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Equilibrium dividend strategies in the dual model with a random time horizon

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Abstract. This paper investigates the dividend problem with non-exponential discounting in a dual model. We assume that the dividends can only be paid at a bounded rate and that the surplus process is killed by an exponential random variable. Since the non-exponential discount function leads to a time inconsistent control problem, we study the equilibrium HJB-equation and give the associated verification theorem. For the case of a mixture of exponential discount functions and exponential gains, we obtain the explicit equilibrium dividend strategy and the corresponding equilibrium value function. Besides, numerical examples are shown to illustrate our results.

§1 Introduction

In this paper, the surplus of a company is described as a dual model

$$dX_t = -\mu dt + dS_t, \quad t \geq 0, \tag{1.1}$$

where the constant $\mu > 0$ is the rate of expenses, and the compound Poisson process $\{S_t = \sum_{k=1}^{N_t} Y_k\}$ is the income process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions. The Poisson process $\{N_t\}$ with the tensity $\lambda > 0$ represents the number of income; and $\{Y_k\}$, which is a sequence of nonnegative, independent and identically distributed variables, represents the amounts of income. We assume that Y_1 has the probability density function $p(y)$ and the expectation $\nu = \int_0^\infty yp(y)dy < \infty$. In addition, the net profit condition holds, i.e.,

$$\theta = -\mu + \lambda\nu > 0. \tag{1.2}$$

The optimization of dividend payments has been studied by many researchers since it was proposed by [8]. In [11] and [14], the authors considered the optimal dividend problems in the classical risk model. In the context of the dual model, the optimal dividend problems were studied in [2], [3], [19] – [21] and so on.

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In the above papers, a common assumption is that the discount rate is constant, namely, the discount function is exponential. However, the assumption of the constant discount rate is unrealistic in some empirical studies of human behavior. The non-exponential discount function leads the time inconsistent problem which was first studied by [13] in a game theoretic framework. [4] and [15] developed the general theories of time inconsistent control problem. [16], [10] and [5] studied the optimal portfolio problems with non-exponential discounting. The time inconsistent dividend problems were considered in [6] for a compound Poisson model and in [17] for a diffusion model. [7] discussed the optimal dividend strategies with a quasi-hyperbolic discount function in a dual model. In the same model, [12] studied the equilibrium dividend strategy with a pseudo exponential discount function. Recently, [18] considered the equilibrium dividend strategy with non-exponential discounting for spectrally negative Lévy processes.

In this paper, we consider the equilibrium dividend problem in the dual model with a random time. Motivated by [4] and [15], we give the equilibrium HJB-equation by a heuristic discussion and prove it by a verification theorem. In a special case, we obtain the explicit equilibrium dividend strategy and the associated value function. When the random time horizon tends to infinity and the discount function is a mixture of two exponential functions, our results are reduced to those in [12]. Moreover, the main method in this work is different from that in [12].

This paper is organized as follows. Section 2 provides the formulations of the control problem. The equilibrium HJB-equation and the verification theorem are presented in Section 3. In Section 4, we study the case of a mixture of exponential discount functions and show some numerical examples.

§2 Model and Control Problem

A dividend strategy is described by a stochastic process $\{u_t\}$. Here, u_t is the rate of dividend payout at time t, which is assumed to be bounded by a constant $M > 0$. We restrict ourselves to the feedback control strategies, i.e., at time t, the control u_t is given by

$$u_t = \mathbf{u}(t, x), \quad t \geq 0, \tag{2.1}$$

where $x \geq 0$ is the surplus level at time t, and the control mapping $\mathbf{u} : [0, \infty) \times [0, \infty) \to [0, M]$ is a Borel measurable function. In order to distinguish between functions and function values, we will always denote a control strategy (i.e., a mapping) by using boldface, like $\mathbf{u}$, whereas a possible value of the mapping will be denoted without boldface, like $u \in [0, M]$. The control strategy $\mathbf{u}$ defined in (2.1) is called admissible strategy. The set of all admissible strategies is denoted by Π.

For an admissible strategy $\mathbf{u}$ and the initial time $t \in [0, \infty)$, the controlled surplus process denoted by $\{X_t^{\mathbf{u}}\}$ evolves according to

$$\begin{cases} dX_s^{\mathbf{u}} = -\mu ds + dS_s - \mathbf{u}(s, X_s^{\mathbf{u}})ds, & s \geq t, \\ X_t^{\mathbf{u}} = x. \end{cases} \tag{2.2}$$

Let $T_t^{\mathbf{u}} = \inf\{s \geq t : X_s^{\mathbf{u}} \leq 0\}$ be the time of ruin under the control strategy $\mathbf{u}$ and the initial time $t \geq 0$. Unlike the conventional exponential discount function, we consider the non-exponential discounting. Inspired by [9] and [17], we assume that the discount function φ: $[0, \infty) \to [0, \infty)$ is continuous differentiable and satisfies

$$\varphi(0) = 1, \quad \varphi(t) \geq 0, \quad \varphi'(t) \leq 0. \tag{2.3}$$

Besides the dividend strategy $\mathbf{u}$, like [1] and [20], the surplus process $\{X_t^{\mathbf{u}}\}$ is killed randomly by an exponential variable ζ with the parameter $\gamma > 0$, which is independent of $\{X_t^{\mathbf{u}}\}$. Furthermore, if $t \le \zeta < T_t^{\mathbf{u}}$, we assume that the surplus at the stopping time is paid out as dividend. For an admissible strategy $\mathbf{u}$, an initial state (t, x) and a non-exponential discount function φ, we define the performance function

$$\widetilde{V}(t,x;\mathbf{u}) = E_{t,x}\left[\int_t^{T_t^{\mathbf{u}}\wedge\zeta} \varphi(z-t)\mathbf{u}(z,X_z^{\mathbf{u}})\mathbf{I}_{\{\zeta\ge t\}}dz + \varphi(\zeta-t)X_\zeta^{\mathbf{u}}\mathbf{I}_{\{t\le\zeta<T_t^{\mathbf{u}}\}}\right],$$

where $E_{t,x}[\cdot]$ is the expectation conditioned on the event $\{X_t^{\mathbf{u}} = x\}$, and $\mathbf{I}_A$ is an indicator function of a set A. By the law of total probability, we have

$$\begin{aligned}\widetilde{V}(t,x;\mathbf{u}) =& E_{t,x}\left[\int_t^{T_t^{\mathbf{u}}} \varphi(z-t)\mathbf{u}(z,X_z^{\mathbf{u}})\mathbf{I}_{\{\zeta\ge t,\zeta\ge z\}}dz + \varphi(\zeta-t)X_\zeta^{\mathbf{u}}\mathbf{I}_{\{t\le\zeta<T_t^{\mathbf{u}}\}}\right]\\
=& E_{t,x}\left[\int_t^{\infty}\left(\int_t^{T_t^{\mathbf{u}}} \varphi(z-t)\mathbf{u}(z,X_z^{\mathbf{u}})\mathbf{I}_{\{r\ge z\}}dz + \varphi(r-t)X_r^{\mathbf{u}}\mathbf{I}_{\{r<T_t^{\mathbf{u}}\}})\right)\gamma e^{-\gamma r}dr\right]\\
=& E_{t,x}\left[\int_t^{T_t^{\mathbf{u}}} \varphi(z-t)\mathbf{u}(z,X_z^{\mathbf{u}})\int_z^{\infty}\gamma e^{-\gamma r}drdz + \int_t^{T_t^{\mathbf{u}}}\varphi(r-t)X_r^{\mathbf{u}}\gamma e^{-\gamma r}dr\right]\\
=& E_{t,x}\left[\int_t^{T_t^{\mathbf{u}}} \varphi(z-t)\mathbf{u}(z,X_z^{\mathbf{u}})e^{-\gamma z}dz + \int_t^{T_t^{\mathbf{u}}}\varphi(r-t)X_r^{\mathbf{u}}\gamma e^{-\gamma r}dr\right]\\
=& e^{-\gamma t}V(t,x;\mathbf{u}),\end{aligned}$$

where

$$V(t,x;\mathbf{u}) = E_{t,x}\left[\int_t^{T_t^{\mathbf{u}}} \varphi(z-t)e^{-\gamma(z-t)}[\mathbf{u}(z,X_z^{\mathbf{u}}) + \gamma X_z^{\mathbf{u}}]dz\right]. \tag{2.4}$$

Then we only need to study the function $V(t, x; \mathbf{u})$, for convenience, which is still called the performance function.

Proposition 2.1. *The performance function $V(t, x; \mathbf{u})$ in (2.4) satisfies*

$$0 \le V(t,x;\mathbf{u}) \le x + \frac{M+\lambda\nu}{\gamma}. \tag{2.5}$$

Proof. Note that, for $X_t^{\mathbf{u}} = x$ and $z \ge t$,

$$X_z^{\mathbf{u}} \le x + S_z - S_t, \quad a.s.. \tag{2.6}$$

By Fubini's Theorem, we obtain

$$\begin{aligned}V(t,x;\mathbf{u}) &\le \int_t^{\infty} \varphi(z-t)e^{-\gamma(z-t)}(M + \gamma E_{t,x}[X_z^{\mathbf{u}}])dz\\
&\le \int_t^{\infty} \varphi(z-t)e^{-\gamma(z-t)}[M + \gamma x + \gamma\lambda\nu(z-t)]dz.\end{aligned}$$

Noting that $\varphi(t) \le \varphi(0) = 1$, we get the second inequality in (2.5). It is easy to show the first inequality by (2.5). □

In the classical control theory, the optimal dividend strategy $\mathbf{u}^*$ is generally defined by $V(t, x; \mathbf{u}^*) = \sup_{\mathbf{u}\in\Pi} V(t, x; \mathbf{u})$. However, since $\varphi(t)$ is non-exponential, the above optimization problem is time-inconsistent in the sense that the Bellman optimality principle fails. Similar to [4], we view the entire problem as a non-cooperative game and look for Nash equilibria for the game. More precisely, we consider a game with one player for each time t, where player t can be regarded as the future incarnation of the decision maker at time t. In the following sections, we will study the subgame perfect Nash equilibrium strategies from the game theoretic point of view.

Definition 2.1. Consider a control strategy $\hat{\mathbf{u}} \in \Pi$, a fixed constant $u \in [0, M]$ and a fixed real number $h > 0$. For any fixed initial point $(t, x) \in [0, \infty) \times [0, \infty)$, we define the control strategy $\mathbf{u}_h$ by

$$\mathbf{u}_h(s, y) = \begin{cases} u, & t \le s < t + h, \quad y \ge 0, \\ \hat{\mathbf{u}}(s, y), & t + h \le s < \infty \quad y \ge 0. \end{cases} \tag{2.7}$$

If

$$\liminf_{h \to 0} \frac{V(t, x; \hat{\mathbf{u}}) - V(t, x; \mathbf{u}_h)}{h} \ge 0,$$

for all $u \in [0, M]$, we say that $\hat{\mathbf{u}}$ is an equilibrium control strategy. The corresponding equilibrium value function $V(t, x)$ is defined by $V(t, x) = V(t, x; \hat{\mathbf{u}})$.

§3 Equilibrium Hamilton-Jacobi-Bellman Equation

In this section, we first, in a heuristic way, derive the equilibrium HJB-equation. We then show that this equation is correct by proving a rigorous verification theorem.

For convenience, we introduce some notations. Let $\mathcal{C}^{1,1}([0, \infty) \times [0, \infty))$ denote the set of all functions on $[0, \infty) \times [0, \infty)$ whose first order partial derivatives are continuous with respect to the each variable. For a constant $u \in [0, M]$ and a control strategy $\mathbf{u} \in \Pi$, we define the infinitesimal generators applying a function $f(t, x) \in \mathcal{C}^{1,1}([0, \infty) \times [0, \infty))$, respectively,

$$\mathcal{A}^u f(t, x) = \frac{\partial f}{\partial t}(t, x) - (\mu + u)\frac{\partial f}{\partial x}(t, x) + \lambda \int_0^\infty [f(t, x + y) - f(t, x)]p(y)dy,$$

$$\mathcal{A}^{\mathbf{u}} f(t, x) = \frac{\partial f}{\partial t}(t, x) - [\mu + \mathbf{u}(t, x)]\frac{\partial f}{\partial x}(t, x) + \lambda \int_0^\infty [f(t, x + y) - f(t, x)]p(y)dy.$$

Let $\mathcal{D}[0, \infty) = \{(s, t) : 0 \le s \le t\}$ and $\mathcal{C}^{0,1,1}(\mathcal{D}[0, \infty) \times [0, \infty))$ be the set of all functions on $\mathcal{D}[0, \infty) \times [0, \infty)$ which are continuous with respect to the first variable, continuously differentiable with respect to the second and third variables.

We consider the following function, for $\mathbf{u} \in \Pi$,

$$J(s, t, x; \mathbf{u}) = E_{t,x}\left[\int_t^{T_t^{\mathbf{u}}} C(s, z, X_z^{\mathbf{u}}, \mathbf{u}(z, X_z^{\mathbf{u}}))dz\right], \tag{3.1}$$

where $(s, t, x) \in \mathcal{D}[0, \infty) \times [0, \infty)$ and

$$C(s, z, x, \mathbf{u}(z, x)) = \varphi(z - s)e^{-\gamma(z-s)}[\gamma x + \mathbf{u}(z, x)]. \tag{3.2}$$

If there exists an equilibrium strategy $\hat{\mathbf{u}} \in \Pi$, we denote $J(s, t, x; \hat{\mathbf{u}}) = J(s, t, x)$. Then

$$J(t, t, x) = J(t, t, x; \hat{\mathbf{u}}) = V(t, x; \hat{\mathbf{u}}) = V(t, x).$$

For the control strategy $\mathbf{u}_h$ in (2.7), if h is small enough, we expect to have

$$J(t, t, x; \mathbf{u}_h) = V(t, x; \mathbf{u}_h) \le V(t, x; \hat{\mathbf{u}}) = J(t, t, x; \hat{\mathbf{u}}) = J(t, t, x),$$

and in the limit as h tends to zero, we should have equality if $u = \hat{\mathbf{u}}(t, x)$.

Since $\mathbf{u}_h(z, y) = \hat{\mathbf{u}}(z, y)$ for $z \ge t + h$ and any $u \in [0, M]$, we have

$$J(s, t + h, X_{t+h}^u; \mathbf{u}_h) = J(s, t + h, X_{t+h}^u; \hat{\mathbf{u}}) = J(s, t + h, X_{t+h}^u).$$

Taking expectations and applying Dynkin's formula to $J(s,\cdot,\cdot)$, we have the approximation

$$\begin{aligned}&E_{t,x}[J(s,t+h,X^u_{t+h};\mathbf{u}_h)]=E_{t,x}[J(s,t+h,X^u_{t+h})]\\&=J(s,t,x)+\mathcal{A}^uJ^s(t,x)\cdot h+o(h),\end{aligned}\tag{3.3}$$

where $J^s(t,x)=J(s,t,x)$ is viewed as a function of two variables t and x for a fixed s. Note that

$$E_{t,x}\left[\int_t^{t+h}C(s,z,X^u_z,u)dz\right]=C(s,t,x,u)\cdot h+o(h),\tag{3.4}$$

where $C(s,z,x,u)=\varphi(z-s)e^{-\gamma(z-s)}(\gamma x+u)$. Then by (3.3) and (3.4), we have

$$\begin{aligned}J(s,t,x;\mathbf{u}_h)&=E_{t,x}\left[\int_t^{t+h}C(s,z,X^u_z,u)dz\right]+E_{t,x}[J(s,t+h,X^u_{t+h};\mathbf{u}_h)]\\&=[C(s,t,x,u)+\mathcal{A}^uJ^s(t,x)]\cdot h+J(s,t,x)+o(h).\end{aligned}\tag{3.5}$$

By the definition of the equilibrium strategy, we have

$$\begin{aligned}&\mathcal{A}^uJ^t(t,x)+C(t,t,x,u)\le 0,\quad \forall u\in[0,M],\\&\mathcal{A}^{\hat{\mathbf{u}}}J^t(t,x)+C(t,t,x,\hat{\mathbf{u}}(t,x))=0.\end{aligned}$$

That is, for $\hat{\mathbf{u}}(t,x)=\operatorname{argmax}_{u\in[0,M]}\{\mathcal{A}^uJ^t(t,x)+C(t,t,x,u)\}$,

$$\mathcal{A}^{\hat{\mathbf{u}}}J^t(t,x)+C(t,t,x,\hat{\mathbf{u}}(t,x))=0.$$

For conveniently solving the equilibrium control problem, we strengthen the above condition by assuming

$$\mathcal{A}^{\hat{\mathbf{u}}}J^s(t,x)+C(s,t,x,\hat{\mathbf{u}}(t,x))=0,\quad s\le t.$$

Collecting all results, we arrive at the proposed equilibrium HJB-equation which is stated the equilibrium HJB-equation in the following definition.

Definition 3.1. For a function $c(s,t,x)\in\mathcal{C}^{0,1,1}(\mathcal{D}[0,\infty)\times[0,\infty))$, we define the equilibrium HJB-equation for the Nash equilibrium problem as follows

$$\begin{cases}\mathcal{A}^{\hat{\mathbf{u}}}c^s(t,x)+C(s,t,x,\hat{\mathbf{u}}(t,x))=0,\quad (s,t,x)\in\mathcal{D}[0,\infty)\times(0,\infty)\\ c(s,t,0)=0,\quad \forall(s,t)\in\mathcal{D}[0,\infty),\end{cases}\tag{3.6}$$

where $c^s(t,x)=c(s,t,x)$ and

$$\hat{\mathbf{u}}(t,x)=\operatorname*{argmax}_{u\in[0,M]}\{\mathcal{A}^uc^t(t,x)+C(t,t,x,u)\}.\tag{3.7}$$

Theorem 3.1. *(Verification Theorem) Assume that a nonnegative function $c(s,t,x)\in\mathcal{C}^{0,1,1}$ $(\mathcal{D}[0,\infty)\times[0,\infty))$ is increasing and concave with respect to x, and solves the equilibrium HJB-equation (3.6). If for any fixed $s\le t$, it holds that*

$$\lim_{t,x\to\infty}c(s,t,x)=0.\tag{3.8}$$

Then $\hat{\mathbf{u}}$ given by (3.7) is an equilibrium strategy, and the associated equilibrium value function is

$$V(t,x)=V(t,x;\hat{\mathbf{u}})=c(t,t,x).$$

Proof. We first show that $c(t,t,x)$ is the performance function corresponding to $\hat{\mathbf{u}}$, i.e., $c(t,t,x)=V(t,x;\hat{\mathbf{u}})$. Applying Dynkin's formula and noting (3.6), we have

$$c^s(t,x)=E_{t,x}[c(s,T_n,X^{\hat{\mathbf{u}}}_{T_n})]+E_{t,x}\left[\int_t^{T_n}C(s,z,X^{\hat{\mathbf{u}}}_z,\hat{\mathbf{u}}(z,X^{\hat{\mathbf{u}}}_z))dz\right],\tag{3.9}$$

where $T_n = T_t^{\hat{\mathbf{u}}} \wedge n$ for $n \geq t$, $n = 1, 2, \cdots$. By $X^{\hat{\mathbf{u}}}_{T_t^{\hat{\mathbf{u}}}} = 0$ $(a.s.)$ for $T_t^{\hat{\mathbf{u}}} < \infty$, we have

$$E_{t,x}\left[c\left(s, T_t^{\hat{\mathbf{u}}}, X^{\hat{\mathbf{u}}}_{T_t^{\hat{\mathbf{u}}}}\right)\mathbf{I}_{\{T_t^{\hat{\mathbf{u}}}<n\}}\right] = 0.$$

Noting that $X_n^{\hat{\mathbf{u}}} \leq x + S_n - S_t$ $(a.s.)$ for the initial state (t, x) and the increase of the function $c(s, t, \cdot)$, we derive

$$\begin{aligned} E_{t,x}[c(s, T_n, X^{\hat{\mathbf{u}}}_{T_n})] &= E_{t,x}\left[c\left(s, n, X_n^{\hat{\mathbf{u}}}\right)\mathbf{I}_{\{T_t^{\hat{\mathbf{u}}}\geq n\}}\right] + E_{t,x}\left[c\left(s, T_t^{\hat{\mathbf{u}}}, X^{\hat{\mathbf{u}}}_{T_t^{\hat{\mathbf{u}}}}\right)\mathbf{I}_{\{T_t^{\hat{\mathbf{u}}}<n\}}\right] \\ &\leq E_{t,x}\left[c\left(s, n, x + S_n - S_t\right)\right]. \end{aligned}$$

By the concavity of $c(s, t, \cdot)$ and Jensen's inequality, we obtain

$$E_{t,x}[c(s, T_n, X^{\hat{\mathbf{u}}}_{T_n})] \leq c\left(s, n, x + E_{t,x}[S_n - S_t]\right) = c(s, n, x + \lambda\nu(n - t)).$$

Then we conclude

$$0 \leq \lim_{n\to\infty} E_{t,x}[c(s, T_n, X^{\hat{\mathbf{u}}}_{T_n})] \leq \lim_{n\to\infty} c(s, n, x + \lambda\nu(n - t)) = 0. \tag{3.10}$$

By (3.2), we have, for the initial state (t, x),

$$C(s, z, X_z^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}(z, X_z^{\hat{\mathbf{u}}})) \leq \varphi(z - s)e^{-\gamma(z-s)}\left[M + \gamma x + \gamma(S_z - S_t)\right].$$

By the dominated convergence theorem, we get

$$\lim_{n\to\infty} E_{t,x}\left[\int_t^{T_n} C(s, z, X_z^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}(z, X_z^{\hat{\mathbf{u}}}))dz\right] = E_{t,x}\left[\int_t^{T_t^{\hat{\mathbf{u}}}} C(s, z, X_z^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}(z, X_z^{\hat{\mathbf{u}}}))dz\right]$$

Letting $n \to \infty$ in (3.9) and combining (3.10), we obtain

$$c^s(t, x) = E_{t,x}\left[\int_t^{T_t^{\hat{\mathbf{u}}}} C(s, z, X_z^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}(z, X_z^{\hat{\mathbf{u}}}))dz\right] = J(s, t, x; \hat{\mathbf{u}}),$$

where $J(s, t, x; \mathbf{u})$ is defined in (3.1). Hence, $c(t, t, x) = V(t, x; \hat{\mathbf{u}})$.

Now we prove that $\hat{\mathbf{u}}$ in (3.7) is indeed an equilibrium control strategy. For any $u \in [0, M]$ and $\mathbf{u}_h$ given by (2.7), we have, from (3.5),

$$V(t, x; \mathbf{u}_h) = [C(t, t, x, u) + \mathcal{A}^u c^t(t, x)]h + c(t, t, x) + o(h).$$

By the HJB-equation (3.6), we obtain

$$c(t, t, x) - V(t, x; \mathbf{u}_h) + o(h) = -\left[C(t, t, x, u) + \mathcal{A}^u c^t(t, x)\right]h \geq 0.$$

Therefore we get

$$\liminf_{h\to 0} \frac{c(t, t, x) - V(t, x; \mathbf{u}_h)}{h} = \liminf_{h\to 0} \frac{V(t, x; \hat{\mathbf{u}}) - V(t, x; \mathbf{u}_h)}{h} \geq 0.$$

The proof is completed. □

Remark 3.1. *By (3.10), we can weaken the condition (3.8) by assuming*

$$\lim_{n\to\infty} c(s, n, x + \lambda\nu(n - t)) = 0.$$

When the killing rate $\gamma = 0$, we have, by the above proof,

$$c^s(t, x) = E_{t,x}\left[\int_t^{T_t^{\hat{\mathbf{u}}}} C(s, z, X_z^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}(z, X_z^{\hat{\mathbf{u}}}))dz\right] = E_{t,x}\left[\int_t^{T_t^{\hat{\mathbf{u}}}} \varphi(z - s)\hat{\mathbf{u}}(z, X_z^{\hat{\mathbf{u}}})dz\right].$$

Furthermore,

$$\mathcal{A}^{\hat{\mathbf{u}}} c(t, t, x) - \mathcal{A}^{\hat{\mathbf{u}}} c^s(t, x)|_{s=t} = -E_{t,x}\left[\int_t^{T_t^{\mathbf{u}}} \varphi'(z - t)\hat{\mathbf{u}}(z, X_z^{\hat{\mathbf{u}}})dz\right].$$

Combining Definition 3.1, we can derive Proposition 3.1 in [12] by the above theorem.

§4 Explicit Solution for a Mixture of Exponential Discount Function

In this section, we try to find a solution of the equilibrium HJB-equation in (3.6) for a specific discount function. Since

$$\mathcal{A}^u c^s(t,x) + C(s,t,x,u) = \mathcal{A}^0 c^s(t,x) + \gamma\phi(t-s)x + u\left[\phi(t-s) - \frac{\partial c}{\partial x}(s,t,x)\right],$$

where $\phi(t) = \varphi(t)e^{-\gamma t}$, we have $\hat{\mathbf{u}}(t,x) = 0$ for $\frac{\partial c}{\partial x}(t,t,x) > 1$ and $\hat{\mathbf{u}}(t,x) = M$ for $\frac{\partial c}{\partial x}(t,t,x) \leq 1$. If there exists a constant $b \geq 0$ such that $\frac{\partial c}{\partial x}(t,t,x) > 1$ for $0 \leq x < b$ and $\frac{\partial c}{\partial x}(t,t,x) \leq 1$ for $x \geq b$, we obtain $\hat{\mathbf{u}}(t,x) = 0$ for $0 \leq x < b$ and $\hat{\mathbf{u}}(t,x) = M$ for $x \geq b$. Then the equilibrium HJB-equation (3.6) becomes

$$\begin{cases} \mathcal{A}^0 c^s(t,x) + C(s,t,x,0) = 0, & (s,t,x) \in \mathcal{D}[0,\infty) \times (0,b), \\ \mathcal{A}^M c^s(t,x) + C(s,t,x,M) = 0, & (s,t,x) \in \mathcal{D}[0,\infty) \times [b,\infty), \\ c(s,t,0) = 0, & (s,t) \in \mathcal{D}[0,\infty). \end{cases} \tag{4.1}$$

We consider a discount function defined by

$$\varphi(t) = \sum_{i=1}^{m} w_i e^{-\delta_i t}, \quad t \geq 0, \tag{4.2}$$

where $\delta_i > 0$, $\delta_i \neq \delta_j$ for $i \neq j$, $w_i > 0$ and $\sum_{i=1}^m w_i = 1$. Then the performance function $V(t,x;\mathbf{u})$ becomes

$$V(t,x;\mathbf{u}) = \sum_{i=1}^{m} E_{t,x}\left[\int_0^{T_t^{\mathbf{u}}} w_i e^{-(\delta_i+\gamma)(z-t)}[\gamma X_z^{\mathbf{u}} + \mathbf{u}(z, X_z^{\mathbf{u}})]dz\right],$$

which is viewed as the case where dividends are proportionally paid to m shareholders who have different discount rate.

We consider the following candidate function

$$c(s,t,x) = \sum_{i=1}^{m} w_i e^{-(\delta_i+\gamma)(s-t)} V_i(x), \quad (s,t,x) \in \mathcal{D}[0,\infty) \times [0,\infty), \tag{4.3}$$

where the functions $V_i(x)$, $i = 1, 2, \cdots, m$, satisfy the following integro-differential equations

$$-\mu V_i'(x) + \lambda \int_0^\infty V_i(x+y)p(y)dy - (\delta_i + \gamma + \lambda)V_i(x) + \gamma x = 0, \quad 0 \leq x < b,$$

$$-(\mu + M)V_i'(x) + \lambda \int_0^\infty V_i(x+y)p(y)dy - (\delta_i + \gamma + \lambda)V_i(x) + \gamma x + M = 0, \quad x \geq b.$$

Furthermore, we assume that the income amount Y is exponential distributed, i.e., $p(y) = \beta e^{-\beta y}$ $(y \geq 0)$. Applying the operator $(\frac{d}{dx} - \beta)$ to the above equations yields

$$\mu V_i''(x) - (\beta\mu - \delta_i - \gamma - \lambda)V_i'(x) - \beta(\delta_i + \gamma)V_i(x) + \gamma\beta x - \gamma = 0, \ 0 < x < b, \tag{4.4}$$

and

$$\begin{aligned} (\mu + M)V_i''(x) - [\beta(\mu + M) - \delta_i - \gamma - \lambda]V_i'(x) - \beta(\delta_i + \gamma)V_i(x) \\ + \beta(\gamma x + M) - \gamma = 0, \ x \geq b. \end{aligned} \tag{4.5}$$

We assume that one particular solution of the inhomogeneous differential equation (4.4) is $A_{i0}x + B_{i0}$. Substituting this solution into (4.4), we obtain

$$A_{i0} = \frac{\gamma}{\delta_i + \gamma} > 0, \quad B_{i0} = \frac{\gamma}{(\delta_i + \gamma)^2}(\frac{\lambda}{\beta} - \mu) = \frac{\gamma\theta}{(\delta_i + \gamma)^2} > 0, \tag{4.6}$$

where $\theta > 0$ is defined by (1.2). Combining the homogenous and particular solutions, we get the general solution of (4.4)

$$V_i(x) = A_{i1}e^{\xi_{i1}x} + A_{i2}e^{\xi_{i2}x} + A_{i0}x + B_{i0}, \quad 0 \le x < b, \tag{4.7}$$

where $\xi_{i1} < 0$ and $\xi_{i2} > 0$ are the solutions to the following equation

$$\mu\xi^2 - (\beta\mu - \delta_i - \gamma - \lambda)\xi - \beta(\delta_i + \gamma) = 0.$$

Similarly, the general solution of (4.5) is

$$V_i(x) = B_{i1}e^{\xi_{i3}x} + B_{i2}e^{\xi_{i4}x} + A_{i0}x + B_{i0} + \frac{\delta_i}{(\delta_i + \gamma)^2}M, \quad x \ge b,$$

where $\xi_{i3} < 0$ and $\xi_{i4} > 0$ are the solutions to the following equation

$$(\mu + M)\xi^2 - [\beta(\mu + M) - \delta_i - \gamma - \lambda]\xi - \beta(\delta_i + \gamma) = 0.$$

By Proposition 2.1, we know that $V_i(x)$ is linearly bounded. Letting $B_i = B_{i1}$, we have

$$V_i(x) = B_i e^{\xi_{i3}x} + A_{i0}x + B_{i0} + \frac{\delta_i}{(\delta_i + \gamma)^2}M, \quad x \ge b. \tag{4.8}$$

According to the principle of smooth fitting and the boundary condition, we derive the following equations

$$\begin{aligned}
&V_i(b-) = V_i(b+), \quad i = 1, 2, \cdots, m;\\
&V_i'(b-) = V_i'(b+), \quad i = 1, 2, \cdots, m;\\
&V_i(0) = 0, \quad i = 1, 2, \cdots, m;\\
&\frac{\partial c}{\partial x}(t, t, b-) = \frac{\partial c}{\partial x}(t, t, b+) = 1.
\end{aligned}$$

We rewrite the above equations as

$$\begin{aligned}
&A_{i1}e^{\xi_{i1}b} + A_{i2}e^{\xi_{i2}b} = B_i e^{\xi_{i3}b} + \frac{\delta_i}{(\delta_i + \gamma)^2}M,\\
&A_{i1}\xi_{i1}e^{\xi_{i1}b} + A_{i2}\xi_{i2}e^{\xi_{i2}b} = B_i\xi_{i3}e^{\xi_{i3}b},\\
&A_{i1} + A_{i2} + B_{i0} = 0,
\end{aligned} \tag{4.9}$$

and

$$\sum_{i=1}^{m} w_i(A_{i1}\xi_{i1}e^{\xi_{i1}b} + A_{i2}\xi_{i2}e^{\xi_{i2}b} + A_{i0}) = 1. \tag{4.10}$$

By (4.9), we get A_{i1}, A_{i2} and B_i in the expression of b

$$A_{i1}(b) = \frac{1}{E_i(b)}\left[B_{i0}(\xi_{i3} - \xi_{i2})e^{\xi_{i2}b} + \frac{\delta_i}{(\delta_i + \gamma)^2}\xi_{i3}M\right], \tag{4.11}$$

$$A_{i2}(b) = -\frac{1}{E_i(b)}\left[B_{i0}(\xi_{i3} - \xi_{i1})e^{\xi_{i1}b} + \frac{\delta_i}{(\delta_i + \gamma)^2}\xi_{i3}M\right], \tag{4.12}$$

$$B_i(b) = \frac{1}{\xi_{i3}e^{\xi_{i3}b}}\left[A_{i1}(b)\xi_{i1}e^{\xi_{i1}b} + A_{i2}(b)\xi_{i2}e^{\xi_{i2}b}\right], \tag{4.13}$$

where

$$E_i(b) = e^{\xi_{i1}b}(\xi_{i3} - \xi_{i1}) - e^{\xi_{i2}b}(\xi_{i3} - \xi_{i2}). \tag{4.14}$$

Lemma 4.1. *Let $\mathbb{I} = \{1, 2, \cdots, m\}$. If $0 \le \gamma \le \min_{i\in\mathbb{I}}\left\{\frac{\delta_i\xi_{i3}M}{(\xi_{i1}-\xi_{i3})\theta}\right\}$, we have, for any $b > 0$ and all $i \in \mathbb{I}$,*

$$A_{i1}(b) < 0, \quad A_{i2}(b) > 0, \quad B_i(b) < 0.$$

Proof. Due to $\xi_{i1} < \xi_{xi3} < 0 < \xi_{i2}$, we have $E_i(b) > 0$. Furthermore, we get

$$B_{i0}(\xi_{i3} - \xi_{i2})e^{\xi_{i2}b} + \frac{\delta_i}{(\delta_i + \gamma)^2}\xi_{i3}M < 0.$$

Then we know $A_{i1}(b) < 0$ for $i = 1, 2, \cdots, m$. Since

$$B_{i0}(\xi_{i3} - \xi_{i1})e^{\xi_{i1}b} + \frac{\delta_i}{(\delta_i+\gamma)^2}\xi_{i3}M < B_{i0}(\xi_{i3} - \xi_{i1}) + \frac{\delta_i}{(\delta_i+\gamma)^2}\xi_{i3}M$$
$$= \frac{1}{(\delta_i+\gamma)^2}[\gamma\theta(\xi_{i3} - \xi_{i1}) + \delta_i\xi_{i3}M] < 0,$$

we obtain $A_{i2}(b) > 0$ for $i = 1, 2, \cdots, m$. Furthermore,

$$A_{i1}(b)\xi_{i1}e^{\xi_{i1}b} + A_{i2}(b)\xi_{i2}e^{\xi_{i2}b} > 0.$$

Hence $B_i(b) < 0$ for $i = 1, 2, \cdots, m$. □

Let

$$F(b) = \sum_{i=1}^{m} w_i \left(A_{i1}(b)\xi_{i1}e^{\xi_{i1}b} + A_{i2}(b)\xi_{i2}e^{\xi_{i2}b} + A_{i0}\right). \tag{4.15}$$

Lemma 4.2. *If* $\sum_{i=1}^m w_i \left[\frac{\gamma}{\delta_i+\gamma} - \frac{\xi_{i3}}{(\delta_i+\gamma)^2}(\gamma\theta + \delta_i M)\right] > 1$, *then the equation* $F(b) = 1$ *has a unique positive solution* $b > 0$.

Proof. Letting

$$\Delta_i(b) = A_{i1}(b)\xi_{i1}e^{\xi_{i1}b} + A_{i2}(b)\xi_{i2}e^{\xi_{i2}b}$$
$$= \frac{1}{E_i(b)}\left[B_{i0}\xi_{i3}(\xi_{i1} - \xi_{i2})e^{(\xi_{i1}+\xi_{i2})b} + \frac{\xi_{i3}\delta_i M}{(\delta_i+\gamma)^2}(\xi_{i1}e^{\xi_{i1}b} - \xi_{i2}e^{\xi_{i2}b})\right],$$

where $E_i(b)$ is defined by (4.14), we have

$$\Delta_i(0) = -\frac{\xi_{i3}}{(\delta_i+\gamma)^2}(\gamma\theta + \delta_i M), \quad \lim_{b\to\infty} \Delta_i(b) = \frac{\xi_{i2}}{\xi_{i3} - \xi_{i2}} \cdot \frac{\xi_{i3}\delta_i M}{(\delta_i+\gamma)^2}.$$

Hence, $F(0) = \sum_{i=1}^m w_i \left(\Delta_i(0) + \frac{\gamma}{\delta_i+\gamma}\right) > 1$. By $\frac{M}{\delta_i+\gamma} - \frac{1}{\xi_{i2}} + \frac{1}{\xi_{i3}} < 0$ in Lemma 4.2 of [12], we have

$$\lim_{b\to\infty}(F(b) - 1) = \sum_{i=1}^{m} w_i(\lim_{b\to\infty} \Delta_i(b) - \frac{\delta_i}{\delta_i+\gamma})$$
$$= \sum_{i=1}^{m} w_i \frac{\delta_i\xi_{i2}}{\delta_i+\gamma} \cdot \frac{\xi_{i3}}{\xi_{i3} - \xi_{i2}}\left(\frac{M}{\delta_i+\gamma} - \frac{1}{\xi_{i2}} + \frac{1}{\xi_{i3}}\right) < 0.$$

Differentiating $\Delta_i(b)$ and after some simplification, we obtain $\Delta_i'(b) = \frac{D_i(b)}{(E_i(b))^2}$ where

$$D_i(b) = B_{i0}\xi_{i3}(\xi_{i1} - \xi_{i2})e^{(\xi_{i1}+\xi_{i2})b}\left[\xi_{i2}(\xi_{i3} - \xi_{i1})e^{\xi_{i1}b} - \xi_{i1}(\xi_{i3} - \xi_{i2})e^{\xi_{i2}b}\right]$$
$$- \frac{\delta_i\xi_{i3}^2}{(\delta_i+\gamma)^2} \cdot (\xi_{i1} - \xi_{i2})^2 e^{(\xi_{i1}+\xi_{i2})b}.$$

Due to $\xi_{i2}(\xi_{i3} - \xi_{i1}) > 0$, $e^{\xi_{i1}b} \le 1$, $-\xi_{i1}(\xi_{i3} - \xi_{i2}) < 0$ and $e^{\xi_{i2}b} \ge 1$, we have

$$\xi_{i2}(\xi_{i3} - \xi_{i1})e^{\xi_{i1}b} - \xi_{i1}(\xi_{i3} - \xi_{i2})e^{\xi_{i2}b}$$
$$< \xi_{i2}(\xi_{i3} - \xi_{i1}) - \xi_{i1}(\xi_{i3} - \xi_{i2}) = \xi_{i3}(\xi_{i2} - \xi_{i1}) < 0.$$

Hence, we obtain $\Delta_i'(b) < 0$, and so $F'(b) = \sum_{i=1}^m w_i\Delta_i'(b) < 0$. Therefore, the equation $F(b) = 1$ admits a unique solution on $(0, \infty)$. □

Lemma 4.3. *Given the discount function (4.2) and the killing rate* $0 < \gamma \le \min_{i\in\mathbb{I}}\left(\frac{\delta_i\xi_{i3}M}{(\xi_{i1}-\xi_{i3})\theta}\right)$, *there exits a function* $c(s,t,x) \in \mathcal{C}^{0,1,1}(\mathcal{D}[0,\infty) \times [0,\infty)$ *satisfying the equilibrium HJB-equation (3.6).*

(i) If $\sum_{i=1}^m w_i \left[\frac{\gamma}{\delta_i+\gamma} - \frac{\xi_{i3}}{(\delta_i+\gamma)^2}(\gamma\theta+\delta_i M)\right] \le 1$*, then the function* $c(s,t,x)$ *is given by, for* $(s,t,x) \in \mathcal{D}[0,\infty)\times[0,\infty)$,

$$c(s,t,x) = \sum_{i=1}^m w_i e^{-(\delta_i+\gamma)(t-s)} \left[\frac{\gamma\theta+\delta_i M}{(\delta_i+\gamma)^2}(1-e^{\xi_{i3}x}) + \frac{\gamma}{\delta_i+\gamma}x\right]. \tag{4.16}$$

(ii) If $\sum_{i=1}^m w_i \left[\frac{\gamma}{\delta_i+\gamma} - \frac{\xi_{i3}}{(\delta_i+\gamma)^2}(\gamma\theta+\delta_i M)\right] > 1$*, then the function* $c(s,t,x)$ *is given by, for* $(s,t) \in \mathcal{D}[0,\infty)$,

$$c(s,t,x) = \begin{cases} \sum_{i=1}^m w_i e^{-(\delta_i+\gamma)(t-s)} \left[A_{i1}(b)e^{\xi_{i1}x} + A_{i2}(b)e^{\xi_{i2}x} + A_{i0}x + B_{i0}\right], & x\in[0,b), \\ \sum_{i=1}^m w_i e^{-(\delta_i+\gamma)(t-s)} \left[B_i(b)e^{\xi_{i3}x} + A_{i0}x + B_{i0} + \frac{\delta_i M}{(\delta_i+\gamma)^2}\right], & x\in[b,\infty), \end{cases} \tag{4.17}$$

where $A_{i1}(b)$, $A_{i2}(b)$, $B_i(b)$ $(i=1,2,\cdots,m)$ *are given respectively by (4.11)-(4.13),* $b>0$ *is determined by* $F(b)=1$*, and* A_{i0}, B_{i0} $(i=1,2,\cdots,m)$ *are given by (4.6).*

Proof. (i) Letting $b=0$ in (4.11)-(4.13) and combining (4.8), we obtain (4.16). Since

$$\frac{\partial c}{\partial x}(t,t,0) = \sum_{i=1}^m w_i\left[\frac{\gamma}{\delta_i+\gamma} - \frac{\xi_{i3}}{(\delta_i+\gamma)^2}(\gamma\theta+\delta_i M)\right] \le 1,$$

$$\frac{\partial^2 c}{\partial x^2}(t,t,x) = -\sum_{i=1}^m w_i \frac{\xi_{i3}^2}{(\delta_i+\gamma)^2}(\gamma\theta+\delta_i M)e^{\xi_{i3}x} < 0,$$

we have $\frac{\partial c}{\partial x}(t,t,0)\le 1$ for $x\ge 0$. Hence, $c(s,t,x)$ in (4.16) solves the equilibrium HJB-equation (3.6).

(ii) By Lemma 4.2, there exists a unique $b>0$ such that $F(b)=0$. It is sufficient to show $\frac{\partial c}{\partial x}(t,t,x)>1$ if $0<x<b$ and $\frac{\partial c}{\partial x}(t,t,x)\le 1$ if $x\ge b$.

For $x\ge b$, since

$$\frac{\partial^2 c}{\partial x^2}(t,t,x) = \sum_{i=1}^m w_i B_i(b)\xi_{i3}^2 e^{\xi_{i3}x} < 0,$$

we obtain $\frac{\partial c}{\partial x}(t,t,x) \le \frac{\partial c}{\partial x}(t,t,b+) = 1$. By (4.4) and (4.5), we get

$$\mu\frac{\partial^2 c}{\partial x^2}(t,t,b-) = (\mu+M)\frac{\partial^2 c}{\partial x^2}(t,t,b+) < 0.$$

For $0<x<b$, we have

$$\frac{\partial^3 c}{\partial x^3}(t,t,x) = \sum_{i=1}^m w_i(A_{i1}(b)\xi_{i1}^3 e^{\xi_{i1}x} + A_{i2}(b)\xi_{i2}^3 e^{\xi_{i2}x}) > 0.$$

Hence, for $0<x<b$,

$$\frac{\partial^2 c}{\partial x^2}(t,t,x) \le \frac{\partial^2 c}{\partial x^2}(t,t,b-) < 0.$$

Furthermore, $\frac{\partial c}{\partial x}(t,t,x) \ge \frac{\partial c}{\partial x}(t,t,b-) = 1$. The results are proved. □

By the above proof, we know that the functions $c(s,t,x)$ in (4.16) and (4.17) are increasing and concave with respective to x, and $\lim_{n\to\infty} c(s,n,x+\lambda\nu(n-t)) = 0$. By Theorem 3.1 and Remark 3.1, we obtain the following theorem.

Theorem 4.1. *Consider the discount function (4.2) and the killing rate* $0<\gamma\le \min_{i\in\mathbb{I}}\left(\frac{\delta_i\xi_{i3}M}{(\xi_{i1}-\xi_{i3})\theta}\right)$.

(i) If $\sum_{i=1}^m w_i \left[\frac{\gamma}{\delta_i+\gamma} - \frac{\xi_{i3}}{(\delta_i+\gamma)^2}(\gamma\theta+\delta_i M)\right] \le 1$, *then*

$$\hat{\mathbf{u}}(t,x) = M, \quad (t,x) \in [0,\infty)\times[0,\infty),$$

is an equilibrium strategy and for $(t,x) \in [0,\infty)\times[0,\infty)$,

$$V(t,x) = c(t,t,x) = \sum_{i=1}^m w_i \left[\frac{\gamma\theta+\delta_i M}{(\delta_i+\gamma)^2}(1-e^{\xi_{i3}x}) + \frac{\gamma}{\delta_i+\gamma}x\right]. \tag{4.18}$$

is the corresponding equilibrium value function.

(ii) If $\sum_{i=1}^m w_i \left[\frac{\gamma}{\delta_i+\gamma} - \frac{\xi_{i3}}{(\delta_i+\gamma)^2}(\gamma\theta+\delta_i M)\right] > 1$, *then*

$$\hat{\mathbf{u}}(t,x) = \begin{cases} 0, & (t,x) \in [0,\infty)\times[0,b), \\ M, & (t,x) \in [0,\infty)\times[b,\infty), \end{cases}$$

is an equilibrium strategy and the corresponding equilibrium value function is

$$V(t,x) = c(t,t,x) = \begin{cases} \sum_{i=1}^m w_i \left[A_{i1}(b)e^{\xi_{i1}x} + A_{i2}(b)e^{\xi_{i2}x} + A_{i0}x + B_{i0}\right], & x \in [0,b), \\ \sum_{i=1}^m w_i \left[B_i(b)e^{\xi_{i3}x} + A_{i0}x + B_{i0} + \frac{\delta_i M}{(\delta_i+\gamma)^2}\right], & x \in [b,\infty), \end{cases}$$

where $A_{i1}(b)$, $A_{i2}(b)$, $B_i(b)$ $(i = 1,2,\cdots,m)$ *are given respectively by (4.11)-(4.13),* $b > 0$ *is determined by* $F(b) = 1$, *and* A_{i0}, B_{i0} $(i = 1,2,\cdots,m)$ *are given by (4.6).*

Example Let $m = 2$, $\mu = 1$, $\lambda = 2$, $\beta = 1.5$, $M = 0.5$, $\gamma = 0.1$, $\delta_1 = 0.2$ and $\delta = 0.5$. We can verify that $\gamma \le \min_{i=1,2}\left(\frac{\delta_i\xi_{i3}M}{(\xi_{i1}-\xi_{i3})\theta}\right)$ holds and obtain

$$f_1 := \frac{\gamma}{\delta_1+\gamma} - \frac{\xi_{13}}{(\delta_1+\gamma)^2}\cdot(\gamma\theta+\delta_1 M) = 1.1698,$$

$$f_2 := \frac{\gamma}{\delta_2+\gamma} - \frac{\xi_{23}}{(\delta_2+\gamma)^2}\cdot(\gamma\theta+\delta_2 M) = 0.8750.$$

Therefore, the equilibrium dividend threshold $b > 0$ if and only if $w_1 f_1 + w_2 f_2 > 1$. For example, for $w_1 = \frac{1}{3}, \frac{2}{3}$, we have the threshold $b = 0$ and $b = 0.1393$, respectively. The corresponding equilibrium value functions are, respectively,

$$V(t,x) = -0.4938e^{-0.5646x} - 0.5247e^{-0.9000x} + 0.2222x + 1.0185, \quad x \ge 0,$$

and

$$V(t,x) = \begin{cases} Z_1(x) + 0.2778(x+1), & x \in [0, 0.1393), \\ Z_2(x) + 0.2778x + 1.2500, & x \in [0.1393,\infty), \end{cases}$$

where $Z_1(x) = -0.4371e^{-1.1810x} + 0.1902e^{0.3810x} - 0.1191e^{-1.6466x} + 0.0882e^{0.5466x}$ and $Z_2(x) = -0.9855e^{-0.5646x} - 0.2618e^{-0.9000x}$.

Finally, we discuss the impact of the killing rate γ to the control problem by some numerical examples. We still let $m = 2$, $\mu = 1$, $\lambda = 2$, $\beta = 1.5$, $M = 0.5$, $\delta_1 = 0.2$ and $\delta = 0.5$. Fig.1 shows the equilibrium dividend threshold b as a function of the killing rate γ for different values of w_1. We find that the larger w_1 leads to the higher threshold. Specially, in the case of $w_1 = 1$, the dividend problem is time-consistent. With the increase of γ, the level of b rises. Large γ means that it is earlier to kill the surplus process. Hence, it is better to enhance the dividend threshold in order to increase the surplus at the stopping time. In other words, it is advisable to retain more surplus at the stopping time rather than paying early out dividend. Accordingly, the equilibrium value function is larger as it is earlier to kill the surplus process, which is illustrated in Fig.2. When $w_1 = 0.6$, the dividend threshold are 0, 0.0292, 0.0983 and 0.1503 for the associated killing rate 0, 0.05, 0.1 and 0.15, respectively. Fig.2 implies that the

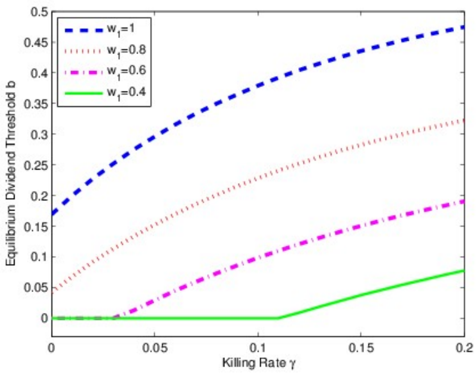

Fig. 1. Influence of γ on threshold b.

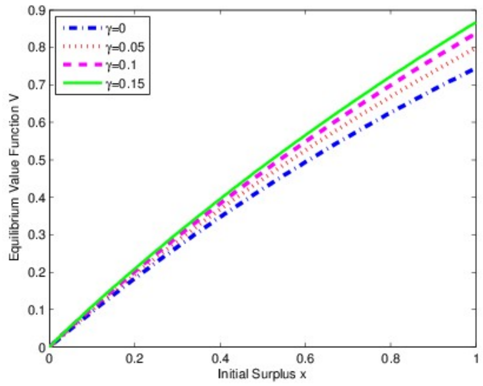

Fig. 2. Influence of γ on value function V.

equilibrium value function is increasing and concave with the initial surplus.

Declarations

Conflict of interest The authors declare no conflict of interest.

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