \leq \dots \leq O_{id}^{(1)}(Y) = \max_{1 \leq k \leq d} Y_{ik}, \quad 1 \leq i \leq n.$$

Similarly, we write $(O_{1d}^{(r)}(Z), \dots, O_{nd}^{(r)}(Z))$ which is generated by \mathcal{Z} . Then for any real numbers u_1, \dots, u_n , and any $1 \leq r \leq d$

$$\begin{aligned} & \left| P\left(O_{id}^{(r)}(Y) \leq u_i, i = 1, \dots, n\right) - P\left(O_{id}^{(r)}(Z) \leq u_i, i = 1, \dots, n\right) \right| \\ & \leq K \sum_{k=1}^d \sum_{1 \leq i < j \leq n} \frac{|\sigma_{ik,jk}^{(1)} - \sigma_{ik,jk}^{(0)}|}{(u_i u_j)^{r-1}} \exp\left(-\frac{r(u_i^2 + u_j^2)}{2(1 + \sigma_{i,j})}\right). \end{aligned}$$

Proof. Let (Z_i, Z_j) be a bivariate standard normal random vector with correlation $|\delta_{ij}|$. By a similar argument as the proof on p. 225 of [17], we can show for $u_j \geq u_i$

$$P\{Z_i > u_i, Z_j > u_j\} \leq \frac{2(1 - \delta_{ij}^2)}{u_i(u_j - |\delta_{ij}|u_i)} \phi(u_i, u_j; |\delta_{ij}|).$$

where $\phi(u, v; \delta)$ is the probability density function of a two dimensional normal random variables. If $u_i = O(u_j)$, we have

$$\frac{2(1 - \delta_{ij}^2)}{u_i(u_j - |\delta_{ij}|u_i)} \leq \frac{2(1 - \delta_{ij}^2)}{u_i u_j |1 - O(1)|\delta_{ij}|} \leq \frac{K}{u_i u_j}.$$

If $u_i = o(u_j)$, obviously,

$$\frac{2(1 - \delta_{ij}^2)}{u_i(u_j - |\delta_{ij}|u_i)} = O(1) \frac{2(1 - \delta_{ij}^2)}{u_i u_j} \leq \frac{K}{u_i u_j}.$$

By the same arguments, we can get the same bound for the case $u_j \leq u_i$. Thus, we have

$$P\{Z_i > u_i, Z_j > u_j\} \leq \frac{K}{u_i u_j} \phi(u_i, u_j; |\delta_{ij}|).$$

Using the above inequality to replace (4.28) in the proof of Theorem 2.4 of [4], we can prove the lemma. □

3.1 Proof of Theorem 2.1

We need the following lemma to prove Theorem 2.1.

Lemma 3.2. *Let $\{\xi_i, i \in \mathbb{Z}\}$ be a sequence of independent standard Gaussian random variable and $\{O_{id}^{(r)}(\xi), i \in \mathbb{Z}\}$ be a sequence of Gaussian r -th order statistics generated by ξ . We have for any $x, y \in \mathbb{R}$*

$$\lim_{\beta \rightarrow 0} P\left(\Upsilon(O_{\xi}^{(r)}) \geq -u_{\beta}(x), \tau(O_{\xi}^{(r)}) \leq u_{\beta}(y)\right) = \exp(-(e^{-x} + e^{-y})), \tag{14}$$

where $u_{\beta}(x)$ is defined in (5).

Proof. First note that $\beta u_{\beta}(x) \rightarrow \infty$ as $\beta \rightarrow 0$. From Lemma 2 of [3], we have

$$P(O_{1d}^{(r)}(\xi) \geq i\beta) = C_d^r (1 - \Phi(i\beta))^r (1 + o(1)) \tag{15}$$

as $i\beta \rightarrow \infty$. Since $O_{id}^{(r)}(\xi)$ are independent, using the following well-known fact

$$1 - \Phi(x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} (1 + o(1)) \tag{16}$$

as $x \rightarrow \infty$, we have as $\beta \rightarrow 0$

$$\begin{aligned} -\ln P\left(\Upsilon(O_{\xi}^{(r)}) \geq -u_{\beta}(x)\right) &= -\ln \prod_{i=u_{\beta}(x)}^{\infty} P(O_{1d}^{(r)}(\xi) \leq i\beta) \\ &= -\sum_{i=u_{\beta}(x)}^{\infty} \ln(1 - C_d^r (1 - \Phi(i\beta))^r (1 + o(1))) \\ &= C_d^r (2\pi)^{-r/2} r^{-1} \beta^{-1} (\beta u_{\beta}(x))^{-(r+1)} e^{-\frac{1}{2}r(\beta u_{\beta}(x))^2} (1 + o(1)). \end{aligned}$$

Since $(\beta u_{\beta}(x))^2 = (a(\beta, r))^2 + 2x/r + o(1)$, in view of (6), we have

$$-\ln P\left(\Upsilon(O_{\xi}^{(r)}) \geq -u_{\beta}(x)\right) = e^{-x} (1 + o(1))$$

as $\beta \rightarrow 0$. Similarly, we have

$$-\ln P\left(\tau(O_{\xi}^{(r)}) \leq u_{\beta}(y)\right) = e^{-y} (1 + o(1))$$

as $\beta \rightarrow 0$. By the independence of $\{O_{id}^{(r)}(\xi), i \in \mathbb{Z}\}$, we can see that

$$P\left(\Upsilon(O_{\xi}^{(r)}) \geq -u_{\beta}(x), \tau(O_{\xi}^{(r)}) \leq u_{\beta}(y)\right)$$

$$\begin{aligned}
 &= P\left(O_{jd}^{(r)}(\xi) \leq -j\beta, j \leq -u_\beta(x), O_{id}^{(r)}(\xi) \leq i\beta, i \geq u_\beta(y)\right) \\
 &= P\left(O_{jd}^{(r)}(\xi) \leq -j\beta, j \leq -u_\beta(x)\right) P\left(O_{id}^{(r)}(\xi) \leq i\beta, i \geq u_\beta(y)\right)
 \end{aligned}$$

and therefore the lemma follows. □

Proof of Theorem 2.1. By Lemma 3.2, to prove Theorem 2.1, we only need to show that $\left|P\left(\Upsilon(O_X^{(r)}) \geq -u_\beta(x), \tau(O_X^{(r)}) \leq u_\beta(y)\right) - P\left(\Upsilon(O_\xi^{(r)}) \geq -u_\beta(x), \tau(O_\xi^{(r)}) \leq u_\beta(y)\right)\right| \rightarrow 0$ as $\beta \rightarrow 0$. We use Lemma 3.1 to bound this difference by

$$\begin{aligned}
 &K \sum_{i=1}^{\infty} |r_i| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1+|r_i|)}\right) \\
 &+ K \sum_{i=1}^{\infty} |r_i| \sum_{j=u_\beta(y)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1+|r_i|)}\right) \\
 &+ K \sum_{i=u_\beta(x)}^{\infty} \sum_{j=-\infty}^{-u_\beta(y)} |r_{|i-j|}| (i\beta)^{-(r-1)} (-j\beta)^{-(r-1)} \exp\left(-\frac{r((i\beta)^2 + (j\beta)^2)}{2(1+|r_{|i-j|}|)}\right) \\
 &=: K(I_\beta + J_\beta + K_\beta).
 \end{aligned}$$

Define $\delta = \sup_{i \geq 1} |r_i|$ and $\delta_n = \sup_{i \geq n} |r_i|$. Since $r_n \rightarrow 0$, as $n \rightarrow \infty$, we have $\delta_n \leq \delta < 1$, see e.g., p. 86 in [17]. Choose ε such that $0 < \varepsilon < (1 - \delta)/(1 + \delta)$. Split the above sum I_β into two parts as

$$\begin{aligned}
 I_\beta &= \sum_{i=1}^{[\beta^{-\varepsilon}]} |r_i| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1+|r_i|)}\right) \\
 &+ \sum_{i=[\beta^{-\varepsilon}]+1}^{\infty} |r_i| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1+|r_i|)}\right) \\
 &=: I_{\beta,1} + I_{\beta,2}.
 \end{aligned}$$

For the first term $I_{\beta,1}$, using (5) and (6), we have

$$\begin{aligned}
 I_{\beta,1} &\leq \beta^{-\varepsilon} \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r(j\beta)^2}{1+\delta}\right) \\
 &\ll \beta^{-(\varepsilon+1)} (u_\beta(x)\beta)^{-(2r-1)} \exp\left(-\frac{r(u_\beta(x)\beta)^2}{1+\delta}\right) \\
 &\ll \beta^{-\varepsilon + \frac{1-\delta}{1+\delta}} |\ln \beta|^{\frac{1}{2} + \frac{1-r\delta}{1+\delta}},
 \end{aligned}$$

which tends to 0 as $\beta \rightarrow 0$ by the choice of ε . For the term $I_{\beta,2}$, recall that $\delta_{[\beta^{-\varepsilon}]} = \sup_{i \geq [\beta^{-\varepsilon}]} |r_i|$ and note that $r_n \ln n \rightarrow 0$ as $n \rightarrow \infty$ implies that $\delta_{[\beta^{-\varepsilon}]} |\ln \beta| \rightarrow 0$ as $\beta \rightarrow 0$. We have in view of (5) and (6) again

$$\begin{aligned}
 I_{\beta,2} &\leq \delta_{[\beta^{-\varepsilon}]} \sum_{i=[\beta^{-\varepsilon}]}^{\infty} \exp\left(-\frac{r((i+u_\beta(x))\beta)^2}{2(1+\delta_{[\beta^{-\varepsilon}]})}\right) \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r(j\beta)^2}{2(1+\delta_{[\beta^{-\varepsilon}]})}\right) \\
 &\ll \delta_{[\beta^{-\varepsilon}]} \beta^{-2} (\beta u_\beta(x))^{-2r} \exp\left(-\frac{r(u_\beta(x)\beta)^2}{1+\delta_{[\beta^{-\varepsilon}]}}\right) \\
 &\ll \delta_{[\beta^{-\varepsilon}]} \beta^{-2\delta_{[\beta^{-\varepsilon}]}} |\ln \beta|^{1-(1+r)\delta_{[\beta^{-\varepsilon}]}}.
 \end{aligned}$$

which tends to 0 as $\beta \rightarrow 0$ since $\delta_{[\beta^{-\epsilon}]}\ln|\beta| \rightarrow 0$ as $\beta \rightarrow 0$. Thus, $I_\beta \rightarrow \infty$ as $\beta \rightarrow 0$.

Similarly, we can show that $J_\beta \rightarrow \infty$ as $\beta \rightarrow 0$.

For the third term K_β , note that $u_\beta(x) \sim u_\beta(y) \sim u_\beta := \beta^{-1}(\frac{2}{r} \ln \beta^{-1})^{1/2}$ as $\beta \rightarrow 0$. Thus, by the same arguments as for $I_{\beta,2}$, we have

$$\begin{aligned} K_\beta &\leq \delta_{[u_\beta]} \sum_{i=[u_\beta(x)]}^{\infty} (i\beta)^{-(r-1)} \exp\left(-\frac{r(i\beta)^2}{2(1+\delta_{[u_\beta]})}\right) \sum_{j=[u_\beta(y)]}^{\infty} (j\beta)^{-(r-1)} \exp\left(-\frac{r(j\beta)^2}{2(1+\delta_{[u_\beta]})}\right) \\ &\ll \delta_{[u_\beta]} \beta^{-2} (\beta u_\beta)^{-2r} \exp\left(-\frac{r(u_\beta(x)\beta)^2}{2(1+\delta_{[u_\beta]})}\right) \exp\left(-\frac{r(u_\beta(y)\beta)^2}{2(1+\delta_{[u_\beta]})}\right) \\ &\ll \delta_{[u_\beta]} \beta^{-2\delta_{[u_\beta]}} |\ln \beta|^{1-(1+r)\delta_{[u_\beta]}}, \end{aligned}$$

which tends to 0 as $\beta \rightarrow 0$ since $\delta_{[u_\beta]}\ln|\beta| \rightarrow 0$ as $\beta \rightarrow 0$. Thus, we have $K_\beta \rightarrow 0$ as $\beta \rightarrow 0$. The proof of Theorem 2.1 is complete. \square

In the following two subsection, let $\mathbb{A}_{\beta,x}$ denote the set $(-u_\beta(x) - \beta^{-1}, -u_\beta(x)] \cap \mathbb{Z}$ and $\mathbb{B}_{\beta,y}$ denote the set $[u_\beta(y), u_\beta(y) + \beta^{-1}) \cap \mathbb{Z}$.

3.2 Proof of Theorem 2.2

To prove Theorem 2.2, we need the following two lemmas.

Lemma 3.3. Let ρ_β be a function of β with $0 < \rho_\beta < c < 1$ and $Y_{ij} = \sqrt{1 - \rho_\beta} \xi_{ij} + \sqrt{\rho_\beta} U_j$, $i \in \mathbb{N}, j = 1, 2, \dots, d$, where ξ_{ij} are independent Gaussian random vectors and $U_j, j = 1, 2, \dots, d$ are a standard normal random variables independent with ξ_{ij} . Denote by $O_{id}^{(r)}(Y)$ the Gaussian order statistics sequences generated by Y_{ij} . Suppose that $\rho_\beta |\ln \beta| \rightarrow \gamma > 0$ as $\beta \rightarrow 0$.

1). If $r = 1$, we have for any $x, y \in \mathbb{R}$

$$\begin{aligned} &\lim_{\beta \rightarrow 0} P\left(O_{id}^{(1)}(Y) \leq -i\beta, i \in \mathbb{A}_{\beta,x}, O_{kd}^{(1)}(Y) \leq k\beta, k \in \mathbb{B}_{\beta,y}\right) \\ &= \int_{\mathbb{R}^d} \exp\left(-\sum_{j=1}^d (e^{-\sqrt{2\gamma}(x-z_j)} + e^{-\sqrt{2\gamma}(y-z_j)})\right) d\Phi(z_1) \cdots d\Phi(z_d), \end{aligned}$$

where $\mathbb{A}_{\beta,x} = (-u_\beta(x) - \beta^{-1}, -u_\beta(x)] \cap \mathbb{Z}$ and $\mathbb{B}_{\beta,y} = [u_\beta(y), u_\beta(y) + \beta^{-1}) \cap \mathbb{Z}$ with $u_\beta(x)$ defined in (9) with ρ_β satisfying the above assumptions.

2). If $r = d$, we have for any $x, y \in \mathbb{R}$

$$\begin{aligned} &\lim_{\beta \rightarrow 0} P\left(O_{id}^{(d)}(Y) \leq -i\beta, i \in \mathbb{A}_{\beta,x}, O_{kd}^{(d)}(Y) \leq k\beta, k \in \mathbb{B}_{\beta,y}\right) \\ &= \int_{\mathbb{R}^d} \exp\left(-\left(e^{\sqrt{2d^{-1}\gamma}(x-\sum_{j=1}^d z_j)} + e^{\sqrt{2d^{-1}\gamma}(y-\sum_{j=1}^d z_j)}\right)\right) d\Phi(z_1) \cdots d\Phi(z_d), \end{aligned}$$

where $\mathbb{A}_{\beta,x} = (-u_\beta(x) - \beta^{-1}, -u_\beta(x)] \cap \mathbb{Z}$ and $\mathbb{B}_{\beta,y} = [u_\beta(y), u_\beta(y) + \beta^{-1}) \cap \mathbb{Z}$ with $u_\beta(x)$ defined in (12) with ρ_β satisfying the above assumptions.

Proof. We show first the case $r = 1$. By the definition of Y_{ij} and the independence of ξ_{ij} , we have

$$\begin{aligned} &P\left(O_{id}^{(1)}(Y) \leq -i\beta, i \in \mathbb{A}_{\beta,x}, O_{kd}^{(1)}(Y) \leq k\beta, k \in \mathbb{B}_{\beta,y}\right) \\ &= P\left(\max_{j=1}^d \{\sqrt{1 - \rho_\beta} \xi_{ij} + \sqrt{\rho_\beta} U_j\} \leq -i\beta, -u_\beta(x) - \beta^{-1} < i \leq -u_\beta(x), \right. \end{aligned}$$

$$\begin{aligned}
 & \max_{j=1}^d \{ \sqrt{1 - \rho_\beta} \xi_{kj} + \sqrt{\rho_\beta} U_j \} \leq k\beta, u_\beta(y) \leq k < u_\beta(y) + \beta^{-1} \\
 = & \int_{\mathbb{R}^d} P \left(\max_{j=1}^d \{ \sqrt{1 - \rho_\beta} \xi_{ij} + \sqrt{\rho_\beta} z_j \} \leq -i\beta, -u_\beta(x) - \beta^{-1} < i \leq -u_\beta(x), \right. \\
 & \left. \max_{j=1}^d \{ \sqrt{1 - \rho_\beta} \xi_{kj} + \sqrt{\rho_\beta} z_j \} \leq k\beta, u_\beta(y) \leq k < u_\beta(y) + \beta^{-1} \right) d\Phi(z_1) \cdots d\Phi(z_d) \\
 = & \int_{\mathbb{R}^d} \prod_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} \prod_{j=1}^d P \left(\sqrt{1 - \rho_\beta} \xi_{ij} + \sqrt{\rho_\beta} z_j \leq -i\beta \right) \\
 & \times \prod_{k=u_\beta(y)}^{u_\beta(y)+[\beta^{-1}]+1} \prod_{j=1}^d P \left(\sqrt{1 - \rho_\beta} \xi_{kj} + \sqrt{\rho_\beta} z_j \leq k\beta \right) d\Phi(z_1) \cdots d\Phi(z_d) \\
 =: & \int_{\mathbb{R}^d} H(\beta, x, z_1, \dots, z_d) H(\beta, y, z_1, \dots, z_d) d\Phi(z_1) \cdots d\Phi(z_d). \tag{17}
 \end{aligned}$$

By the same arguments as for the proof of Lemma 3.2, we have

$$\begin{aligned}
 & -\ln H(\beta, x, z_1, \dots, z_d) \\
 = & -\ln \prod_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} \prod_{j=1}^d \left[1 - P \left(\xi_{ij} \geq \frac{-(i\beta + \sqrt{\rho_\beta} z_j)}{\sqrt{1 - \rho_\beta}} \right) \right] \\
 = & -\sum_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} \sum_{j=1}^d \ln \left[1 - P \left(\xi_{ij} \geq \frac{-(i\beta + \sqrt{\rho_\beta} z_j)}{\sqrt{1 - \rho_\beta}} \right) \right] \\
 = & \sum_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} \sum_{j=1}^d \left(1 - \Phi \left(\frac{-(i\beta + \sqrt{\rho_\beta} z_j)}{\sqrt{1 - \rho_\beta}} \right) \right) (1 + o(1)) \\
 = & \sum_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} \sum_{j=1}^d (2\pi)^{-1/2} (1 - \rho_\beta)^{1/2} |i\beta|^{-1} \exp \left(-\frac{(i\beta)^2 + 2i\beta\sqrt{\rho_\beta} z_j}{2(1 - \rho_\beta)} \right) (1 + o(1)) \\
 = & (2\pi)^{-1/2} (1 - \rho_\beta)^{1/2} \beta^{-1} (\beta u_\beta(x))^{-2} \exp \left(-\frac{(\beta u_\beta(x))^2}{2(1 - \rho_\beta)} \right) \\
 & \times \sum_{j=1}^d \exp \left(\frac{\beta u_\beta(x) \sqrt{\rho_\beta} z_j}{1 - \rho_\beta} \right) (1 + o(1)).
 \end{aligned}$$

Now, using (9) and the fact that $\rho_\beta |\ln \beta| \rightarrow \gamma > 0$ as $\beta \rightarrow 0$, we have

$$-\ln H(\beta, x, z_1, \dots, z_d) \rightarrow \sum_{j=1}^d \exp \left(-\sqrt{2\gamma}(x - z_j) \right), \tag{18}$$

as $\beta \rightarrow 0$. Similarly, we have

$$-\ln H(\beta, y, z_1, \dots, z_d) \rightarrow \sum_{j=1}^d \exp \left(-\sqrt{2\gamma}(y - z_j) \right), \tag{19}$$

as $\beta \rightarrow 0$. Now, (17-19) combined with the dominated convergence theorem completes the proof for the case $r = 1$.

Next, we prove the case $r = d$. By the definition of Y_{ij} and the independence of ξ_{ij} again, we

have

$$\begin{aligned}
 & P\left(O_{id}^{(d)}(Y) \leq -i\beta, i \in \mathbb{A}_{\beta,x}, O_{kd}^{(d)}(Y) \leq k\beta, k \in \mathbb{B}_{\beta,y}\right) \\
 &= P\left(\min_{j=1}^d \{\sqrt{1-\rho_\beta}\xi_{ij} + \sqrt{\rho_\beta}U_j\} \leq -i\beta, -u_\beta(x) - \beta^{-1} < i \leq -u_\beta(x)\right. \\
 &\quad \left. \min_{j=1}^d \{\sqrt{1-\rho_\beta}\xi_{kj} + \sqrt{\rho_\beta}U_j\} \leq k\beta, u_\beta(y) \leq k < u_\beta(y) + \beta^{-1}\right) \\
 &= \int_{\mathbb{R}^d} P\left(\min_{j=1}^d \{\sqrt{1-\rho_\beta}\xi_{ij} + \sqrt{\rho_\beta}z_j\} \leq -i\beta, -u_\beta(x) - \beta^{-1} < i \leq -u_\beta(x),\right. \\
 &\quad \left. \min_{j=1}^d \{\sqrt{1-\rho_\beta}\xi_{kj} + \sqrt{\rho_\beta}z_j\} \leq k\beta, u_\beta(y) \leq k < u_\beta(y) + \beta^{-1}\right) d\Phi(z_1) \cdots d\Phi(z_d) \\
 &= \int_{\mathbb{R}^d} \prod_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} P\left(\min_{j=1}^d \{\sqrt{1-\rho_\beta}\xi_{ij} + \sqrt{\rho_\beta}z_j\} \leq -i\beta\right) \\
 &\quad \times \prod_{k=u_\beta(y)}^{u_\beta(y)+[\beta^{-1}]+1} P\left(\min_{j=1}^d \{\sqrt{1-\rho_\beta}\xi_{kj} + \sqrt{\rho_\beta}z_j\} \leq k\beta\right) d\Phi(z_1) \cdots d\Phi(z_d) \\
 &=: \int_{\mathbb{R}^d} G(\beta, x, z_1, \dots, z_d) G(\beta, y, z_1, \dots, z_d) d\Phi(z_1) \cdots d\Phi(z_d). \tag{20}
 \end{aligned}$$

By the same arguments as for the proof of Lemma 3.2, we have

$$\begin{aligned}
 & -\ln G(\beta, x, z_1, \dots, z_d) \\
 &= -\ln \prod_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} \left[1 - \prod_{j=1}^d P\left(\xi_{ij} \geq \frac{-(i\beta + \sqrt{\rho_\beta}z_j)}{\sqrt{1-\rho_\beta}}\right) \right] \\
 &= -\sum_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} \ln \left[1 - \prod_{j=1}^d P\left(\xi_{ij} \geq \frac{-(i\beta + \sqrt{\rho_\beta}z_j)}{\sqrt{1-\rho_\beta}}\right) \right] \\
 &= \sum_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} \prod_{j=1}^d \left(1 - \Phi\left(\frac{-(i\beta + \sqrt{\rho_\beta}z_j)}{\sqrt{1-\rho_\beta}}\right) \right) (1 + o(1)) \\
 &= \sum_{i=-u_\beta(x)-[\beta^{-1}]+1}^{-u_\beta(x)} (2\pi)^{-d/2} (1-\rho_\beta)^{d/2} |i\beta|^{-d} \exp\left(-\frac{d(i\beta)^2 + 2i\beta\sqrt{\rho_\beta}\sum_{j=1}^d z_j}{2(1-\rho_\beta)}\right) (1 + o(1)) \\
 &= (2\pi)^{-d/2} d^{-1} (1-\rho_\beta)^{d/2} \beta^{-1} (\beta u_\beta(x))^{-(d+1)} \exp\left(-\frac{d(\beta u_\beta(x))^2}{2(1-\rho_\beta)}\right) \\
 &\quad \times \exp\left(\frac{\beta u_\beta(x)\sqrt{\rho_\beta}\sum_{j=1}^d z_j}{1-\rho_\beta}\right) (1 + o(1)).
 \end{aligned}$$

Now, using (12) and the fact $\rho_\beta |\ln \beta| \rightarrow \gamma > 0$ as $\beta \rightarrow 0$, we have

$$-\ln G(\beta, x, z_1, \dots, z_d) \rightarrow \exp\left(-\sqrt{2d^{-1}\gamma}(x - \sum_{j=1}^d z_j)\right), \tag{21}$$

as $\beta \rightarrow 0$. Similarly, we have

$$-\ln G(\beta, y, z_1, \dots, z_d) \rightarrow \exp\left(-\sqrt{2d^{-1}}\gamma\left(y - \sum_{j=1}^d z_j\right)\right), \tag{22}$$

as $\beta \rightarrow 0$. Now, combining (20-22) with the dominated convergence theorem, we complete the proof for the case $r = d$. \square

Lemma 3.4. *Under the conditions of Theorem 2.2, we have for $r = 1, d$*

$$\left| P\left(\Upsilon(O_X^{(r)}) \geq -u_\beta(x), \tau(O_X^{(r)}) \leq u_\beta(y)\right) - P\left(O_{id}^{(r)}(X) \leq -i\beta, i \in \mathbb{A}_{\beta,x}, O_{kd}^{(r)}(X) \leq k\beta, k \in \mathbb{B}_{\beta,y}\right) \right| \rightarrow 0$$

as $\beta \rightarrow 0$.

Proof. Obviously, the absolute value in Lemma 3.4 is bound above by

$$\begin{aligned} & P\left(\bigcup_{i=-\infty}^{-u_\beta(x)-[\beta^{-1}]} (O_{id}^{(r)}(X) > -i\beta) \bigcup_{k=u_\beta(y)+[\beta^{-1}]}^{\infty} (O_{kd}^{(r)}(X) > k\beta)\right) \\ & \leq \sum_{i=-\infty}^{-u_\beta(x)-[\beta^{-1}]} P(O_{id}^{(r)}(X) > -i\beta) + \sum_{k=u_\beta(y)+[\beta^{-1}]}^{\infty} P(O_{kd}^{(r)}(X) > k\beta) \\ & = \sum_{i=u_\beta(x)+[\beta^{-1}]}^{\infty} P(O_{id}^{(r)}(X) > i\beta) + \sum_{k=u_\beta(y)+[\beta^{-1}]}^{\infty} P(O_{kd}^{(r)}(X) > k\beta) \\ & = \sum_{i=u_\beta(x)+[\beta^{-1}]}^{\infty} C_d^r (1 - \Phi(i\beta))^r (1 + o(1)) + \sum_{k=u_\beta(y)+[\beta^{-1}]}^{\infty} C_d^r (1 - \Phi(k\beta))^r (1 + o(1)), \end{aligned}$$

where in the last step, we use the fact (15). Applying (16), the last sum is bounded above by

$$\begin{aligned} & \sum_{i=u_\beta(x)+[\beta^{-1}]}^{\infty} \frac{C_d^r}{(2\pi)^{r/2}(i\beta)^r} \exp\left(-\frac{r(i\beta)^2}{2}\right) + \sum_{k=u_\beta(y)+[\beta^{-1}]}^{\infty} \frac{C_d^r}{(2\pi)^{r/2}(k\beta)^r} \exp\left(-\frac{r(k\beta)^2}{2}\right) \\ & \ll \beta^{-1}(\beta u_\beta(x))^{-(r+1)} \exp\left(-\frac{r(\beta u_\beta(x) + 1)^2}{2}\right) \\ & \quad + \beta^{-1}(\beta u_\beta(y))^{-(r+1)} \exp\left(-\frac{r(\beta u_\beta(y) + 1)^2}{2}\right) \\ & \ll \beta^{-\rho\beta} |\ln \beta|^{-\frac{(1+r)\rho\beta}{2}} e^{-K|\ln \beta|^{1/2}}, \end{aligned}$$

which tends to 0, since $\rho\beta \ln \beta^{-1} = \gamma$ under the conditions of Theorem 2.2. \square

Proof of Theorem 2.2. Define $Y_{ij} = \sqrt{1 - \rho\beta}\xi_{ij} + \sqrt{\rho\beta}U_j$ and $O_{id}^{(r)}(Y)$ as in Lemma 3.3. Then by Lemmas 3.3 and 3.4, to prove Theorem 2.2, it suffices to prove for $r = 1$ and $r = d$ that

$$\left| P\left(O_{id}^{(r)}(X) \leq -i\beta, i \in \mathbb{A}_{\beta,x}, O_{kd}^{(r)}(X) \leq k\beta, k \in \mathbb{B}_{\beta,y}\right) - P\left(O_{id}^{(r)}(Y) \leq -i\beta, i \in \mathbb{A}_{\beta,x}, O_{kd}^{(r)}(Y) \leq k\beta, k \in \mathbb{B}_{\beta,y}\right) \right| \rightarrow 0$$

as $\beta \rightarrow 0$. Using Lemma 3.1 again, we can bound the difference as

$$\begin{aligned} & K \sum_{i=1}^{[\beta^{-1}]+1} |r_i - \rho_\beta| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1 + \varpi_{i,\beta})}\right) \\ & + K \sum_{i=1}^{[\beta^{-1}]+1} |r_i - \rho_\beta| \sum_{j=u_\beta(y)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1 + \varpi_{i,\beta})}\right) \\ & + K \sum_{i=-u_\beta(x)-[\beta^{-1}]}^{-u_\beta(x)} \sum_{j=u_\beta(y)}^{u_\beta(y)+[\beta^{-1}]} |r_{|i-j|} - \rho_\beta| (-i\beta)^{-(r-1)} (j\beta)^{-(r-1)} \exp\left(-\frac{r((i\beta)^2 + (j\beta)^2)}{2(1 + \varpi_{|i-j|,\beta})}\right) \\ & =: KR_\beta + KS_\beta + KT_\beta, \end{aligned}$$

where $\varpi_{i,\beta} = \max\{|r_i|, \rho_\beta\}$. Recall that $\delta = \sup_{i \geq 1} |r_i| < 1$ and $\delta_n = \sup_{i \geq n} |r_i| < 1$. Choose also ε such that $0 < \varepsilon < (1 - \delta)/(1 + \delta)$ and $\varepsilon < 1 - |\ln \beta|^{-1/2} =: \theta$. Split the above sum R_β into three parts as

$$\begin{aligned} R_\beta &= \sum_{i=1}^{[\beta^{-\varepsilon}]} |r_i - \rho_\beta| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1 + \varpi_{i,\beta})}\right) \\ &+ \sum_{i=[\beta^{-\varepsilon}]+1}^{[\beta^{-\theta}]} |r_i - \rho_\beta| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1 + \varpi_{i,\beta})}\right) \\ &+ \sum_{i=[\beta^{-\theta}]+1}^{[\beta^{-1}]+1} |r_i - \rho_\beta| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1 + \varpi_{i,\beta})}\right) \\ &=: R_{\beta,1} + R_{\beta,2} + R_{\beta,3}. \end{aligned}$$

By the same arguments as for $I_{\beta,1}$, we can show that $R_{\beta,1} = o(1)$ as $\beta \rightarrow 0$. For the term $R_{\beta,2}$, let $\pi_\beta = \sup_{i \geq [\beta^{-\theta}]} \varpi_{i,\beta}$. Then it is easy to see that $\pi_\beta |\ln \beta| \rightarrow \gamma$ as $\beta \rightarrow 0$. We have

$$\begin{aligned} R_{\beta,2} &\leq (\beta^{-\theta} - \beta^{-\varepsilon}) \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r(j\beta)^2}{1 + \pi_\beta}\right) \\ &\ll (\beta^{-\theta} - \beta^{-\varepsilon}) \beta^{-1} (\beta u_\beta(x))^{-2r+1} \exp\left(-\frac{r(u_\beta(x)\beta)^2}{1 + \pi_\beta}\right) \\ &\ll \beta^{\frac{2(1-\rho_\beta)}{1+\pi_\beta} - (1+\theta)} |\ln \beta|^{\frac{(1-\rho_\beta)(1+r)}{1+\pi_\beta} - \frac{2r-1}{2}} \\ &\ll \beta^{1-\theta} |\ln \beta|^{3/2} \\ &\ll \exp\{-|\ln \beta|^{1/2} + 3/2 \ln |\ln \beta|\}, \end{aligned}$$

which tends to 0 as $\beta \rightarrow 0$. To estimate the term $R_{\beta,3}$, using the following inequality

$$|r_i - \rho_\beta| \leq \left| r_i - \frac{\gamma}{\ln i} \right| + \gamma \left| \frac{1}{\ln i} - \frac{1}{|\ln \beta|} \right| \tag{23}$$

we have

$$R_{\beta,3} \leq \sum_{i=[\beta^{-\theta}]+1}^{[\beta^{-1}]+1} \left| r_i - \frac{\gamma}{\ln i} \right| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1 + \varpi_{i,\beta})}\right)$$

$$\begin{aligned}
 &+ \gamma \sum_{i=[\beta^{-\theta}]+1}^{[\beta^{-1}]+1} \left| \frac{1}{\ln i} - \frac{1}{|\ln \beta|} \right| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(r-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1 + \varpi_{i,\beta})}\right) \\
 &=: R_{\beta,31} + R_{\beta,32}.
 \end{aligned}$$

By the same arguments as for the proof of $I_{\beta,2}$ in Theorem 2.1, we have

$$\begin{aligned}
 R_{\beta,31} &\ll |\ln \beta|^{-1} \sum_{i=[\beta^{-\theta}]+1}^{[\beta^{-1}]+1} |r_i \ln i - \gamma| \sum_{j=u_\beta(x)}^{\infty} (j\beta)^{-2(d-1)} \exp\left(-\frac{r((j\beta)^2 + ((j+i)\beta)^2)}{2(1 + \varpi_{i,\beta})}\right) \\
 &\leq \sup_{i \geq \beta^{-\theta}} \{ |r_i \ln i - \gamma| \} \beta^{-2(\pi_\beta + \rho_\beta)} |\ln \beta|^{-(1+r)(\rho_\beta + \pi_\beta)},
 \end{aligned}$$

which tends to 0, since $r_i \ln i \rightarrow \gamma$. Using the inequality for $[\beta^{-\theta}] + 1 \leq i \leq [\beta^{-1}] + 1$

$$\left| \frac{1}{\ln i} - \frac{1}{|\ln \beta|} \right| \ll |\ln \beta|^{-3/2}$$

and by the same arguments as for the proof of $R_{\beta,31}$, we have

$$R_{\beta,32} \leq \beta^{-2(\pi_\beta + \rho_\beta)} |\ln \beta|^{-(1+r)(\pi_\beta + \rho_\beta) - 1/2},$$

which also tends to 0.

By the same arguments, we get $S_\beta \rightarrow 0$ as $\beta \rightarrow 0$.

For the term T_β , note that $u_\beta(x) \sim u_\beta(y) \sim u_\beta := \beta^{-1}(\frac{2}{r} \ln \beta^{-1})^{1/2}$ as $\beta \rightarrow 0$. Define $\kappa_\beta = \sup_{i \geq 2[u_\beta]} \varpi_{i,\beta}$. Using (23) again, we have

$$\begin{aligned}
 K_\beta &\leq \sum_{i=-u_\beta(x)-[\beta^{-1}]}^{-u_\beta(x)} \sum_{j=u_\beta(y)}^{u_\beta(y)+[\beta^{-1}]} \left| r_{|i-j|} - \frac{\gamma}{\ln|i-j|} \right| (-ij\beta^2)^{-(r-1)} \exp\left(-\frac{r(i\beta)^2 + r(j\beta)^2}{2(1 + \kappa_\beta)}\right) \\
 &+ \sum_{i=-u_\beta(x)-[\beta^{-1}]}^{-u_\beta(x)} \sum_{j=u_\beta(y)}^{u_\beta(y)+[\beta^{-1}]} \left| \frac{\gamma}{\ln|i-j|} - \frac{\gamma}{|\ln \beta|} \right| (-ij\beta^2)^{-(r-1)} \exp\left(-\frac{r(i\beta)^2 + r(j\beta)^2}{2(1 + \kappa_\beta)}\right) \\
 &=: K_{\beta,1} + K_{\beta,2}.
 \end{aligned}$$

By the same arguments as for the proof of $R_{\beta,31}$, we have

$$\begin{aligned}
 K_{\beta,1} &\leq \sup_{i \geq 2[u_\beta]} \{ |r_i \ln i - \gamma| \} |\ln \beta|^{-1} \sum_{i=u_\beta(x)}^{u_\beta(x)+[\beta^{-1}]} (i\beta)^{-(r-1)} \exp\left(-\frac{r(i\beta)^2}{2(1 + \kappa_\beta)}\right) \\
 &\times \sum_{j=u_\beta(y)}^{u_\beta(y)+[\beta^{-1}]} (j\beta)^{-(r-1)} \exp\left(-\frac{r(j\beta)^2}{2(1 + \kappa_\beta)}\right) \\
 &\ll \sup_{i \geq 2[u_\beta]} \{ |r_i \ln i - \gamma| \} \beta^{-2(\kappa_\beta + \rho_\beta)} |\ln \beta|^{-(1+r)(\rho_\beta + \kappa_\beta)},
 \end{aligned}$$

which tends to 0, since $r_n \ln n \rightarrow \gamma$ as $n \rightarrow \infty$. Using the inequality for $u_\beta(x) + u_\beta(y) < |i-j| < u_\beta(x) + u_\beta(y) + 2[\beta^{-1}]$

$$\left| \frac{1}{\ln|i-j|} - \frac{1}{|\ln \beta|} \right| \ll |\ln \beta|^{-3/2},$$

we have

$$K_{\beta,2} \leq |\ln \beta|^{-3/2} \sum_{i=u_\beta(x)}^{u_\beta(x)+[\beta^{-1}]} (i\beta)^{-(r-1)} \exp\left(-\frac{r(i\beta)^2}{2(1 + \kappa_\beta)}\right)$$

$$\begin{aligned} & \times \sum_{j=u_\beta(y)}^{u_\beta(y)+[\beta^{-1}]} (j\beta)^{-(r-1)} \exp\left(-\frac{r(j\beta)^2}{2(1+\kappa_\beta)}\right) \\ & \ll \beta^{-2(\kappa_\beta+\rho_\beta)} |\ln \beta|^{-(1+r)(\rho_\beta+\kappa_\beta)-1/2}, \end{aligned}$$

which tends to 0 as $\beta \rightarrow 0$. Thus, we have $K_\beta \rightarrow 0$ as $\beta \rightarrow 0$. The proof of Theorem 2.2 is complete. \square

3.3 Proof of Theorem 2.3

We need the following two lemmas to prove Theorem 2.3.

Lemma 3.5. *Let ρ_β be a function of β with $0 \leq \rho_\beta < c < 1$ and $Y_{\beta,i} = \sqrt{1-\rho_\beta}\xi_i + \sqrt{\rho_\beta}U$, where ξ_i is a sequence of independent standard Gaussian variables and U is a standard Gaussian random variable independent with ξ_i . If $\rho_\beta |\ln \beta| \rightarrow \infty$ as $\beta \rightarrow 0$, we have for any $x, y \in \mathbb{R}$*

$$\lim_{\beta \rightarrow 0} P(Y_{\beta,i} \leq -i\beta, i \in \mathbb{A}_{\beta,x}, Y_{\beta,j} \leq j\beta, j \in \mathbb{B}_{\beta,y}) = \Phi(\min(x, y)),$$

where $\mathbb{A}_{\beta,x} = (-u_\beta(x) - \beta^{-1}, -u_\beta(x)] \cap \mathbb{Z}$ and $\mathbb{B}_{\beta,y} = [u_\beta(y), u_\beta(y) + \beta^{-1}] \cap \mathbb{Z}$ with $u_\beta(x)$ defined in (12) with $d = 1$ and ρ_β satisfying the above assumptions.

Proof. Since

$$\begin{aligned} & P(Y_{\beta,i} \leq -i\beta, i \in \mathbb{A}_{\beta,x}, Y_{\beta,j} \leq j\beta, j \in \mathbb{B}_{\beta,y}) \\ & = P(\sqrt{1-\rho_\beta}\xi_i + \sqrt{\rho_\beta}U \leq -i\beta, -u_\beta(x) - \beta^{-1} < i \leq -u_\beta(x), \\ & \quad \sqrt{1-\rho_\beta}\xi_j + \sqrt{\rho_\beta}U \leq j\beta, u_\beta(y) \leq j < u_\beta(y) + \beta^{-1}) \\ & = \int_{-\infty}^{+\infty} P\left(\xi_i \leq -\frac{\beta(i + \sqrt{\rho_\beta}z/\beta)}{\sqrt{1-\rho_\beta}}, -u_\beta(x) - \beta^{-1} < i \leq -u_\beta(x), \right. \\ & \quad \left. \xi_j \leq -\frac{\beta(j + \sqrt{\rho_\beta}z/\beta)}{\sqrt{1-\rho_\beta}}, u_\beta(y) \leq j < u_\beta(y) + \beta^{-1}\right) d\Phi(z), \end{aligned}$$

by the definition of ξ_i , we get the above integral equals

$$\begin{aligned} & \int_{-\infty}^{+\infty} P\left(\xi_i \leq -\frac{i\beta}{\sqrt{1-\rho_\beta}}, -(u_\beta(x) - \sqrt{\rho_\beta}z/\beta) - \beta^{-1} < i \leq -(u_\beta(x) - \sqrt{\rho_\beta}z/\beta)\right) \\ & \quad \times P\left(\xi_j \leq \frac{j\beta}{\sqrt{1-\rho_\beta}}, (u_\beta(y) - \sqrt{\rho_\beta}z/\beta) \leq j < (u_\beta(y) - \sqrt{\rho_\beta}z/\beta) + \beta^{-1}\right) d\Phi(z) \\ & =: \int_{-\infty}^{+\infty} F(\beta, x, z)F(\beta, y, z)d\Phi(z). \end{aligned} \tag{24}$$

By the same arguments as in Lemma 3.3 (see also the proof of Lemma 2.2 of [9]), we have

$$-\ln F(\beta, x, z) \rightarrow \begin{cases} 0, & \text{if } x > z; \\ \infty, & \text{if } x < z, \end{cases}$$

and

$$-\ln F(\beta, y, z) \rightarrow \begin{cases} 0, & \text{if } y > z; \\ \infty, & \text{if } y < z, \end{cases}$$

which combining with (24) completes the proof of the lemma. \square

Lemma 3.6. *Under the conditions of Theorem 2.3, we have*

$$|P(\Upsilon(X) \geq -u_\beta(x), \tau(X) \leq u_\beta(x)) - P(X_i \leq -i\beta, i \in \mathbb{A}_{\beta,x}, X_j \leq j\beta, j \in \mathbb{B}_{\beta,y})| \rightarrow 0$$

as $\beta \rightarrow 0$.

Proof. Let $A_X = (\Upsilon(X) \geq -u_\beta(x))$, $A'_X = (X_i \leq -i\beta, -u_\beta(x) - [\beta^{-1}] < i \leq -u_\beta(x))$, $B_X = (\tau(X) \leq u_\beta(x))$ and $B'_X = (X_j \leq j\beta, u_\beta(y) \leq i < u_\beta(y) + [\beta^{-1}])$. Obviously,

$$\begin{aligned} |P(A_X \cap B_X) - P(A'_X \cap B'_X)| &\leq |P(A_X \cap B_X) - P(A'_X \cap B_X)| \\ &\quad + |P(A'_X \cap B_X) - P(A'_X \cap B'_X)| \\ &\leq |P(A_X) - P(A'_X)| + |P(B_X) - P(B'_X)| \\ &\leq P(E_X^c) + P(F_X^c), \end{aligned}$$

where $E_X = (\Upsilon(X) \geq -u_\beta(x) - [\beta^{-1}])$ and $F_X = (\tau(X) \leq u_\beta(x) + [\beta^{-1}])$. From the proof of Lemma 2.3 of [9], we get $P(E_X) \rightarrow 1$, as $\beta \rightarrow 0$. Similarly, we have $P(F_X) \rightarrow 1$, as $\beta \rightarrow 0$. The proof of the lemma is complete. \square

Proof of Theorem 2.3. By Lemma 3.6, it suffices to show that

$$P(X_i \leq -i\beta, i \in \mathbb{A}_{\beta,x}, X_j \leq j\beta, j \in \mathbb{B}_{\beta,y}) \rightarrow \Phi(\min(x, y))$$

as $\beta \rightarrow 0$. Define $Y_{\beta,i} = \sqrt{1 - \rho_\beta} \xi_i + \sqrt{\rho_\beta} U$ as in Lemma 3.5. Since $r_i \geq \rho_\beta$ for all $i \leq \beta^{-1}$, by Slepian's lemma (see, e.g., [17]), we have

$$\begin{aligned} \lim_{\beta \rightarrow 0} P(X_i \leq -i\beta, i \in \mathbb{A}_{\beta,x}, X_j \leq j\beta, j \in \mathbb{B}_{\beta,y}) \\ \geq \lim_{\beta \rightarrow 0} P(Y_{\beta,i} \leq -i\beta, i \in \mathbb{A}_{\beta,x}, Y_{\beta,j} \leq j\beta, j \in \mathbb{B}_{\beta,y}), \end{aligned}$$

which by Lemma 3.5 equals $\Phi(\min(x, y))$. Thus, we only need to show

$$P(X_i \leq -i\beta, i \in \mathbb{A}_{\beta,x}, X_j \leq j\beta, j \in \mathbb{B}_{\beta,y}) \leq \Phi(\min(x, y) + \varepsilon)$$

for all $\varepsilon > 0$. Since r_n is convex, there is a Gaussian sequence $\{Z_i = Z_i(\beta), i \in \mathbb{Z}\}$ with the correlations (see, e.g., [19])

$$\varrho_k = (r_k - \rho_\beta) / (1 - \rho_\beta) \text{ for } k = 1, 2, \dots, [\beta^{-1}].$$

Let V be independent of $\{Z_i, i \in \mathbb{Z}\}$, such that

$$X_i = \sqrt{1 - \rho_\beta} Z_i + \sqrt{\rho_\beta} V.$$

Now

$$\begin{aligned} &P(X_i \leq -i\beta, i \in \mathbb{A}_{\beta,x}, X_j \leq j\beta, j \in \mathbb{B}_{\beta,y}) \\ &= \int_{-\infty}^{\infty} P\left(Z_i \leq \frac{-i\beta}{\sqrt{1 - \rho_\beta}}, -u_\beta(x - z) - [\beta^{-1}] < i \leq -u_\beta(x - z), \right. \\ &\quad \left. Z_j \leq \frac{j\beta}{\sqrt{1 - \rho_\beta}}, u_\beta(y - z) \leq j < u_\beta(y - z) + [\beta^{-1}]\right) d\Phi(z) \\ &\leq \Phi(\min(x, y) + \varepsilon) \\ &+ \int_{\min(x,y)+\varepsilon}^{\infty} P\left(Z_i \leq \frac{-i\beta}{\sqrt{1 - \rho_\beta}}, -u_\beta(x - z) - [\beta^{-1}] < i \leq -u_\beta(x - z), \right. \\ &\quad \left. Z_j \leq \frac{j\beta}{\sqrt{1 - \rho_\beta}}, u_\beta(y - z) \leq j < u_\beta(y - z) + [\beta^{-1}]\right) d\Phi(z) \end{aligned}$$

$$\leq \Phi(\min(x, y) + \varepsilon) + P\left(Z_i \leq \frac{-i\beta}{\sqrt{1-\rho_\beta}}, -u_\beta(-\varepsilon) - [\beta^{-1}] < i \leq -u_\beta(-\varepsilon)\right).$$

Therefore, to complete the proof of the theorem, it suffices to show that the second term tends to 0, which has been done in the proof of Theorem 2.3 in [9]. \square

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Declarations

Conflict of interest The authors declare no conflict of interest.

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