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Inhomogeneous Besov and Triebel-Lizorkin spaces associated with a para-accretive function and their applications

LIAO Fang-hui[1,*] LIU Zong-guang[2] ZHANG Xiao-jin[3]

Abstract. In this paper, using inhomogeneous Calderón's reproducing formulas and the space of test functions associated with a para-accretive function, the inhomogeneous Besov and Triebel-Lizorkin spaces are established. As applications, pointwise multiplier theorems are also obtained.

§1 Introduction

To study the L^2 boundedness of generalized Calderón-Zygmund singular integral operators, David and Journé [2] discovered the $T1$ theorem. However, the $T1$ theorem cannot be directly applied to the Cauchy integral on Lipschitz curves. If the function 1 in the $T1$ theorem is replaced by a bounded complex-valued function b which satisfies $0 < \delta < Re\ b(x)$ *a.e.*, Meyer obtained L^2 boundedness of the Cauchy integral on Lipschitz curves. McIntosh and Meyer [13] verified the Tb theorem, where b is an accretive function.

Replacing the accretive function with a para-accretive function, David, Journé and Semmes [3] proved a Tb theorem. Han and Sawyer [9] got a characterization of para-accretivity in terms of the weak boundedness property and proved a sharp Tb theorem for the Besov and Triebel-Lizorkin spaces. In [7], Han obtained Calderón's reproducing formula associated with a para-accretive function, introduced a class of Besov and Triebel-Lizokin spaces and verified a Tb theorem of the Besov and Triebel-Lizorkin spaces with $Tb = T^*b = 0$. For more results, see [4], [11] and [12].

Yang [14] obtained the inhomogeneous Calderón's reproducing formula associated with a para-accretive function. As usual, using inhomogeneous Calderón's reproducing formula and the space of test functions, one can formally define the inhomogeneous Besov and Triebel-Lizorkin spaces and obtain some properties of these spaces. A natural question arises: are

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*Corresponding author.

these inhomogeneous Besov and Triebel-Lizorkin spaces well defined? And furthermore, do the properties of classical Besov and Triebel-Lizorkin spaces still hold in this setting? The main purpose of this paper is to answer these questions. More precisely, using the inhomogeneous Calderón's reproducing formula and the space of test functions, we establish a class of inhomogeneous Besov and Triebel-Lizorkin spaces. As applications, the pointwise multiplier theorems on these spaces are also presented.

Throughout this paper, we use C to denote positive constants, whose value may change from one occurrence to the next. $f \sim g$ means that there exists a constant $C > 0$ independent of main parameters such that $C^{-1}g \le f \le Cg$. For any $1 < q < \infty$, let q' be its conjugate index, that is, $1/q + 1/q' = 1$. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Denote by M the Hardy-Littlewood maximal operator and by M_b the multiplication operator: $M_b(f) = bf$.

The paper is organized as follows. In Section 2, we introduce the inhomogeneous Besov and Triebel-Lizorkin spaces associated with a para-accretive function and verify that the spaces are independent of the choice of approximations to the identity. Section 3 is devoted to the pointwise multiplier theorems on these spaces.

§2 Definitions of Inhomogeneous Besov and Triebel-Lizorkin Spaces

First, we recall definitions of spaces of homogeneous type. A quasi-metric d on a set X is a function d: $X \times X \to [0, \infty)$ satisfying

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for $x, y \in X$;

(iii) There exists a constant $A \in [1, \infty)$ such that

$$d(x, y) \le A[d(x, z) + d(z, y)] \quad \text{for } x, y, z \in X. \tag{1}$$

A quasi-metric defines a topology for which the sets $\{B(x, r) : x \in X, r > 0\}$ form a basis, where $B(x, r) = \{y \in X : d(y, x) < r\}$.

The following definition of a space of homogeneous type was introduced by Coifman and Weiss [1].

Definition 2.1. A space of homogeneous type (X, ρ, μ) is a non-empty set X with a quasi-metric ρ and a nonnegative Borel regular measure μ on X such that $0 < \mu(B(x, r)) < \infty$ and there exists a constant $C < \infty$ such that

$$\mu(B(x, 2r)) \le C\mu(B(x, r)) \tag{2}$$

for $x \in X$ and $r > 0$, where μ is assumed to be defined on a $\sigma-$algebra which contains all Borel sets and all balls $B(x, r)$.

Throughout this paper, let $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for $x \in X$. Further, suppose that there exist constant $C > 0$ and $0 < \theta < 1$ such that for $0 < r < \infty$ and $x, x', y \in X$,

$$\mu(B(x, r)) \sim r, \tag{3}$$

and

$$|\rho(x,y)-\rho(x',y)| \le C\rho(x,x')^{\theta}[\rho(x,y)+\rho(x',y)]^{1-\theta}. \tag{4}$$

Now we recall definitions of para-accretive functions and the space of test functions.

Definition 2.2.[14] A bounded complex-valued function b defined on X is said to be a para-accretive if there exist constants $C, r > 0$, such that for any cube $Q \subset X$, there is subcube $Q' \subseteq Q$ with $r\mu(Q) \le \mu(Q')$ and

$$\frac{1}{\mu(Q)}\Big|\int_{Q'} b(x)\,d\mu(x)\Big| \ge C > 0.$$

Definition 2.3.[7] Fixed two exponents $0 < \beta \le \theta$ and $\gamma > 0$. A function f defined on X is said to be a test function of type (β,γ) centered at $x_0 \in X$ with width $r > 0$ if f satisfies

$$|f(x)| \le C\frac{r^{\gamma}}{(r+d(x,x_0))^{1+\gamma}} \tag{5}$$

for all $x \in X$, and

$$|f(x)-f(y)| \le C\Big(\frac{d(x,y)}{r+d(x,x_0)}\Big)^{\beta}\frac{r^{\gamma}}{(r+d(x,x_0))^{1+\gamma}} \tag{6}$$

for $d(x,y) \le \frac{1}{2A}(r+d(x,x_0))$.

If f is a test function of type (β,γ) center at $x_0 \in X$ and with width $r > 0$, we write $f \in \mathcal{G}(x_0,r,\beta,\gamma)$ and the norm of f in $\mathcal{G}(x_0,r,\beta,\gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\{C : (5) \;\text{ and }\; (6) \;\text{ hold}\}.$$

We denote $\mathcal{G}(\beta,\gamma) = \mathcal{G}(x_0,1,\beta,\gamma)$. It is easy to check that $\mathcal{G}(x_1,r,\beta,\gamma) = \mathcal{G}(\beta,\gamma)$ with equivalent norms for any $x_1 \in X$ and $r > 0$. Furthermore, $\mathcal{G}(\beta,\gamma)$ is a Banach space with respect to the norm on $\mathcal{G}(\beta,\gamma)$. Also, let the dual space $(\mathcal{G}(\beta,\gamma))'$ be the set of all linear functionals $\pounds$ from $\mathcal{G}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a constant C such that

$$|\pounds(f)| \le C\|f\|_{\mathcal{G}(\beta,\gamma)}$$

for all $f \in \mathcal{G}(\beta,\gamma)$. We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in (\mathcal{G}(\beta,\gamma))'$ and $f \in \mathcal{G}(\beta,\gamma)$. Clearly, for all $h \in (\mathcal{G}(\beta,\gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(\beta,\gamma)$.

For $\epsilon \in (0,\theta]$, let $\widetilde{\mathcal{G}}(\beta,\gamma)$ be the completion of the space $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$ with $0 < \beta,\gamma \le \epsilon$. Obviously, $\widetilde{\mathcal{G}}(\epsilon,\epsilon) = \mathcal{G}(\epsilon,\epsilon)$. Moreover, $f \in \widetilde{\mathcal{G}}(\beta,\gamma)$ if and only if $f \in \mathcal{G}(\beta,\gamma)$ with $0 < \beta,\gamma \le \epsilon$ and there exists $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{G}(\epsilon,\epsilon)$ such that $\|f - f_j\|_{\mathcal{G}(\beta,\gamma)} \to 0$ as $j \to \infty$. If $f \in \widetilde{\mathcal{G}}(\beta,\gamma)$, we define $\|f\|_{\widetilde{\mathcal{G}}(\beta,\gamma)} = \|f\|_{\mathcal{G}(\beta,\gamma)}$. Obviously, $\widetilde{\mathcal{G}}(\beta,\gamma)$ is a Banach space and we also have $\|f\|_{\widetilde{\mathcal{G}}(\beta,\gamma)} = \lim_{j\to\infty}\|f_j\|_{\mathcal{G}(\beta,\gamma)}$ for above $\{f_j\}_{j\in\mathbb{N}}$.

Define

$$b\mathcal{G}(\beta,\gamma) = \Big\{f : f = bg \;\text{ for some }\; g \in \mathcal{G}(\beta,\gamma)\Big\},$$

where b is a para-accretive function. If $f \in b\mathcal{G}(\beta,\gamma)$ and $f = bg$ for $g \in \mathcal{G}(\beta,\gamma)$, then the norm of f is defined by $\|f\|_{b\mathcal{G}(\beta,\gamma)} = \|g\|_{\mathcal{G}(\beta,\gamma)}$. For $f \in \Big(b\widetilde{\mathcal{G}}(\beta,\gamma)\Big)'$, we define $bf \in \Big(\widetilde{\mathcal{G}}(\beta,\gamma)\Big)'$ by $\langle bf, g\rangle = \langle f, bg\rangle$ for $g \in \widetilde{\mathcal{G}}(\beta,\gamma)$. Then it is easy to see that

$$f \in \Big(b\widetilde{\mathcal{G}}(\beta,\gamma)\Big)' \;\text{ if and only if }\; bf \in \Big(\widetilde{\mathcal{G}}(\beta,\gamma)\Big)'. \tag{7}$$

The approximation to the identity associated with a para-accretive function is defined by

Definition 2.4. [7] Let b be a para-accretive function. A sequence $\{S_k\}_{k\in\mathbb{Z}_+}$ of linear operators is said to be an approximation to the identity of order $\epsilon\in(0,\theta]$ associated with b if there exists $C>0$ such that for all $k\in\mathbb{Z}_+$ and all $x,x',y,y'\in X$, $S_k(x,y)$, the kernel of S_k, is a function from $X\times X$ into $\mathbb{C}$ satisfying
(i)

$$|S_k(x,y)|\le C\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}};$$

(ii)

$$|S_k(x,y)-S_k(x',y)|\le C\Big(\frac{d(x,x')}{2^{-k}+d(x,y)}\Big)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}}$$

for $d(x,x')\le\frac{1}{2A}\big(2^{-k}+d(x,y)\big)$;
(iii)

$$|S_k(x,y)-S_k(x,y')|\le C\Big(\frac{d(y,y')}{2^{-k}+d(x,y)}\Big)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}}$$

for $d(y,y')\le\frac{1}{2A}\big(2^{-k}+d(x,y)\big)$;
(iv)

$$\begin{aligned}&|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\\ \le& C\Big(\frac{d(x,x')}{2^{-k}+d(x,y)}\Big)^{\epsilon}\Big(\frac{d(y,y')}{2^{-k}+d(x,y)}\Big)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}}\end{aligned}$$

for $d(x,x')\le\frac{1}{(2A)^2}\big(2^{-k}+d(x,y)\big)$ and $d(y,y')\le\frac{1}{(2A)^2}\big(2^{-k}+d(x,y)\big)$;
(v)

$$\int_X S_k(x,y)b(y)\,d\mu(y)=\int_X S_k(x,y)b(x)\,d\mu(x)=1.$$

The following lemmas are Calderón's reproducing formulas.

Lemma 2.5. *[14] Let b be a para-accretive function, $\epsilon\in(0,\theta]$, $\{S_k\}_{k\in\mathbb{Z}_+}$ be an approximation to the identity of order ϵ. Set $D_k=S_k-S_{k-1}$ when $k\in\mathbb{N}$ and $D_0=S_0$. Then there exist families of linear operators $\widetilde{D}_k$ and $\widetilde{\widetilde{D}}_k$ for $k\in\mathbb{Z}_+$ such that for all $f\in\widetilde{\mathcal{G}}(\beta,\gamma)$ with $0<\beta,\gamma<\epsilon$,*

$$f=\sum_{k=0}^{\infty}\widetilde{D}_kM_bD_kM_b(f)=\sum_{k=0}^{\infty}D_kM_b\widetilde{\widetilde{D}}_kM_b(f),\tag{8}$$

where the series converges in the norm of $\widetilde{\mathcal{G}}(\beta',\gamma')$ for $0<\beta'<\beta$ and $0<\gamma'<\gamma$, and in the norm of L^p with $1<p<\infty$. When $f\in(b\widetilde{\mathcal{G}}(\beta,\gamma))'$, the series (8) converges in the norm of $(b\widetilde{\mathcal{G}}(\beta',\gamma'))'$ with $\beta<\beta'<\theta,\gamma<\gamma'<\theta$. Moreover, $\widetilde{D}_k(x,y)$, the kernel of $\widetilde{D}_k$, satisfies (i) and (ii) in Definition 2.4 with ϵ is replaced by ϵ' for $0<\epsilon'<\epsilon$,

$$\int_X \widetilde{D}_0(x,y)b(y)\,d\mu(y)=\int_X \widetilde{D}_0(x,y)b(x)\,d\mu(x)=1,\tag{9}$$

and for $k\in\mathbb{N}$

$$\int_X \widetilde{D}_k(x,y)b(y)\,d\mu(y)=\int_X \widetilde{D}_k(x,y)b(x)\,d\mu(x)=0.\tag{10}$$

$\widetilde{\widetilde{D}}_k(x,y)$, the kernel of $\widetilde{\widetilde{D}}_k$, satisfies (i) and (iii) in Definition 2.4 with ϵ replaced by ϵ' for

$0 < \epsilon' < \epsilon$, *(9) and (10).*

Lemma 2.6. *[14] Suppose that all the notation as in Lemma 2.5, then for all* $f \in b\widetilde{\mathcal{G}}(\beta,\gamma)$,

$$f = \sum_{k=0}^{\infty} M_b \widetilde{D}_k M_b D_k(f) = \sum_{k=0}^{\infty} M_b D_k M_b \widetilde{\widetilde{D}}_k(f), \tag{11}$$

where the series converges in the norm of $b\widetilde{\mathcal{G}}(\beta',\gamma')$ *with* $0 < \beta' < \beta$, $0 < \gamma' < \gamma$, *and in the norm of* L^p *with* $1 < p < \infty$. *When* $f \in (\widetilde{\mathcal{G}}(\beta,\gamma))'$, *the series (11) converges in the norm of* $(\widetilde{\mathcal{G}}(\beta',\gamma'))'$ *with* $\beta < \beta' < \theta, \gamma < \gamma' < \theta$.

We now introduce the inhomogeneous Besov and Triebel-Lizorkin spaces associated with a para-accretive function.

Definition 2.7. Suppose that b is a para-accretive function. Let $1 < p, q < \infty$, $|s| < \epsilon$ and $\{D_k\}_{k\in\mathbb{Z}_+}$ be defined as in Lemma 2.5. The inhomogeneous Besov spaces $B^{s,q}_{p,b^{-1}}$ and $B^{s,q}_{p,b}$ are defined by

$$B^{s,q}_{p,b^{-1}} = \{f \in (b\widetilde{\mathcal{G}}(\beta,\gamma))' : \|f\|_{B^{s,q}_{p,b^{-1}}} < \infty\}$$

and

$$B^{s,q}_{p,b} = \{f \in (\widetilde{\mathcal{G}}(\beta,\gamma))' : \|f\|_{B^{s,q}_{p,b}} < \infty\},$$

where

$$\|f\|_{B^{s,q}_{p,b^{-1}}} = \|D_0 M_b(f)\|_{L^p} + \left\{\sum_{k=1}^{\infty}\left(2^{ks}\|D_k M_b(f)\|_{L^p}\right)^q\right\}^{1/q}$$

and

$$\|f\|_{B^{s,q}_{p,b}} = \|D_0(f)\|_{L^p} + \left\{\sum_{k=1}^{\infty}\left(2^{ks}\|D_k(f)\|_{L^p}\right)^q\right\}^{1/q}.$$

The inhomogeneous Triebel-Lizorkin spaces $F^{s,q}_{p,b^{-1}}$ and $F^{s,q}_{p,b}$ are defined by

$$F^{s,q}_{p,b^{-1}} = \{f \in (b\widetilde{\mathcal{G}}(\beta,\gamma))' : \|f\|_{F^{s,q}_{p,b^{-1}}} < \infty\}$$

and

$$F^{s,q}_{p,b} = \{f \in (\widetilde{\mathcal{G}}(\beta,\gamma))' : \|f\|_{F^{s,q}_{p,b}} < \infty\},$$

where

$$\|f\|_{F^{s,q}_{p,b^{-1}}} = \|D_0 M_b(f)\|_{L^p} + \left\|\left\{\sum_{k=1}^{\infty}\left(2^{ks}|D_k M_b(f)|\right)^q\right\}^{1/q}\right\|_{L^p}$$

and

$$\|f\|_{F^{s,q}_{p,b}} = \|D_0(f)\|_{L^p} + \left\|\left\{\sum_{k=1}^{\infty}\left(2^{ks}|D_k(f)|\right)^q\right\}^{1/q}\right\|_{L^p}.$$

To see that the Besov and Triebel-Lizorkin spaces are well defined, we need to show that definitions of the spaces $B^{s,q}_{p,b^{-1}}$, $B^{s,q}_{p,b}$, $F^{s,q}_{p,b^{-1}}$ and $F^{s,q}_{p,b}$ are independent of the choice of approximations to the identity. This follows from

Theorem 2.8. *Let* b *be a para-accretive function and* $1 < p, q < \infty$. *Suppose that* $\{S_k\}_{k\in\mathbb{Z}_+}$ *and* $\{P_k\}_{k\in\mathbb{Z}_+}$ *are two approximations to the identity associated with* b. *Let* $D_0 = S_0, D_k = S_k - S_{k-1}$ *and* $E_0 = P_0, E_k = P_k - P_{k-1}$ *for all* $k \in \mathbb{N}$. *Then* *(i) for all* $f \in \left(b\widetilde{\mathcal{G}}(\beta,\gamma)\right)'$,

$$\|E_0M_b(f)\|_{L^p} + \Big\|\Big\{\sum_{l=1}^{\infty}\Big(2^{ls}|E_lM_b(f)|\Big)^q\Big\}^{1/q}\Big\|_{L^p} \sim \|D_0M_b(f)\|_{L^p} + \Big\|\Big\{\sum_{k=1}^{\infty}\Big(2^{ks}|D_kM_b(f)|\Big)^q\Big\}^{1/q}\Big\|_{L^p} \tag{12}$$

and

$$\|E_0M_b(f)\|_{L^p} + \Big\{\sum_{l=1}^{\infty}\Big(2^{ls}\|E_lM_b(f)\|_{L^p}\Big)^q\Big\}^{1/q} \sim \|D_0M_b(f)\|_{L^p} + \Big\{\sum_{k=1}^{\infty}\Big(2^{ks}\|D_kM_b(f)\|_{L^p}\Big)^q\Big\}^{1/q}. \tag{13}$$

(ii) for all $f \in \left(\widetilde{\mathcal{G}}(\beta,\gamma)\right)'$,

$$\|E_0(f)\|_{L^p} + \Big\|\Big\{\sum_{l=1}^{\infty}\Big(2^{ls}|E_l(f)|\Big)^q\Big\}^{1/q}\Big\|_{L^p} \sim \|D_0(f)\|_{L^p} + \Big\|\Big\{\sum_{k=1}^{\infty}\Big(2^{ks}|D_k(f)|\Big)^q\Big\}^{1/q}\Big\|_{L^p} \tag{14}$$

and

$$\|E_0(f)\|_{L^p} + \Big\{\sum_{l=1}^{\infty}\Big(2^{ls}\|E_l(f)\|_{L^p}\Big)^q\Big\}^{1/q} \sim \|D_0(f)\|_{L^p} + \Big\{\sum_{k=1}^{\infty}\Big(2^{ks}\|D_k(f)\|_{L^p}\Big)^q\Big\}^{1/q}. \tag{15}$$

Noticing that $\int_X b(y)\widetilde{D}_0(y,x)d\mu(y) = 1$, similar to the proof of [4] and [8], we get the following almost orthogonality estimate. Here we omit the proof.

Proposition 2.9. *Let b be a para-accretive function, $\epsilon \in (0,\theta]$. Suppose that E_l, $\widetilde{D}_k$ are defined as in Theorem 2.8, Lemma 2.5, respectively. For any $\epsilon' \in (0,\epsilon)$ and $\epsilon'' \in (0,\epsilon')$, there exists a constant $C > 0$ such that*

$$|E_lM_b\widetilde{D}_k(x,y)| \le C2^{-|k-l|\epsilon''}\frac{2^{-(k\wedge l)\epsilon'}}{\left(2^{-(k\wedge l)}+d(x,y)\right)^{1+\epsilon'}} \tag{16}$$

for all $x, y \in X$.

Using Calderón's reproducing formula and the almost orthogonality estimate, we give the *Proof of Theorem 2.8.* We first verify (12). By Lemma 2.5, we have

$$f(x) = \sum_{k=0}^{\infty}\widetilde{D}_kM_bD_kM_b(f)(x) = \sum_{k=0}^{\infty}\int_X \widetilde{D}_k(x,y)b(y)D_kM_b(f)(y)\,d\mu(y),$$

and thus

$$E_lM_bf(x) = \sum_{k=0}^{\infty}\int_X E_lM_b\widetilde{D}_k(x,y)b(y)D_kM_b(f)(y)\,d\mu(y).$$

From Proposition 2.9, then

$$
\begin{aligned}
|E_l M_b f(x)| &\leq C \sum_{k=0}^{\infty} \int_X 2^{-|k-l|\epsilon''} \frac{2^{-(k\wedge l)\epsilon'}}{\big(2^{-(k\wedge l)} + d(x,z)\big)^{1+\epsilon'}} |b(y) D_k M_b(f)(y)| \, d\mu(y) \\
&\leq C \sum_{k=0}^{\infty} 2^{-|k-l|\epsilon''} M(D_k M_b(f))(x).
\end{aligned} \tag{17}
$$

Using Minkowski's inequality, Hölder's inequality, Fefferman-Stein's vector-valued maximal inequality in [5] and (17), we have

$$
\begin{aligned}
\|E_0 M_b f\|_{L^p} &\leq C \Big\| \sum_{k=0}^{\infty} 2^{-k\epsilon''} M(D_k M_b(f)) \Big\|_{L^p} \\
&\leq C \|M(D_0 M_b(f))\|_{L^p} + C \Big\| \sum_{k=1}^{\infty} 2^{-k(s+\epsilon'')} 2^{ks} M(D_k M_b(f)) \Big\|_{L^p} \\
&\leq C \|D_0 M_b(f)\|_{L^p} + C \Big\| \Big\{ \sum_{k=1}^{\infty} \Big(2^{ks} M(D_k M_b(f)) \Big)^q \Big\}^{1/q} \Big\|_{L^p} \\
&\leq C \|D_0 M_b(f)\|_{L^p} + C \Big\| \Big\{ \sum_{k=1}^{\infty} \Big(2^{ks} |D_k M_b(f)| \Big)^q \Big\}^{1/q} \Big\|_{L^p}
\end{aligned}
$$

and

$$
\begin{aligned}
& \Big\| \Big\{ \sum_{l=1}^{\infty} \Big(2^{ls} |E_l M_b(f)| \Big)^q \Big\}^{1/q} \Big\|_{L^p} \\
\leq & C \Big\| \Big\{ \sum_{l=1}^{\infty} \Big(\sum_{k=0}^{\infty} 2^{ls} 2^{-|k-l|\epsilon''} M(D_k M_b(f)) \Big)^q \Big\}^{1/q} \Big\|_{L^p} \\
\leq & C \Big\| \Big\{ \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} 2^{-(k-l)s} 2^{-|k-l|\epsilon''} \Big(2^{ks} M(D_k M_b(f)) \Big)^q \Big\}^{1/q} \Big\|_{L^p} \\
\leq & C \Big\| \Big\{ \sum_{k=0}^{\infty} \Big(2^{ks} M(D_k M_b(f)) \Big)^q \Big\}^{1/q} \Big\|_{L^p} \\
\leq & C \|D_0 M_b(f)\|_{L^p} + \Big\| \Big\{ \sum_{k=1}^{\infty} \Big(2^{ks} |D_k M_b(f)| \Big)^q \Big\}^{1/q} \Big\|_{L^p},
\end{aligned}
$$

where

$$
\begin{aligned}
\sup_{k\in\mathbb{Z}_+} \sum_{l=1}^{\infty} 2^{-(k-l)s} 2^{-|k-l|\epsilon''} &< \infty, \\
\sup_{l\in\mathbb{N}} \sum_{k=0}^{\infty} 2^{-(k-l)s} 2^{-|k-l|\epsilon''} &< \infty
\end{aligned} \tag{18}
$$

with $|s| < \epsilon''$.

We now consider (13). Applying Lemma 2.5, Minkowski's inequality, Hölder's inequality and (17) yields

$$
\|E_0 M_b(f)\|_{L^p} \leq C \sum_{k=0}^{\infty} 2^{-k\epsilon''} \|M(D_k M_b(f))\|_{L^p}
$$

$$\leq C\|D_0M_b(f)\|_{L^p}+C\sum_{k=1}^{\infty}2^{-k(\epsilon''+s)}2^{ks}\|D_kM_b(f)\|_{L^p}$$
$$\leq C\|D_0M_b(f)\|_{L^p}+C\Big\{\sum_{k=1}^{\infty}\Big(2^{ks}\|D_kM_b(f)\|_{L^p}\Big)^q\Big\}^{1/q}.$$

Consequently,

$$\Big\{\sum_{l=1}^{\infty}\Big(2^{ls}\|E_lM_b(f)\|_{L^p}\Big)^q\Big\}^{1/q}$$
$$\leq C\Big\{\sum_{l=1}^{\infty}\Big(2^{ls}\sum_{k=0}^{\infty}2^{-|k-l|\epsilon''}\|M(D_kM_b(f))\|_{L^p}\Big)^q\Big\}^{1/q}$$
$$\leq C\Big\{\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}2^{(l-k)s}2^{-|k-l|\epsilon''}\Big(2^{ks}\|M(D_kM_b(f))\|_{L^p}\Big)^q\Big\}^{1/q}$$
$$\leq C\|D_0M_b(f)\|_{L^p}+C\Big\{\sum_{k=1}^{\infty}\Big(2^{ks}\|D_kM_b(f)\|_{L^p}\Big)^q\Big\}^{1/q},$$

where s satisfies (18).

By Lemma 2.6 and Proposition 2.9, we can get the estimates (14) and (15) by an argument similar to the estimate of (12) and (13). Here we omit the details. □

Remark 2.10. Let $D_k, \tilde{\tilde{D}}_k$ be the same as in Lemma 2.5 for $k\in\mathbb{Z}_+$. Since, in the proof of Theorem 2.8, only the smoothness condition of D_k for the second variable was used, we see that Theorem 2.8 continues to hold when D_k is replaced by $\tilde{\tilde{D}}_k$. The proof of this result is similar to the remark in [10].

The following result gives the relationship between different inhomogeneous Besov and Triebel-Lizorkin spaces.

Proposition 2.11. *Suppose that b is a para-accretive function. Let $|s|<\epsilon$ and $1<p,q<\infty$. Then*
(i) $f\in B^{s,q}_{p,b^{-1}}$ if and only if $bf\in B^{s,q}_{p,b}$. Moreover,
$$\|f\|_{B^{s,q}_{p,b^{-1}}}\sim\|bf\|_{B^{s,q}_{p,b}}.$$
(ii) $f\in F^{s,q}_{p,b^{-1}}$ if and only if $bf\in F^{s,q}_{p,b}$. Moreover,
$$\|f\|_{F^{s,q}_{p,b^{-1}}}\sim\|bf\|_{F^{s,q}_{p,b}}.$$

Proof. We only verify (i), since the proof of (ii) is similar. Let $f\in B^{s,q}_{p,b^{-1}}$, we get $f\in\Big(b\widetilde{\mathcal{G}}(\beta,\gamma)\Big)'$. Thus, $bf\in\Big(\widetilde{\mathcal{G}}(\beta,\gamma)\Big)'$ and

$$\|f\|_{B^{s,q}_{p,b^{-1}}}\sim\|D_0M_b(f)\|_{L^p}+\Big\{\sum_{k=1}^{\infty}\Big(2^{ks}\|D_kM_b(f)\|_{L^p}\Big)^q\Big\}^{1/q}\sim\|M_b(f)\|_{B^{s,q}_{p,b}}.$$

If $bf\in B^{s,q}_{p,b}$, we have $bf\in\Big(\widetilde{\mathcal{G}}(\beta,\gamma)\Big)'$. So we have $f\in\Big(b\widetilde{\mathcal{G}}(\beta,\gamma)\Big)'$ and

$$\|f\|_{B^{s,q}_{p,b}}\sim\|D_0M_b(f)\|_{L^p}+\Big\{\sum_{k=1}^{\infty}\Big(2^{ks}\|D_k(bf)\|_{L^p}\Big)^q\Big\}^{1/q}\sim\|f\|_{B^{s,q}_{p,b^{-1}}},$$

which finishes the proof of (i). □

We now show the density property of the space of test functions in the inhomogeneous Besov and Triebel-Lizorkin spaces associated with a para-accretive function.

Proposition 2.12. *Suppose that b is a para-accretive function. Let $0 < \beta, \gamma < \epsilon$ and $|s| < \epsilon$. Then*
(i) $b\mathcal{G}(\sigma, \sigma)$ is dense in $B^{s,q}_{p,b}$ and $F^{s,q}_{p,b}$ with $|s| < \sigma < \epsilon$.
(ii) $\mathcal{G}(\sigma, \sigma)$ is dense in $B^{s,q}_{p,b^{-1}}$ and $F^{s,q}_{p,b^{-1}}$ with $|s| < \sigma < \epsilon$.

Proof. (i) Let $f \in B^{s,q}_{p,b}$. For $N \in \mathbb{N}$, we put

$$f_N = \sum_{k=0}^{N} M_b D_k M_b \widetilde{\widetilde{D}}_k(f).$$

Similar to the proof of Proposition 2.9, for any $\sigma \in (0, \epsilon)$, we have the following almost orthogonality estimate

$$|E_l M_b D_k(x, y)| \le C 2^{-|k-l|\sigma} \frac{2^{-(k\wedge l)\epsilon}}{(2^{-(k\wedge l)} + d(x, y))^{1+\epsilon}}, \tag{19}$$

where E_l and D_k are defined as in Theorem 2.8 for $k, l \in \mathbb{Z}_+$. By the definition of test functions, we see that $f_N \in b\mathcal{G}(\sigma, \sigma)$, and repeating the proof of the Theorem 2.8, we obtain $\|f_N\|_{B^{s,q}_{p,b}} \le \|f\|_{B^{s,q}_{p,b}}$. It follows from the almost orthogonality estimate (19), Minkowski's inequality and Hölder's inequality that

$$\begin{aligned}
\|f - f_N\|_{B^{s,q}_{p,b}} =& \Big\| \sum_{k=N+1}^{\infty} M_b D_k M_b \widetilde{\widetilde{D}}_k(f) \Big\|_{B^{s,q}_{p,b}} \\
=& \Big\{ \sum_{l=0}^{\infty} \Big(2^{ls} \Big\| E_l \Big(\sum_{k=N+1}^{\infty} M_b D_k M_b \widetilde{\widetilde{D}}_k(f) \Big) \Big\|_{L^p} \Big)^q \Big\}^{1/q} \\
\le& C \Big\{ \sum_{l=0}^{\infty} \Big(2^{ls} \sum_{k=N+1}^{\infty} \|E_l M_b D_k M_b\|_{L^1} \|\widetilde{\widetilde{D}}_k(f)\|_{L^p} \Big)^q \Big\}^{1/q} \\
\le& C \Big\{ \sum_{l=0}^{\infty} \Big(2^{ls} \sum_{k=N+1}^{\infty} 2^{-|k-l|\sigma} \|\widetilde{\widetilde{D}}_k(f)\|_{L^p} \Big)^q \Big\}^{1/q} \\
\le& C \Big\{ \sum_{k=N+1}^{\infty} \Big(2^{ks} \|\widetilde{\widetilde{D}}_k(f)\|_{L^p} \Big)^q \Big\}^{1/q},
\end{aligned}$$

where $|s| < \sigma < \epsilon$. This implies that f_N tends to f in $B^{s,q}_{p,b}$ as N tends to infinity. In the same way, we can prove that $b\mathcal{G}(\sigma, \sigma)$ is dense in $F^{s,q}_{p,b}$ with $|s| < \sigma < \epsilon$. And proof of (ii) is similar to the proof of (i). Here we omit the details. □

§3 Pointwise Multiplier Theorems

In this section, we study the pointwise multiplier theory on the inhomogeneous Besov and Triebel-Lizorkin spaces associated with a para-accretive function b. When $b = 1$, the results in this section are the related results in [15], [16] and [6].

We first recall definitions of Hölder-Zygmund spaces.

Definition 3.1. The Hölder-Zygmund space $\mathcal{C}^\alpha$, $\alpha > 0$, consists of all bounded continuous functions f such that

$$\|f\|_{\mathcal{C}^\alpha} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} < \infty. \tag{20}$$

Now we present the definitions of pointwise multipliers for the inhomogeneous Besov and Triebel-Lizorkin spaces associated with a para-accretive function.

Definition 3.2. A function g on X is called a pointwise multiplier for $B^{s,q}_{p,b^{-1}}$ if $f \to gf$ admits a bounded linear mapping from $B^{s,q}_{p,b^{-1}}$ into itself. Similarly, g is called a pointwise multiplier for $B^{s,q}_{p,b}$ if $f \to gf$ admits a bounded linear mapping from $B^{s,q}_{p,b}$ into itself.

Definition 3.3. A function g defined on X is called a pointwise multiplier for $F^{s,q}_{p,b^{-1}}$ if $f \to gf$ admits a bounded linear mapping from $F^{s,q}_{p,b^{-1}}$ into itself. Similarly, g is called a pointwise multiplier for $F^{s,q}_{p,b}$ if $f \to gf$ admits a bounded linear mapping from $F^{s,q}_{p,b}$ into itself.

The main results in this section are as follows.

Theorem 3.4. *Let $|s| < \epsilon, 1 < p, q < \infty$, then $g \in \mathcal{C}^\alpha$ is a multiplier for $B^{s,q}_{p,b^{-1}}$ with $\max(s,-s) < \alpha < \epsilon$. Moreover, $f \to gf$ yields a bounded linear mapping from $B^{s,q}_{p,b^{-1}}$ into itself and there exists a positive constant C such that*

$$\|gf\|_{B^{s,q}_{p,b^{-1}}} \leq C\|g\|_{\mathcal{C}^\alpha}\|f\|_{B^{s,q}_{p,b^{-1}}}$$

for all $g \in \mathcal{C}^\alpha$ and $f \in B^{s,q}_{p,b^{-1}}$.

Theorem 3.5. *Let all the notation be the same as in Theorem 3.4. Then the conclusion of Theorem 3.4 continues to hold when $B^{s,q}_{p,b^{-1}}$ is replaced by $B^{s,q}_{p,b}$(or $F^{s,q}_{p,b^{-1}}$, or $F^{s,q}_{p,b}$).*

Before presenting the proof of these main results, we first give the following technical version of Theorems 3.4 and 3.5.

Proposition 3.6. *Let $|s| < \epsilon, 1 < p, q < \infty$ and $\max(s,-s) < \beta < \epsilon, 0 < \gamma < \epsilon$. Suppose that $g \in \mathcal{C}^\alpha$ with $\max(s,-s) < \alpha < \epsilon$.*
(i) If $f \in \widetilde{\mathcal{G}}(\beta,\gamma)$, then

$$\|fg\|_{B^{s,q}_{p,b^{-1}}} \leq C\|g\|_{\mathcal{C}^\alpha}\|f\|_{B^{s,q}_{p,b^{-1}}} \tag{21}$$

and

$$\|fg\|_{F^{s,q}_{p,b^{-1}}} \leq C\|g\|_{\mathcal{C}^\alpha}\|f\|_{F^{s,q}_{p,b^{-1}}}; \tag{22}$$

(ii) If $f \in b\widetilde{\mathcal{G}}(\beta,\gamma)$, then

$$\|fg\|_{B^{s,q}_{p,b}} \leq C\|g\|_{\mathcal{C}^\alpha}\|f\|_{B^{s,q}_{p,b}} \tag{23}$$

and

$$\|fg\|_{F^{s,q}_{p,b}} \leq C\|g\|_{\mathcal{C}^\alpha}\|f\|_{F^{s,q}_{p,b}}. \tag{24}$$

In order to prove Proposition 3.6, we need the following almost orthogonality estimate.

Lemma 3.7. *Let $0 < \sigma < \epsilon$ and D_k, E_l be defined as in Theorem 2.8 for $k, l \in \mathbb{Z}_+$. For any $\epsilon \in (0, \theta]$ and $g \in \mathcal{C}^\alpha$ with $0 < \alpha < \epsilon$, we have*

$$|D_k M_b g E_l(x,y)b(y)| \leq C\|g\|_{\mathcal{C}^\alpha} 2^{-|k-l|(\alpha\wedge\sigma)} \frac{2^{-(k\wedge l)\epsilon}}{(2^{-(k\wedge l)} + d(x,y))^{1+\epsilon}}. \tag{25}$$

Proof. We first consider the case $k = l = 0$. By definitions of Hölder-Zygmund spaces and the size conditions of D_0 and E_0, we get

$$\begin{aligned}
|D_0 M_b g E_0(x,y)b(y)| =& \Big| \int_X D_0(x,z)b(z)g(z)E_0(z,y)b(y)\,d\mu(z) \Big| \\
\leq& C\|g\|_{\mathcal{C}^\alpha} \int_X \frac{1}{(1+d(x,z))^{1+\epsilon}} \frac{1}{(1+d(z,y))^{1+\epsilon}}\,d\mu(z) \\
\leq& C\|g\|_{\mathcal{C}^\alpha} \frac{1}{(1+d(x,y))^{1+\epsilon}}.
\end{aligned}$$

When $l \geq k \geq 1$, since $\int_X b(z)E_l(z,y)d\mu(z) = 0$, we have

$$\begin{aligned}
|D_k M_b g E_l(x,y)b(y)| =& \Big| \int_X D_k(x,z)b(z)g(z)E_l(z,y)b(y)\,d\mu(z) \Big| \\
\leq& \int_X |D_k(x,z) - D_k(x,y)||b(z)g(z)||E_l(z,y)b(y)|\,d\mu(z) \\
&+ \int_X |D_k(x,y)||g(z)-g(y)||b(z)||E_l(z,y)b(y)|\,d\mu(z) \\
:=& L_1 + L_2.
\end{aligned}$$

We now estimate L_1 by dividing it into

$$\begin{aligned}
L_1 \leq& \int_{W_1} |D_k(x,z) - D_k(x,y)||b(z)g(z)||E_l(z,y)b(y)|\,d\mu(z) \\
&+ \int_{W_2} |D_k(x,z)||g(z)b(z)||E_l(z,y)b(y)|\,d\mu(z) \\
&+ \int_{W_2} |D_k(x,y)||g(z)b(z)||E_l(z,y)b(y)|\,d\mu(z) \\
:=& L_{1.1} + L_{1.2} + L_{1.3},
\end{aligned}$$

where $W_1 = \{z \in X : d(z,y) \leq \frac{1}{2A}(2^{-k} + d(x,y))\}$ and $W_2 = X \backslash W_1$.

For $L_{1.1}$, we have

$$\begin{aligned}
L_{1.1} \leq& C\|g\|_{\mathcal{C}^\alpha} \int_{W_1} \Big(\frac{d(z,y)}{2^{-k}+d(x,y)}\Big)^\sigma \frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}} \frac{2^{-l\epsilon}}{(2^{-l}+d(z,y))^{1+\epsilon}}\,d\mu(z) \\
\leq& C\|g\|_{\mathcal{C}^\alpha} \frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}} 2^{-(l-k)\sigma} \int_X \frac{2^{-l(\epsilon-\sigma)}}{(2^{-l}+d(z,y))^{1+\epsilon-\sigma}}\,d\mu(z) \\
\leq& C\|g\|_{\mathcal{C}^\alpha} 2^{-(l-k)\sigma} \frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}},
\end{aligned}$$

where $0 < \sigma < \epsilon$.

To estimate $L_{1.2}$, noticing that $d(z,y) \geq \frac{1}{2A}(2^{-k} + d(x,y))$, we then have

$$L_{1.2} \leq C\|g\|_{\mathcal{C}^\alpha} \int_{W_1} \frac{2^{-k\epsilon}}{(2^{-k}+d(x,z))^{1+\epsilon}} \frac{2^{-l\epsilon}}{(2^{-l}+d(z,y))^{1+\epsilon}}\,d\mu(z)$$

$$
\begin{aligned}
&\le C\|g\|_{\mathcal{C}^\alpha}\frac{2^{-l\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}}\int_X\frac{2^{-k\epsilon}}{(2^{-k}+d(x,z))^{1+\epsilon}}d\mu(z)\\
&\le C\|g\|_{\mathcal{C}^\alpha}2^{-(l-k)\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}}.
\end{aligned}
$$

As for $L_{1.3}$, we have

$$
L_{1.3}\le C\|g\|_{\mathcal{C}^\alpha}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}}\int_{W_2}\frac{2^{-l\epsilon}}{(2^{-l}+d(z,y))^{1+\epsilon}}\,d\mu(z).
$$

Denote $a=\frac{1}{2A}(2^{-k}+d(x,y))$. Then

$$
\begin{aligned}
\int_{W_2}\frac{2^{-l\epsilon}}{(2^{-l}+d(z,y))^{1+\epsilon}}\,d\mu(z)=&\sum_{j=0}^{\infty}\int_{2^ja<d(z,y)\le 2^{j+1}a}\frac{2^{-l\epsilon}}{(2^{-l}+d(z,y))^{1+\epsilon}}\,d\mu(z)\\
\le&\sum_{j=0}^{\infty}\frac{2^{-l\epsilon}}{(2^ja)^{1+\epsilon}}\int_{d(z,y)\le 2^{j+1}a}d\mu(z)\\
\le& C\sum_{j=0}^{\infty}2^{-l\epsilon}\frac{2^{j+1}a}{(2^ja)^{1+\epsilon}}\\
\le& C2^{-(l-k)\epsilon}.
\end{aligned}
$$

Thus

$$
L_{1.3}\le C\|g\|_{\mathcal{C}^\alpha}2^{-(l-k)\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}}.
$$

Combining the above estimates for $L_{1.1}$, $L_{1.2}$ and $L_{1.3}$, we have

$$
L_1\le C\|g\|_{\mathcal{C}^\alpha}2^{-(l-k)\sigma}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}},
$$

where $0<\sigma<\epsilon$.

We turn now to estimate L_2. From the size condition of D_k, E_l and the definition of $\mathcal{C}^\alpha$, we have

$$
\begin{aligned}
L_2\le& C\|g\|_{\mathcal{C}^\alpha}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}}\int_X d(z,y)^\alpha\frac{2^{-l\epsilon}}{(2^{-l}+d(z,y))^{1+\epsilon}}\,d\mu(z)\\
\le& C\|g\|_{\mathcal{C}^\alpha}2^{-l\alpha}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}}\\
\le& C\|g\|_{\mathcal{C}^\alpha}2^{-(l-k)\alpha}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}},
\end{aligned}
$$

where $0<\alpha<\epsilon$.

Hence

$$
L_1+L_2\le C\|g\|_{\mathcal{C}^\alpha}2^{-(l-k)(\alpha\wedge\sigma)}\frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{1+\epsilon}},
$$

where $0<\alpha,\sigma<\epsilon$.

When $k\ge l\ge 1$, the above estimate continues to hold by symmetry. The cases $k=0, l\ge 1$ and $l=0, k\ge 1$ can be handled similarly. We thus complete the proof of Lemma 3.7. □

Now we can give the

Proof of Proposition 3.6. We only prove (22) and (23), since the proof of (21) and (24) are similar. For any $g\in\mathcal{C}^\alpha$ with $0<\alpha<\epsilon$, $f\in\widetilde{\mathcal{G}}(\beta,\gamma)$ with $\max(s,-s)<\beta<\epsilon, 0<\gamma<\epsilon$, we have

$$\begin{aligned}\|fg\|_{F^{s,q}_{p,b^{-1}}} =&\|D_0M_b(fg)\|_{L^p} + \Big\|\Big\{\sum_{k=1}^{\infty}\Big(2^{ks}|D_kM_b(fg)|\Big)^q\Big\}^{1/q}\Big\|_{L^p}\\ :=&Y_1+Y_2.\end{aligned}$$

Applying Lemma 2.5, Minkowski's inequality, Hölder's inequality, Fefferman-Stein's vector-valued maximal inequality in [5], Lemma 3.7 and Remark 2.10, we have

$$\begin{aligned}Y_1 \le&\Big\|D_0M_bg\Big(\sum_{l=0}^{\infty}E_lM_b\widetilde{\widetilde{E}}_lM_b(f)\Big)\Big\|_{L^p}\\ \le&\Big\|D_0M_bgE_0M_b\widetilde{\widetilde{E}}_0M_b(f)\Big\|_{L^p} + \Big\|\sum_{l=1}^{\infty}D_0M_bgE_lM_b\widetilde{\widetilde{E}}_lM_b(f)\Big\|_{L^p}\\ \le&C\|g\|_{\mathcal{C}^\alpha}\Big\|\widetilde{\widetilde{E}}_0M_b(f)\Big\|_{L^p} + C\|g\|_{\mathcal{C}^\alpha}\Big\|\sum_{l=1}^{\infty}2^{-l(\alpha\wedge\sigma)}M(\widetilde{\widetilde{E}}_lM_b(f))\Big\|_{L^p}\\ \le&C\|g\|_{\mathcal{C}^\alpha}\Big\|\widetilde{\widetilde{E}}_0M_b(f)\Big\|_{L^p} + C\|g\|_{\mathcal{C}^\alpha}\Big\|\sum_{l=1}^{\infty}2^{-l((\alpha\wedge\sigma)+s)}2^{ls}M(\widetilde{\widetilde{E}}_lM_b(f))\Big\|_{L^p}\\ \le&C\|g\|_{\mathcal{C}^\alpha}\Big\|\widetilde{\widetilde{E}}_0M_b(f)\Big\|_{L^p} + C\|g\|_{\mathcal{C}^\alpha}\Big\|\Big\{\sum_{l=1}^{\infty}\Big(2^{ls}\Big|\widetilde{\widetilde{E}}_lM_b(f)\Big|\Big)^q\Big\}^{1/q}\Big\|_{L^p}\\ \le&C\|g\|_{\mathcal{C}^\alpha}\|f\|_{F^{s,q}_{p,b^{-1}}},\end{aligned}$$

where $(\alpha\wedge\sigma)+s>0$.

By an analogous argument, we obtain

$$\begin{aligned}Y_2 \le&\Big\|\Big\{\sum_{k=1}^{\infty}\Big(2^{ks}D_kM_bg\Big(\sum_{l=0}^{\infty}E_lM_b\widetilde{\widetilde{E}}_lM_b(f)\Big)^q\Big\}^{1/q}\Big\|_{L^p}\\ \le&C\|g\|_{\mathcal{C}^\alpha}\Big\|\Big\{\sum_{k=1}^{\infty}\Big(2^{k(s-(\alpha\wedge\sigma)}M(\widetilde{\widetilde{E}}_0M_b(f))\Big)^q\Big\}^{1/q}\Big\|_{L^p}\\ &+C\|g\|_{\mathcal{C}^\alpha}\Big\|\Big\{\sum_{k=1}^{\infty}\Big(\sum_{l=0}^{\infty}2^{(k-l)s}2^{-|k-l|(\alpha\wedge\sigma)}2^{ls}M(\widetilde{\widetilde{E}}_lM_b(f))\Big)^q\Big\}^{1/q}\Big\|_{L^p}\\ \le&C\|g\|_{\mathcal{C}^\alpha}\|E_0M_b(f)\|_{L^p} + C\|g\|_{\mathcal{C}^\alpha}\Big\|\Big\{\sum_{l=0}^{\infty}\Big(2^{ls}|\widetilde{\widetilde{E}}_lM_b(f)|\Big)^q\Big\}^{1/q}\Big\|_{L^p}\\ \le&C\|g\|_{\mathcal{C}^\alpha}\|f\|_{F^{s,q}_{p,b^{-1}}},\end{aligned}$$

where we have used

$$\begin{aligned}\sup_{k\in\mathbb{N}}\sum_{l\in\mathbb{Z}_+}2^{(k-l)s}2^{-|k-l|(\alpha\wedge\sigma)}<\infty,\\ \sup_{l\in\mathbb{Z}_+}\sum_{k\in\mathbb{N}}2^{(k-l)s}2^{-|k-l|(\alpha\wedge\sigma)}<\infty\end{aligned}\tag{26}$$

with $|s|<(\alpha\wedge\sigma)$.

Hence we obtain

$$Y_1+Y_2\le C\|g\|_{\mathcal{C}^\alpha}\|f\|_{F^{s,q}_{p,b^{-1}}},$$

which finishes the proof of (22).

We turn to prove (23). For any $g \in \mathcal{C}^\alpha$ with $0 < \alpha < \epsilon$, $f \in b\widetilde{\mathcal{G}}(\beta,\gamma)$ with $\max(s,-s) < \beta < \epsilon, 0 < \gamma < \epsilon$, we have

$$\|fg\|_{B^{s,q}_{p,b}} = \|D_0(fg)\|_{L^p} + \Big\{\sum_{k=1}^{\infty}\Big(2^{ks}\|D_k(fg)\|_{L^p}\Big)^q\Big\}^{1/q} := Z_1 + Z_2.$$

Applying Lemma 2.6, Minkowski's inequality, Hölder's inequality, Lemma 3.7 and Remark 2.10, we conclude that

$$\begin{aligned}
Z_1 \leq &\Big\|D_0 g\Big(\sum_{l=0}^{\infty} M_b E_l M_b \widetilde{\widetilde{E}}_l(f)\Big)\Big\|_{L^p} \\
\leq &\Big\|D_0 g M_b E_0 M_b \widetilde{\widetilde{E}}_0(f)\Big\|_{L^p} + \sum_{l=1}^{\infty}\Big\|D_0 g M_b E_l M_b \widetilde{\widetilde{E}}_l(f)\Big\|_{L^p} \\
\leq &C\|g\|_{\mathcal{C}^\alpha}\Big\|\widetilde{\widetilde{E}}_0(f)\Big\|_{L^p} + C\|g\|_{\mathcal{C}^\alpha}\sum_{l=1}^{\infty}2^{-l(\alpha\wedge\sigma)}\Big\|\widetilde{\widetilde{E}}_l(f)\Big\|_{L^p} \\
\leq &C\|g\|_{\mathcal{C}^\alpha}\Big\|\widetilde{\widetilde{E}}_0(f)\Big\|_{L^p} + C\|g\|_{\mathcal{C}^\alpha}\Big(\sum_{l=1}^{\infty}2^{-l((\alpha\wedge\sigma)+s)q'}\Big)^{1/q'}\Big\{\sum_{l=1}^{\infty}\Big(2^{ls}\Big\|\widetilde{\widetilde{E}}_l(f)\Big\|_{L^p}\Big)^q\Big\}^{1/q} \\
\leq &C\|g\|_{\mathcal{C}^\alpha}\|f\|_{B^{s,q}_{p,b}},
\end{aligned}$$

where $s + (\alpha\wedge\sigma) > 0$. Similarly,

$$\begin{aligned}
Z_2 \leq &\Big\{\sum_{k=1}^{\infty}\Big(2^{ks}\Big\|D_k g\Big(\sum_{l=0}^{\infty}M_b E_l M_b \widetilde{\widetilde{E}}_l(f)\Big)\Big\|_{L^p}\Big)^q\Big\}^{1/q} \\
\leq &C\|g\|_{\mathcal{C}^\alpha}\Big\{\sum_{k=1}^{\infty}\Big(2^{k(s-(\alpha\wedge\sigma))}\Big\|\widetilde{\widetilde{E}}_0(f)\Big\|_{L^p}\Big)^q\Big\}^{1/q} \\
&+ C\|g\|_{\mathcal{C}^\alpha}\Big\{\sum_{k=1}^{\infty}\Big(\sum_{l=1}^{\infty}2^{(k-l)s}2^{-|k-l|(\alpha\wedge\sigma)}2^{ls}\Big\|\widetilde{\widetilde{E}}_l(f)\Big\|_{L^p}\Big)^q\Big\}^{1/q} \\
\leq &C\|g\|_{\mathcal{C}^\alpha}\Big\|\widetilde{\widetilde{E}}_0(f)\Big\|_{L^p} + C\|g\|_{\mathcal{C}^\alpha}\Big\{\sum_{l=1}^{\infty}\Big(2^{ls}\Big\|\widetilde{\widetilde{E}}_l(f)\Big\|_{L^p}\Big)^q\Big\}^{1/q} \\
\leq &C\|g\|_{\mathcal{C}^\alpha}\|f\|_{B^{s,q}_{p,b}},
\end{aligned}$$

where s satisfies (26) with $|s| < (\alpha\wedge\sigma)$. This concludes the proof of (23). □

For $f \in B^{s,q}_{p,b^{-1}}(B^{s,q}_{p,b})$ or $f \in F^{s,q}_{p,b^{-1}}(F^{s,q}_{p,b})$, f could be, in general, a distribution. Thus, the multiplication of gf may not make sense even for $g \in \mathcal{C}^\alpha$. In the following lemmas, we define gf as a distribution acting on test functions.

Lemma 3.8. *Let $|s| < \epsilon, 0 < \epsilon' < \epsilon$ and $1 < p,q < \infty$. For any $f \in B^{s,q}_{p,b^{-1}}$ or $f \in F^{s,q}_{p,b^{-1}}$ and $g \in \mathcal{C}^\alpha$ with $\max(s,-s) < \alpha < \epsilon$, there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ such that $f_j \in \widetilde{\mathcal{G}}(\epsilon',\epsilon')$, $\|f_j\|_{B^{s,q}_{p,b^{-1}}} \lesssim \|f\|_{B^{s,q}_{p,b^{-1}}}$ or $\|f_j\|_{F^{s,q}_{p,b^{-1}}} \lesssim \|f\|_{F^{s,q}_{p,b^{-1}}}$ and $\lim_{j\to\infty}\langle gf_j, h\rangle$ converges for any $h \in \widetilde{\mathcal{G}}(\beta,\gamma)$ with β,γ satisfying $\max(s,-s) < \beta < \epsilon', 0 < \gamma < \epsilon'$.*

Proof. Let $A^{s,q}_{p,b^{-1}} = B^{s,q}_{p,b^{-1}}$ or $A^{s,q}_{p,b^{-1}} = F^{s,q}_{p,b^{-1}}$. For any $f \in A^{s,q}_{p,b^{-1}}$ with $1 < p,q < \infty, |s| < \sigma$, denote

$$f_M = \sum_{k=0}^{M} M_b \widetilde{D}_k M_b D_k(f).$$

By Proposition 2.12 and the definitions of test functions, we can show that $f_M \in \widetilde{\mathcal{G}}(\epsilon', \epsilon')$ and $\|f_M\|_{A^{s,q}_{p,b^{-1}}} \le \|f\|_{A^{s,q}_{p,b^{-1}}}$.

Now we claim that $\lim_{n\to\infty} \langle gf_n, h\rangle$ converges for any $h \in \widetilde{\mathcal{G}}(\beta,\gamma)$ with $\max(s,-s) < \beta < \epsilon', 0 < \gamma < \epsilon'$. For $n, m \in \mathbb{N}, m < n$, by Propositions 3.6 and duality spaces properties of Besov and Triebel-Lizorkin spaces (see Propositions 3.10 and 3.11 below), we have

$$\begin{aligned}|\langle f_n - f_m, gh\rangle| \le& \|f_n - f_m\|_{A^{s,q}_{p,b^{-1}}} \|gh\|_{A^{-s,q'}_{p',b^{-1}}}\\ \le& C\|g\|_{\mathcal{C}^\alpha} \|f_n - f_m\|_{A^{s,q}_{p,b^{-1}}} \|h\|_{A^{-s,q'}_{p',b^{-1}}}.\end{aligned}$$

Notcing that $\|h\|_{A^{-s,q'}_{p',b^{-1}}} \le C\|h\|_{\widetilde{\mathcal{G}}(\beta,\gamma)}$ and $\|f_n - f_m\|_{A^{s,q}_{p,b^{-1}}}$ tends to zero as n, m tend to infinity. This implies that $|\langle f_n - f_m, gh\rangle| \to 0$ as $n, m \to \infty$. The proof of Lemma 3.8 is concluded. □

Lemma 3.9. *Let $|s| < \epsilon, 0 < \epsilon' < \epsilon$ and $1 < p, q < \infty$. For any $f \in B^{s,q}_{p,b}$ or $f \in F^{s,q}_{p,b}$ and $g \in \mathcal{C}^\alpha$ with $\max(s,-s) < \alpha < \epsilon$, there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ such that $f_j \in b\widetilde{\mathcal{G}}(\epsilon', \epsilon')$, $\|f_j\|_{B^{s,q}_{p,b}} \lesssim \|f\|_{B^{s,q}_{p,b}}$ or $\|f_j\|_{F^{s,q}_{p,b}} \lesssim \|f\|_{F^{s,q}_{p,b}}$ and $\lim_{j\to\infty}\langle gf_j, h\rangle$ converges for any $h \in b\widetilde{\mathcal{G}}(\beta,\gamma)$ with β, γ satisfying $\max(s,-s) < \beta < \epsilon', 0 < \gamma < \epsilon'$.*

Proof. The proof is similar to that of Lemma 3.8. We leave the details to the interested reader. □

We are ready to give the

Proof of Theorems 3.4 and 3.5. For any $g \in \mathcal{C}^\alpha$ with $\max(-s,s) < \epsilon$, $f \in B^{s,q}_{p,b^{-1}}$ with $1 < p, q < \infty$, by Lemma 3.8, $\lim_{n\to\infty}\langle gf_n, h\rangle$ exists. We thus define

$$\langle gf, h\rangle = \lim_{n\to\infty}\langle gf_n, h\rangle$$

for $h \in \widetilde{\mathcal{G}}(\beta,\gamma)$ with β,γ satisfying $\max(s,-s) < \beta < \epsilon', 0 < \gamma < \epsilon'$ for $0 < \epsilon' < \epsilon$, and the limit is independent of the choice of f_n. From Fatou's lemma and Proposition 3.6, we have

$$\|gf\|_{B^{s,q}_{p,b^{-1}}} \le \liminf_{n\to\infty} \|gf_n\|_{B^{s,q}_{p,b^{-1}}} \le \|g\|_{\mathcal{C}^\alpha}\|f\|_{B^{s,q}_{p,b^{-1}}},$$

which concludes the proof of Theorem 3.4. Theorem 3.5 can be verified similarly. We omit the details. □

In the last part of this section, we identify the dual spaces of the inhomogeneous Besov and Triebel-Lizorkin spaces. Using Proposition 2.12, the proof is similar to that of Theorem 7.1 in [10] and the details are omitted.

Proposition 3.10. *Let $|s| < \epsilon, 1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.*
(i) $\left(B^{s,q}_{p,b^{-1}}\right)^* = B^{-s,q'}_{p',b^{-1}}$. More precisely, given $g \in B^{-s,q'}_{p',b^{-1}}$, then $\pounds_g(f) = \langle f, g\rangle$ defines a linear functional on $B^{s,q}_{p,b^{-1}} \cap \mathcal{G}(\epsilon', \epsilon')$ with $|s| < \epsilon' < \epsilon$ such that

$$|\pounds_g(f)| \le C\|f\|_{B^{s,q}_{p,b^{-1}}}\|g\|_{B^{-s,q'}_{p',b^{-1}}},$$

and this linear functional can be extended to $B^{s,q}_{p,b^{-1}}$ with norm at most $C\|g\|_{B^{-s,q'}_{p',b^{-1}}}$.

Conversely, if $\pounds$ is a linear functional on $B^{s,q}_{p,b^{-1}}$, then there exists a unique $g \in B^{-s,q'}_{p',b^{-1}}$ such that

$$\pounds_g(f) = \langle f, g\rangle$$

defines a linear functional on $B^{s,q}_{p,b^{-1}} \cap \mathcal{G}(\epsilon',\epsilon')$ *with* $|s| < \epsilon' < \epsilon$*, and* $\pounds$ *is the extension of* $\pounds_g$ *with* $\|g\|_{B^{-s,q'}_{p',b^{-1}}} \le C\|\pounds\|$.

(ii) $\left(B^{s,q}_{p,b}\right)^* = B^{-s,q'}_{p',b}$*. More precisely, given* $g \in B^{-s,q'}_{p',b}$*, then* $\pounds_g(f) = \langle f, g\rangle$ *defines a linear functional on* $B^{s,q}_{p,b} \cap b\mathcal{G}(\epsilon',\epsilon')$ *with* $|s| < \epsilon' < \epsilon$ *such that*

$$|\pounds_g(f)| \le C\|f\|_{B^{s,q}_{p,b}}\|g\|_{B^{-s,q'}_{p',b}},$$

and this linear functional can be extended to $B^{s,q}_{p,b}$ *with norm at most* $C\|g\|_{B^{-s,q'}_{p',b}}$.

Conversely, if $\pounds$ *is a linear functional on* $B^{s,q}_{p,b}$*, then there exists a unique* $g \in B^{-s,q'}_{p',b}$ *such that*

$$\pounds_g(f) = \langle f, g\rangle$$

defines a linear functional on $B^{s,q}_{p,b} \cap b\mathcal{G}(\epsilon',\epsilon')$ *with* $|s| < \epsilon' < \epsilon$*, and* $\pounds$ *is the extension of* $\pounds_g$ *with* $\|g\|_{B^{-s,q'}_{p',b}} \le C\|\pounds\|$.

Proposition 3.11. *Let all notation be the same as in Proposition 3.10, then Proposition 3.10 also holds when* $B^{s,q}_{p,b^{-1}}$, $B^{-s,q'}_{p',b^{-1}}$, $B^{s,q}_{p,b}$, $B^{-s,q'}_{p',b}$ *are replaced by* $F^{s,q}_{p,b^{-1}}$, $F^{-s,q'}_{p',b^{-1}}$, $F^{s,q}_{p,b}$, $F^{-s,q'}_{p',b}$*, respectively.*

Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China.
Email: liaofanghui1028@163.com

[2]Department of Mathematics, China University of Mining and Technology, Beijing 100083, China.
Email: liuzg@cumtb.edu.cn

[3]Department of basic curriculum, North China Institute of Science and Technology, Langfang 065201, China.
Email: zxj800225@126.com