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Inference for accelerated bivariate dependent competing risks model based on Archimedean copulas under progressive censoring

ZHANG Chun-fang[1] SHI Yi-min[2] WANG Liang[3]

Abstract. Dependent competing risks model is a practical model in the analysis of lifetime and failure modes. The dependence can be captured using a statistical tool to explore the relationship among failure causes. In this paper, an Archimedean copula is chosen to describe the dependence in a constant-stress accelerated life test. We study the Archimedean copula based dependent competing risks model using parametric and nonparametric methods. The parametric likelihood inference is presented by deriving the general expression of likelihood function based on assumed survival Archimedean copula associated with the model parameter estimation. Combining the nonparametric estimation with progressive censoring and the nonparametric copula estimation, we introduce a nonparametric reliability estimation method given competing risks data. A simulation study and a real data analysis are conducted to show the performance of the estimation methods.

§1 Introduction

Statistical analysis for competing risks model has recently attracted much attention with the development of scientific technology. When the tested products suffer several different failure causes and the lifetime of the product is the latent failure time of the first failure cause among all the possible failure causes, we call those possible failure causes competing risks. The concept of competing risks [5] is important for products. For example, the failures of rolling bearings may be responsible for outer ring failure, inner ring failure, rolling element failure and cage failure. As many competing risks products nowadays have a long lifetime, in order to observe enough failure times and failure causes, it will cost much time and expenses. By extrapolating

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the reliability indexes under use conditions, the accelerated life test (ALT) can be applied to shorten the total test time and costs for those products with long lifetimes.

The maximum likelihood estimation, optimal Fisher information and Bayesian prediction for independent competing risks data have been studied respectively from Weibull distribution [28], Lomax distribution [4] and half-logistic distribution [1]. Mao et al. [20] derived the exact conditional distributions of unknown parameters for generalized type-I hybrid censoring data with an exponential failure model. In accelerated life tests, Roy and Mukhopadhyay [29] presented the maximum likelihood analysis by using the expectation-maximization algorithm of accelerated life test data with independent log-normal causes of failure. Han and Kundu [12] considered the point estimates, and approximate confidence intervals of the step-stress model when the failure factors were from generalized exponential lifetime distributions. The literature mentioned above was based on the assumption that the competing risks are statistically independent. In practice, the competing risks may interact with each other. They are usually dependent. The assumption of independence among competing risks is not practical and the estimation results of reliability indexes may be incorrect. Therefore, it is of significance to focus on the statistical inference for accelerated dependent competing risks.

To model the dependence among failure causes, the Marshall-Olkin bivariate Weibull distribution [7] and bivariate Birnbaum-Saunders distribution [32] have been used for bivariate competing risks. However, the bivariate distribution has the same margins and can not capture the tail dependence. Therefore, the copula function [22] is introduced to improve the bivariate distribution. It has been developed as an important tool to model dependent competing risks in biostatistics, econometrics and engineering ([15], [17-18], [27]). In ALT, Xu and Tang [31] used Gumbel family to assess the exponential competing risks model and gave the maximum likelihood estimates. Zhang et al. [34] introduced general copula theory to derive likelihood function and parameter estimation in two-dimensional case and in multi-dimensional case. They also presented a simple engineering-based multi-dimensional copula construction method. Wu et al. [30] discussed the maximum likelihood estimators, approximate confidence intervals and percentile bootstrap confidence intervals of dependent competing risks model in accelerated life testing under progressively hybrid censoring using the Gumbel family.

Since Archimedean copulas, such as Clayton, Frank and Gumbel copulas, can be constructed easily and describe the symmetric and nonsymmetric dependence structure, and the different tail dependence ([10], [13-14], [25]), we derive the general expressions of likelihood function and likelihood equations based on Archimedean copulas in constant-stress accelerated competing risks model with progressive censoring scheme as a generalization of dependent competing risks model in [30-31]. To avoid the model misspecification in this model, we further introduce a nonparametric estimation method.

The remainder of this paper is organized as follows. In Section 2, we present the bivariate dependent competing risks model in constant-stress accelerated life tests (CSALT) and make the basic assumptions. In Section 3, we derive the likelihood function and obtain the likelihood based point estimates and interval estimates of model parameters. The nonparametric estimation method of the accelerated dependent competing risks model is proposed in Section 4. Section 5 presents a simulation study, and a real data set is analyzed in Section 6. Finally, conclusions are drawn in Section 7.

§2 CSALT model with bivariate competing risks

In this paper, we analyze CSALT with bivariate competing risks. Consider the case of two competing risks (X_1, X_2) and $k \geq 2$ accelerated stress levels in CSALT with $S_0 < S_1 < \cdots < S_k$. S_0 is the common use stress level. Under accelerated stress level S_i, n_i products are tested and r_i failure times are observed. To study other product features besides the lifetime, it is necessary to remove the tested products. Then at each observation time (t_{ij}, δ_{ij}) of $T_i = \min(X_1, X_2)$ and $\delta_i = I(X_{i1} < X_{i2})$, $\mathcal{R}_{ij} \geq 0$ samples are removed. Here $n_i = \sum_{j=1}^{r_i} (1 + \mathcal{R}_{ij})$, $i = 1, 2, \ldots, k, j = 1, 2, \ldots, r_i$.

The likelihood inference of competing risks model in CSALT is studied in need of the following foundational assumptions.

A1: The cumulative probability function and survival function of each competing risk under stress level S_i are denoted as $F_{il}(x; \theta_{il})$ and $S_{il}(x; \theta_{il})$ for $i = 1, 2, \ldots, k, l = 1, 2$. The distribution type will not be changed by the stress levels. The accelerated stress levels have influence on the distribution parameters.

A2: The two competing risks X_1, X_2 are dependent. The dependence structure between the two competing risks is described by an Archimedean copula with the joint survival function given by

$$S(x_{i1}, x_{i2}; \theta_{i1}, \theta_{i2}, \theta_c) = {\varphi_{\theta_c}}^{[-1]}(\varphi_{\theta_c}[S_{i1}(x_{i1}; \theta_{i1})] + \varphi_{\theta_c}[S_{i2}(x_{i2}; \theta_{i2})]), \tag{1}$$

where θ_c is the parameter of generator function. θ_c is assumed to be equal so that the dependence structure of the two competing risks will not be affected with stress levels.

A3: There exists a functional relationship between the distribution parameters θ_{il} and each stress level S_i. That is $\theta_{il} = \psi_l(S_i; \boldsymbol{\theta}_{\psi_l})$, where $\psi_l(\cdot)$ is a known function and $\boldsymbol{\theta}_{\psi_l}$ is the parameter vector of the function $\psi_l(\cdot)$. This is a general function for acceleration model. Usually, Arrhenius model, inverse power law model and Eyring model are specified in ALT.

Note that θ_c can be specified by the known Kendall's tau τ. In the bivariate Archimedean copula, there are several common families with one-parameter generators, namely Gumbel, Ali-Mikhail-Haq, Clayton, Frank copulas. The relationships between Kendall's tau and θ_c of the mentioned copulas above are given in Table 1 where $D_1(\theta) = \int_0^\theta t/[\exp(t) - 1]dt/\theta$ is the Debye function of order one.

Table 1. Relationships between copula parameter and Kendall's tau of bivariate Archimedean copulas with one-parameter generators.

Family	$\varphi_\theta(t)$	Parameter	Kendall's tau
Gumbel	$\{-\log(t)\}^\theta$	$\theta \in [1, \infty)$	$(\theta - 1)/\theta$
Ali-Mikhail-Haq	$\log\{[1 - \theta(1 - t)]/t\}$	$\theta \in [1, \infty)$	$1 - 2[\theta + (1 - \theta)]/(3\theta^2)$
Clayton	$(t^{-\theta} - 1)/\theta$	$\theta \in [0, \infty)$	$\theta/(\theta + 2)$
Frank	$-\log\{(e^{-\theta t} - 1)/(e^{-\theta} - 1)\}$	$\theta \in (0, \infty)$	$1 + 4[D_1(\theta) - 1)]/\theta$

§3 Likelihood inference

3.1 Likelihood function

Having the observed data $(t_{ij}, \delta_{ij}) = (\min(X_{i1}, X_{i2}), I(X_{i1} < X_{i2}))$ and $\mathcal{R}_{ij}$ removals at the time t_{ij} for $i = 1, 2, \ldots, k, j = 1, 2, \ldots, r_i$, the likelihood function for the observed data in ALT can be obtained. In addition to above assumptions, two theorems with assumed survival copula are needed to derive the likelihood function.

Theorem 3.1. *Let X_{i1} and X_{i2} be continuous random variables with distribution functions F_{i1} and F_{i2}, density functions $f_{i1}(x_{i1})$ and $f_{i2}(x_{i2})$. There exists a continuous generator $\varphi \in \Omega$ and an absolutely continuous Archimedean survival copula $\hat{C}$ such that $\hat{C}(1-F_{i1}(x_{i1}), 1-F_{i2}(x_{i2})) = \varphi^{-1}(\varphi(1 - F_{i1}(x_{i1})) + \varphi(1 - F_{i2}(x_{i2}))$. Then the joint density function $f(x_{i1}, x_{i2})$ for X_{i1} and X_{i2} is given by*

$$f(x_{i1}, x_{i2}) = \left\{ -\frac{\varphi''(\hat{C}(u,v))\varphi'(u)\varphi'(v)}{[\varphi'(\hat{C}(u,v))]^3}\bigg|_{u=1-F_{i1}(x_{i1}),v=1-F_{i2}(x_{i2})} \right\} \times f_{i1}(x_{i1})f_{i2}(x_{i2}). \quad (2)$$

Proof: for $i = 1, 2, \ldots, k, j = 1, 2, \ldots, r_i$,

$$C(F_{i1}(x_{i1}), F_{i2}(x_{i2})) = F_{i1}(x_{i1}) + F_{i2}(x_{i2}) - 1 + \varphi^{-1}(\varphi(1 - F_{i1}(x_{i1})) + \varphi(1 - F_{i2}(x_{i2})),$$

$$\frac{d\varphi^{-1}(x)}{dx} = \frac{1}{\varphi'(\varphi^{-1}(x))}.$$

$$\frac{\partial C(F_{i1}(x_{i1}), F_{i2}(x_{i2}))}{\partial x_{i1}} = f_{i1}(x_{i1}) - \frac{f_{i1}(x_{i1})}{\varphi'(\hat{C}(1 - F_{i1}(x_{i1}), 1 - F_{i2}(x_{i2}))},$$

$$\begin{aligned} f(x_1, x_2) &= \frac{\partial^2 C(F_{i1}(x_{i1}), F_{i2}(x_{i2}))}{\partial x_1 \partial x_2} \\ &= \left\{ -\frac{\varphi''(\hat{C}(u,v))\varphi'(u)\varphi'(v)}{[\varphi'(\hat{C}(u,v))]^3}\bigg|_{u=1-F_{i1}(x_{i1}),v=1-F_{i2}(x_{i2})} \right\} \\ &\quad \times f_{i1}(x_{i1})f_{i2}(x_{i2}). \end{aligned}$$

Therefore, Theorem 3.1 holds. □

In bivariate competing risks model under the stress level S_i, the likelihood function is specified for the sub-densities $f^{il}(t_{ij}), i = 1, 2, \ldots, k, j = 1, 2, \ldots, r_i, l = 1, 2$. Based on Theorem 3.1, the sub-densities are given in the following theorem.

Theorem 3.2. *Let X_{i1} and X_{i2} be continuous random variables with distribution functions F_{i1} and F_{i2}, density functions $f_{i1}(x_{i1})$ and $f_{i2}(x_{i2})$. There exists a continuous generator $\varphi \in \Omega$ and an absolutely continuous Archimedean survival copula $\hat{C}$ such that $\hat{C}(1-F_{i1}(x_{i1}), 1-F_{i2}(x_{i2})) = \varphi^{-1}(\varphi(1-F_{i1}(x_{i1}))+\varphi(1-F_{i2}(x_{i2}))$. If $|\varphi'(u)| \to \infty$ when $u \to 0$, then the sub-density functions $f^{il}(t_{ij})$ are expressed as*

$$f^{il}(t_{ij}) = f_{il}(t_{ij})\frac{\varphi'[1 - F_{il}(t_{ij})]}{\varphi'[S(t_{ij})]}, \quad (3)$$

where

$$S(t_{ij}) = \hat{C}(1 - F_{i1}(x_{i1}), 1 - F_{i2}(x_{i2}))|_{x_{i1}=x_{i2}=t_{ij}},$$

and $i = 1, 2, \ldots, k, j = 1, 2, \ldots, r_i, l = 1, 2$.

The proof is given based on Zheng [33] as follow.

$$F^{i1}(t_{ij}) = P(X_{i1} < t_{ij}, X_{i1} < X_{i2}) = \int_0^{t_{ij}} \left[\int_{x_{i1}}^{\infty} f(x_{i1}, x_{i2}) dx_{i2}\right] dx_{i1}$$
$$= \int_0^{t_{ij}} \left[\int_{x_{i1}}^{\infty} \frac{\partial^2 C(F_{i1}(x_{i1}), F_{i2}(x_{i2}))}{\partial x_1 \partial x_2} dx_{i2}\right] dx_{i1}$$
$$= \int_0^{F_1(t_{ij})} \left[\int_{F_2 F_1{}^{-1}(u)}^{1} \frac{\partial^2 C(u,v)}{\partial u \partial v} dv\right] du$$
$$f^{i1}(t_{ij}) = \frac{dF^{i1}(t_{ij})}{dt_{ij}} = f_{i1}(t_{ij}) \int_{F_2(t_{ij})}^{1} \left.\frac{\partial^2 C(u,v)}{\partial u \partial v}\right|_{u=F_1(t_{ij})} dv$$
$$= f_{i1}(t_{ij}) \left[\left.\frac{\partial C(u,v)}{\partial u}\right|_{u=F_1(t_{ij}), v=1} - \left.\frac{\partial C(u,v)}{\partial u}\right|_{u=F_1(t_{ij}), v=F_2(t_{ij})} \right]$$
$$\frac{\partial C(u,v)}{\partial u} = 1 - \frac{\partial \hat{C}(1-u, 1-v)}{\partial u} = 1 - \frac{\varphi'(1-u)}{\varphi'[\varphi^{[-1]}(\varphi(1-u)+\varphi(1-v))]}$$

As $\lim\limits_{v\to 1} \varphi(1-v) = \varphi(0)$, $\lim\limits_{v\to 1} \varphi^{[-1]}(\varphi(1-u)+\varphi(1-v)) = 0$, and $|\varphi'(u)| \to \infty$ when $u \to 0$, $\lim\limits_{v\to 1} \frac{\varphi'(1-u)}{\varphi'[\varphi^{[-1]}(\varphi(1-u)+\varphi(1-v))]} = 0$. Then

$$f^{i1}(t_{ij}) = f_{i1}(t_{ij}) \frac{\varphi'(1-F_1(t_{ij}))}{\varphi'[\varphi^{[-1]}(\varphi(1-F_1(t_{ij})) + \varphi(1-F_2(t_{ij})))]}.$$

The sub-density function $f^{i2}(t_{ij})$ can be similarly derived, and this theorem is proved. □

Corollary 3.1. *Let $X_{i1}, \ldots, X_{ip}, p > 2$ be continuous random variables with distribution functions $F_{i1}, \ldots, F_{ip}$, density functions $f_{i1}(x_{i1}), \ldots, f_{ip}(x_{ip})$. There exists a continuous generator $\varphi \in \Omega$ and an absolutely continuous multivariate Archimedean survival copula $\hat{C}$ such that*

$$\hat{C}(1-F_{i1}(x_{i1}), \ldots, 1-F_{ip}(x_{ip})) = \varphi^{-1}(\varphi(1-F_{i1}(x_{i1})) + \ldots + \varphi(1-F_{ip}(x_{ip})).$$

If $|\varphi'(u)| \to \infty$ when $u \to 0$, then the sub-density functions $f^{il}(t_{ij})$ are expressed as

$$f^{il}(t_{ij}) = f_{il}(t_{ij}) \frac{\varphi'[1-F_{il}(t_{ij})]}{\varphi'[S(t_{ij})]}, \tag{4}$$

where

$$S(t_{ij}) = \hat{C}(1-F_{i1}(x_{i1}), \ldots, 1-F_{ip}(x_{ip}))|_{x_{i1}=\ldots=x_{ip}=t_{ij}},$$

and $i = 1, 2, \ldots, k, j = 1, 2, \ldots, r_i, l = 1, 2, \ldots, p$.

Corollary 3.1 is a multivariate extension of Theorem 3.2. The proof is similar with Theorem 3.2 using multivariate copula to replace bivariate copula. Thus we omit it here, and what follows is the likelihood inference for the bivariate competing risks model.

Let $\underline{\boldsymbol{t}}_i = \{(t^*_{i1}, \delta^*_{i1}, \mathcal{R}_{i1}), (t^*_{i2}, \delta^*_{i2}, \mathcal{R}_{i2}), \ldots, (t^*_{ir_i}, \delta^*_{ir_i}, \mathcal{R}_{ir_i})\}$ at stress level S_i, $\underline{\boldsymbol{t}} = \{\underline{\boldsymbol{t}}_1, \underline{\boldsymbol{t}}_2, \ldots, \underline{\boldsymbol{t}}_k\}$, $\boldsymbol{\theta}_i = (\theta_{i1}, \theta_{i2}), i = 1, 2 \ldots, k$. According to the theorems and assumptions above, the likelihood function for the observed progressive censored data in CSALT with bivariate dependent competing risks is expressed as

$$L(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \cdots, \boldsymbol{\theta}_k; \underline{\boldsymbol{t}})$$
$$= \prod_{i=1}^{k} L_i(\boldsymbol{\theta}_i; \underline{\boldsymbol{t}}_i) = \prod_{i=1}^{k} B_i \prod_{j=1}^{r_i} [f^{i1}(t_{ij})]^{\delta_{ij}} [f^{i2}(t_{ij})]^{1-\delta_{ij}} [S(t_{ij})]^{\mathcal{R}_{ij}} \prod_{j=2}^{r_i} I(t_{i,j-1} < t_{ij})$$

$$
\begin{aligned}
&= \prod_{i=1}^{k} B_i \prod_{j=1}^{r_i} \left[f_{i1}(t_{ij}) \frac{\varphi'[1 - F_{i1}(t_{ij})]}{\varphi'[S(t_{ij})]} \right]^{\delta_{ij}} \\
&\times [f_{i2}(t_{ij}) \frac{\varphi'[1 - F_{i2}(t_{ij})]}{\varphi' \times [S(t_{ij})]}]^{1-\delta_{ij}} [S(t_{ij})]^{\mathcal{R}_{ij}} \prod_{j=2}^{r_i} I(t_{i,j-1} < t_{ij}),
\end{aligned} \tag{5}
$$

where $B_i = \prod_{j=1}^{r_i} \sum_{m=j}^{r_i} (1 + \mathcal{R}_{im})$. The log-likelihood function under S_i is

$$
\begin{aligned}
\log L_i(\boldsymbol{\theta}_i; \underline{\boldsymbol{t}}_i) &= \log B_i + \sum_{j=1}^{r_i} \delta_{ij} [\log f_{i1}(t_{ij}) + \log \frac{\varphi'[1 - F_{i1}(t_{ij})]}{\varphi'[S(t_{ij})]}] \\
&+ \sum_{j=1}^{r_i} (1 - \delta_{ij}) [\log f_{i2}(t_{ij}) + \log \frac{\varphi'[1 - F_{i2}(t_{ij})]}{\varphi'[S(t_{ij})]}] + \sum_{j=1}^{r_i} \mathcal{R}_{ij} \log S(t_{ij}).
\end{aligned} \tag{6}
$$

Suppose the marginal distribution function has one parameter. Let $u_{ij} = 1 - F_{i1}(t_{ij}; \theta_{i1}), v_{ij} = 1 - F_{i2}(t_{ij}; \theta_{i2}), C_{ij} = \varphi^{[-1]}(\varphi(u_{ij}) + \varphi(v_{ij}))$. and

$$
{}_{\theta_{i1}}u'_{ij} = \frac{\partial u_{ij}}{\partial \theta_{i1}}, {}_{t_{ij}}u'_{ij} = \frac{\partial u_{ij}}{\partial t_{ij}}, {}_{t_{ij}\theta_{i1}}u''_{ij} = \frac{\partial^2 u_{ij}}{\partial t_{ij} \partial \theta_{i1}},
$$

$$
{}_{\theta_{i2}}v'_{ij} = \frac{\partial v_{ij}}{\partial \theta_{i2}}, {}_{t_{ij}}v'_{ij} = \frac{\partial v_{ij}}{\partial t_{ij}}, {}_{t_{ij}\theta_{i2}}v''_{ij} = \frac{\partial^2 v_{ij}}{\partial t_{ij} \partial \theta_{i2}}.
$$

Under stress level S_i, the log-likelihood equations is given by

$$
\begin{aligned}
\frac{\partial \log L_i}{\partial \theta_{i1}} &= \sum_{j=1}^{r_i} \delta_{ij} \left\{ \frac{{}_{t_{ij}\theta_{i1}}u''_{ij}}{{}_{t_{ij}}u'_{ij}} + {}_{\theta_{i1}}u'_{ij} \varphi'(u_{ij}) \frac{\varphi(u_{ij})}{[\varphi'(u_{ij})]^2} \right\} \\
&- \sum_{j=1}^{r_i} {}_{\theta_{i1}}u'_{ij} \varphi'(u_{ij}) \frac{C_{ij}\varphi''(C_{ij}) - \mathcal{R}_{ij}\varphi'(C_{ij})}{[\varphi'(C_{ij})]^2 C_{ij}} = 0, \\
\frac{\partial \log L_i}{\partial \theta_{i2}} &= \sum_{j=1}^{r_i} (1 - \delta_{ij}) \left\{ \frac{{}_{t_{ij}\theta_{i2}}v''_{ij}}{{}_{t_{ij}}v'_{ij}} + {}_{\theta_{i2}}v'_{ij} \varphi'(v_{ij}) \frac{\varphi(v_{ij})}{[\varphi'(v_{ij})]^2} \right\} \\
&- \sum_{j=1}^{r_i} {}_{\theta_{i2}}v'_{ij} \varphi'(v_{ij}) \frac{C_{ij}\varphi''(C_{ij}) - \mathcal{R}_{ij}\varphi'(C_{ij})}{[\varphi'(C_{ij})]^2 C_{ij}} = 0.
\end{aligned} \tag{7}
$$

The likelihood function and likelihood equations are general expressions for competing risks data with specified dependence structure of Archimedean copulas. In the parametric estimation method, the family of Archimedean copulas, the marginal distribution function and density function are assumed to be known. Using quasi-Newton method [21] for the log-likelihood equations, the maximum likelihood estimates (MLEs) $\hat{\boldsymbol{\theta}}_i = (\hat{\theta}_{i1}, \hat{\theta}_{i2}), i = 1, 2, \ldots, k$ can be obtained.

To extrapolate the parameter $\theta_{0l}, l = 1, 2$ under the use stress level S_0, we can evaluate $\hat{\theta}_{0l}, l = 1, 2$ based on the accelerated function $\theta_{il} = \psi_l(S_i; \theta_{\psi_l})$ in Assumption A3 by constructing a regression model. When $\log \theta_{il} = \log \psi_l(\cdot)$ is a linear function, it means that, $\log \theta_{il} = a_l + b_l \phi(S_i)$. Least square method can be used to estimate a_l, b_l as

$$
\hat{a}_l = \frac{A \sum_{i=1}^{k} \log \hat{\theta}_{il} - B \sum_{i=1}^{k} \phi_i \log \hat{\theta}_{il}}{kA - B^2}, \hat{b}_l = \frac{k \sum_{i=1}^{k} \phi_i \log \hat{\theta}_{il} - B \sum_{i=1}^{k} \log \hat{\theta}_{il}}{kA - B^2},
$$

where $\phi_i = \phi(S_i), A = \sum_{i=1}^{k} {\phi_i}^2, B = \sum_{i=1}^{k} \phi_i$.

3.2 Clayton and Frank copulas

Assume competing failures follow the exponential distribution with parameter $\theta_{il} = \lambda_{il} > 0$ for $i = 0, 1, \ldots, k$ and $l = 1, 2$. Then, we have the simple derivatives

$$\begin{cases} u_{ij} = \exp\{-\lambda_{i1} t_{ij}\}, \\ {}_{t_{ij}}u'_{ij} = -\lambda_{i1} \exp\{-\lambda_{i1} t_{ij}\}, \\ {}_{\lambda_{i1}}u'_{ij} = -t_{ij} \exp\{-\lambda_{i1} t_{ij}\}, \\ {}_{t_{ij}\lambda_{i1}}u''_{ij} = (\lambda_{i1} t_{ij} - 1) \exp\{-\lambda_{i1} t_{ij}\}, \end{cases} \qquad \begin{cases} v_{ij} = \exp\{-\lambda_{i2} t_{ij}\}, \\ {}_{t_{ij}}v'_{ij} = -\lambda_{i2} \exp\{-\lambda_{i2} t_{ij}\}, \\ {}_{\lambda_{i2}}v'_{ij} = -t_{ij} \exp\{-\lambda_{i2} t_{ij}\}, \\ {}_{t_{ij}\lambda_{i2}}v''_{ij} = (\lambda_{i2} t_{ij} - 1) \exp\{-\lambda_{i2} t_{ij}\}. \end{cases}$$

For Clayton family, the generator function is $\varphi(t) = (t^{-\theta_c} - 1)/\theta_c$. The copula is expressed as

$$C_{ij} = [(\exp\{\theta_c \lambda_{i1} t_{ij}\} + \exp\{\theta_c \lambda_{i2} t_{ij}\} - 1)]^{-1/\theta_c} I(\exp\{\theta_c \lambda_{i1} t_{ij}\} + \exp\{\theta_c \lambda_{i2} t_{ij}\} > 1).$$

Thus the exact likelihood equations under the stress level S_i are expressed as

$$\begin{aligned} &\sum_{j=1}^{r_i} \delta_{ij} \left[\frac{{}_{t_{ij}\lambda_{i1}}u''_{ij}}{{}_{t_{ij}}u'_{ij}} - (\theta_c + 1)_{\lambda_{i1}}u'_{ij} u_{ij}^{-1} \right] \\ =& \sum_{j=1}^{r_i} [-{}_{\lambda_{i1}}u'_{ij}(\theta_c + 1) - \mathcal{R}_{ij}] \\ &\quad \times [(u_{ij}^{-\theta_c} + v_{ij}^{-\theta_c} - 1) I(u_{ij}^{-\theta_c} + v_{ij}^{-\theta_c} > 1)]^{-1/\theta_c}, \\ &\sum_{j=1}^{r_i} (1 - \delta_{ij}) \left[\frac{{}_{t_{ij}\lambda_{i2}}v''_{ij}}{{}_{t_{ij}}v'_{ij}} - (\theta_c + 1)_{\lambda_{i2}}v'_{ij} v_{ij}^{-1} \right] \\ =& \sum_{j=1}^{r_i} [-{}_{\lambda_{i2}}v'_{ij}(\theta_c + 1) - \mathcal{R}_{ij}] \\ &\quad \times [(u_{ij}^{\theta_c} + v_{ij}^{-\theta_c} - 1) I(u_{ij}^{-\theta_c} + v_{ij}^{-\theta_c} > 1)]^{-1/\theta_c}. \end{aligned} \tag{8}$$

When Frank family is considered,

$$\varphi(t) = -\log\{(e^{-\theta_c t} - 1)/(e^{-\theta_c} - 1)\}, C_{ij} = -\frac{1}{\theta_c} \log \left[1 + \frac{(e^{-\theta_c u_{ij}} - 1)(e^{-\theta_c v_{ij}} - 1)}{e^{-\theta_c} - 1} \right].$$

We derive the likelihood equations under S_i as

$$\begin{aligned} &\sum_{j=1}^{r_i} \delta_{ij} \left\{ \frac{{}_{t_{ij}\lambda_{i1}}u''_{ij}}{{}_{t_{ij}}u'_{ij}} + {}_{\lambda_{i1}}u'_{ij}\theta_c (e^{-\theta_c u_{ij}} - 1)^{-1} \right\} \\ =& \sum_{j=1}^{r_i} {}_{\lambda_{i1}}u'_{ij}\theta_c (e^{-\theta_c u_{ij}} - 1)^{-1} \\ &+ \sum_{j=1}^{r_i} {}_{\lambda_{i1}}u'_{ij} \frac{\theta_c \mathcal{R}_{ij}\{1 + (e^{-\theta_c} - 1)/[(e^{-\theta_c u_{ij}} - 1)(e^{-\theta_c v_{ij}} - 1)]\}}{e^{-\theta_c u_{ij}} (e^{-\theta_c u_{ij}} - 1)^{-1} C_{ij}}, \\ &\sum_{j=1}^{r_i} (1 - \delta_{ij}) \left\{ \frac{{}_{t_{ij}\lambda_{i2}}v''_{ij}}{{}_{t_{ij}}v'_{ij}} + {}_{\lambda_{i2}}v'\theta_c (e^{-\theta_c v_{ij}} - 1)^{-1} \right\} \\ =& \sum_{j=1}^{r_i} {}_{\lambda_{i2}}v'_{ij}\theta_c (e^{-\theta_c v_{ij}} - 1)^{-1} \end{aligned} \tag{9}$$

$$+\sum_{j=1}^{r_i}\lambda_{i2}v'_{ij}\theta_c\frac{\theta_c\mathcal{R}_{ij}\{1+(e^{-\theta_c}-1)/[(e^{-\theta_c u_{ij}}-1)(e^{-\theta_c v_{ij}}-1)]\}}{e^{-\theta_c v_{ij}}(e^{-\theta_c v_{ij}}-1)^{-1}C_{ij}}.$$

Having the MLEs of parameters $\boldsymbol{\theta}_i=(\theta_{i1},\theta_{i2}), i=1,2,\ldots,k$, the approximate confidence interval, the parametric percentile bootstrap confidence interval and the parametric bootstrap-t confidence interval of parameters from exponential competing risks can be obtained.

3.3 Asymptotic confidence interval

The approximate confidence intervals for the parameters are derived based on the asymptotic distributions of the maximum likelihood estimates using large sample approximation. Under S_i, the asymptotic normality of $\hat{\boldsymbol{\theta}}_i=(\hat\theta_{i1},\hat\theta_{i2})$ is $I^{1/2}(\boldsymbol{\theta}_i)(\hat{\boldsymbol{\theta}}_i-\boldsymbol{\theta}_i)\xrightarrow{d}N(\mathbf{0},I_2)$ as $r_i\to\infty$, where I_2 denotes the identity matrix in $\mathbf{R}^{2\times 2}$ and $I(\theta_i)$ is the observed information matrix. The $(1-\alpha)\%$ confidence interval of θ_{il} is approximated as $[\hat\theta_{il}-u_{\alpha/2}\sqrt{\mathrm{var}(\hat\theta_{il})},\hat\theta_{il}+u_{\alpha/2}\sqrt{\mathrm{var}(\hat\theta_{il})}], l=1,2$. $u_{\alpha/2}$ is the $\alpha/2$th upper quantile of the standard normal distribution function and $\mathrm{var}(\hat\theta_{il})$ is denoted by the diagonal value of the inverse of $I(\theta_i)$. That is

$$I^{-1}(\boldsymbol{\theta}_i)=\left.\begin{pmatrix}-\frac{\partial^2\log L_i}{\partial\theta_{i1}{}^2} & -\frac{\partial^2\log L_i}{\partial\theta_{i1}\partial\theta_{i2}}\\ -\frac{\partial^2\log L_i}{\partial\theta_{i2}\partial\theta_{i1}} & -\frac{\partial^2\log L_i}{\partial\theta_{i2}{}^2}\end{pmatrix}^{-1}\right|_{(\theta_{i1},\theta_{i2})=(\hat\theta_{i1},\hat\theta_{i2})}=\begin{pmatrix}\mathrm{var}(\hat\theta_{i1}) & \mathrm{cov}(\hat\theta_{i1},\hat\theta_{i2})\\ \mathrm{cov}(\hat\theta_{i1},\hat\theta_{i2}) & \mathrm{var}(\hat\theta_{i2})\end{pmatrix}. \quad (10)$$

The second partial derivatives for Clayton copula are given by

$$\frac{\partial^2\log L_i}{\partial\theta_{i1}{}^2}=\sum_{j=1}^{r_i}[-\delta_{ij}\lambda_{i1}^{-2}+(\theta_c+1)t_{ij}^2e^{-\lambda_{i1}t_{ij}}(e^{\theta_c\lambda_{i1}t_{ij}}+e^{\theta_c\lambda_{i2}t_{ij}}-1)^{-\theta_c^{-1}}]$$
$$+\sum_{j=1}^{r_i}t_{ij}e^{-\theta_c\lambda_{i1}t_{ij}}[(\theta_c+1)t_{ij}e^{-\lambda_{i1}t_{ij}}-\mathcal{R}_{ij}](e^{\theta_c\lambda_{i1}t_{ij}}+e^{\theta_c\lambda_{i2}t_{ij}}-1)^{-\theta_c^{-1}-1},$$

$$\frac{\partial^2\log L_i}{\partial\theta_{i1}\partial\theta_{i2}}=\sum_{j=1}^{r_i}t_{ij}e^{-\theta_c\lambda_{i2}t_{ij}}[(\theta_c+1)t_{ij}e^{-\lambda_{i1}t_{ij}}-\mathcal{R}_{ij}](e^{\theta_c\lambda_{i1}t_{ij}}+e^{\theta_c\lambda_{i2}t_{ij}}-1)^{-\theta_c^{-1}-1},$$

$$\frac{\partial^2\log L_i}{\partial\theta_{i2}{}^2}=\sum_{j=1}^{r_i}[-(1-\delta_{ij})\lambda_{i2}^{-2}+(\theta_c+1)t_{ij}^2e^{-\lambda_{i2}t_{ij}}(e^{\theta_c\lambda_{i1}t_{ij}}+e^{\theta_c\lambda_{i2}t_{ij}}-1)^{-\theta_c^{-1}}]$$
$$+\sum_{j=1}^{r_i}t_{ij}e^{-\theta_c\lambda_{i2}t_{ij}}[(\theta_c+1)t_{ij}e^{-\lambda_{i1}t_{ij}}-\mathcal{R}_{ij}](e^{\theta_c\lambda_{i1}t_{ij}}+e^{\theta_c\lambda_{i2}t_{ij}}-1)^{-\theta_c^{-1}-1}.$$

The second partial derivatives for Frank copula can be similarly obtained. We then omit the derivatives here.

3.4 Bootstrap confidence intervals

Different from approximate confidence intervals based on the property of the asymptotic normality, parametric bootstrap confidence intervals are given by bootstrapping. There are parametric percentile bootstrap confidence interval and parametric bootstrap-t confidence interval. Given the MLEs $\hat{\boldsymbol{\theta}}_i=(\hat\theta_{i1},\hat\theta_{i2}), i=1,2,\ldots,k$, the bootstrap confidence intervals can be calculated using the following steps.

3.4.1 Percentile bootstrap confidence interval

(1) Under the specified progressive censoring $(\mathcal{R}_{i1}, \ldots, \mathcal{R}_{ir_i})$, apply $\hat{\boldsymbol{\theta}}_i$ to generate a bootstrap progressive censoring competing risk sample $\{(t^*_{i1}, \delta^*_{i1}, \mathcal{R}_{i1}), (t^*_{i2}, \delta^*_{i2}, \mathcal{R}_{i2}), \ldots, (t^*_{ir_i}, \delta^*_{ir_i}, \mathcal{R}_{ir_i})\}$ at stress level $S_i, i = 1, 2, \ldots, k$. Then we obtain the bootstrap estimates $\hat{\boldsymbol{\theta}}^*_i$ of $\boldsymbol{\theta}_i$ using maximum likelihood method.
(2) Repeat Step (1) N times.
(3) Compute the empirical cumulative distribution function $\hat{F}(x) = P(\hat{\theta}^*_{il} \leq x)$. Let $\hat{\theta}^{il}_{Boot-p}(p) = \hat{F}^{-1}(p)$ be the p-th quantile of cumulative distribution function. Then, the $100(1-\alpha)\%$ confidence interval for θ_{il} can be approximated by $\left(\hat{\theta}^{il}_{Boot-p}(\alpha/2), \hat{\theta}^{il}_{Boot-p}(1-\alpha/2)\right)$.

3.4.2 Bootstrap-t confidence interval

(1) Generate a bootstrap progressive censoring competing risk sample $\{(t^*_{i1}, \delta^*_{i1}, \mathcal{R}_{i1}), (t^*_{i2}, \delta^*_{i2}, \mathcal{R}_{i2}), \ldots, (t^*_{ir_i}, \delta^*_{ir_i}, \mathcal{R}_{ir_i})\}$ using $\hat{\boldsymbol{\theta}}_i$ for $i = 1, 2, ..., k$, and obtain the bootstrap MLE $\hat{\boldsymbol{\theta}}^*_i$ of $\boldsymbol{\theta}_i$.
(2) Resample the competing risks data $\{(\tilde{t}_{i1}, \tilde{\delta}_{i1}, \mathcal{R}_{i1}), (\tilde{t}_{i2}, \tilde{\delta}_{i2}, \mathcal{R}_{i2}), \ldots, (\tilde{t}_{ir_i}, \tilde{\delta}_{ir_i}, \mathcal{R}_{ir_i})\}$ using $\hat{\boldsymbol{\theta}}^*_i$, and compute a new bootstrap estimate $\tilde{\boldsymbol{\theta}}_i$.
(3) Repeat Step (2) N_1 times. The variance of $\hat{\theta}^*_{il}$ is given by $Var(\hat{\theta}^*_{il}) = (\hat{\theta}^*_{il} - \tilde{\theta}_{il})^2/N_1$.
(4) Determine the $T^*_i = (T^*_{i1}, T^*_{i2})$ statistic where $T^*_{il} = \frac{\hat{\theta}^*_{il} - \hat{\theta}_{il}}{\sqrt{Var(\hat{\theta}^*_{il})}}, l = 1, 2$.
(5) Repeat Step (2)–(4) N times.
(6) Compute the empirical cumulative distribution function $\hat{F}(x) = P(\hat{\theta}^*_{il} \leq x)$. Let $\hat{\theta}^{T_p}_{il}$ be the p-th quantile of $\hat{\theta}^*_{il}$. The $100(1-\alpha)\%$ confidence interval for θ_{il} can be approximated by $\left(\hat{\theta}^{il}_{Boot-t}(\alpha/2), \hat{\theta}^{il}_{Boot-t}(1-\alpha/2)\right)$ with

$$\hat{\theta}^{il}_{Boot-t}(p) = \hat{\theta}^{T_p}_{il} - \sqrt{Var(\hat{\theta}^*_{il})}\hat{F}^{-1}(1-p).$$

§4 Nonparametric reliability estimation

The likelihood based reliability estimates under the normal stress level S_0 can be given by the parameter estimates in the accelerated dependent competing risks model above. For comparison with parametric estimation, we introduce the nonparametric reliability estimation method under S_0.

The failure times $\underline{t}_i = \{t_{i1}, \ldots, t_{ir_i}\}$ at each stress level $S_i, i = 1, 2, \ldots, k$ are not the real failure times under S_0. To estimate the reliability under S_0, we transform the failure times from S_i to S_0. It is also assumed that the failure cause for each failure time retain the same from S_i to S_0. Let $\underline{t}'_i = \{t'_{i1}, \ldots, t'_{ir_i}\}$, δ'_{ij} and $\mathcal{R}'_{ij}$ be the transformed time, failure cause and progressive censoring scheme under S_0, respectively. We have the transformation expression of competing risks data $(t'_{ij}, \delta'_{ij}, \mathcal{R}'_{ij})$ given by

$$t'_{ij} = \begin{cases} \alpha_1(S_i)t_{ij}, \text{ if } \delta_{ij} = 1 \\ \alpha_2(S_i)t_{ij}, \text{ if } 1 - \delta_{ij} = 1 \end{cases}, \qquad \delta'_{ij} = \delta_{ij}, \qquad \mathcal{R}'_{ij} = \mathcal{R}_{ij}, \tag{11}$$

where $j = 1, 2, \ldots, r_i$. The functions $\alpha_l(S), l = 1, 2$ are the acceleration functions for each competing risk. We assume that the derivative of $\log \alpha_l(S)$ with regard to S is of linear form, which is expressed as $\frac{d\log\alpha_l(S)}{dS}=\sum_{m=1}^{M}c_{lm}h_m(S)$. Then the acceleration function is given by

$$\log\alpha_l(S)=\sum_{m=1}^{M}c_{lm}H_m(S),$$

where $H_m(S)=\int_{S_0}^{S}h_m(s)ds$. More details of the transformation setting refers to [35].

Considering the progressive censoring scheme, we construct two linear models for bivariate competing risks between the log p-th quantile $\log\xi_{i,p}^l$ and $\log\alpha_l(S)$as

$$\log\xi_{i,p}^l=c_{l0}-\sum_{m=1}^{M}c_{lm}H_m(S)+\varepsilon_i, i=1,2,\ldots,k, l=1,2. \quad (12)$$

The least square estimates $\hat{\underline{c}}_l=(c_{l0},c_{l1},\ldots,c_{lM})^{\mathrm{T}}$ is given by

$$\hat{\underline{\beta}}_l=(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}\underline{v}_p^l$$

where

$$\underline{v}_p^l=\begin{bmatrix}\log\xi_{1,p}^l\\ \log\xi_{2,p}^l\\ \vdots\\ \log\xi_{k,p}^l\end{bmatrix}, X=\begin{bmatrix}1 & H_1(S_1) & \ldots & H_M(S_1)\\ 1 & H_1(S_2) & \ldots & H_M(S_2)\\ \vdots & \vdots & \vdots & \vdots\\ 1 & H_1(S_k) & \ldots & H_M(S_k)\end{bmatrix}, \underline{\beta}_l=\begin{bmatrix}\hat{c}_{l0}\\ -\hat{c}_{l1}\\ \vdots\\ -\hat{c}_{lM}\end{bmatrix}.$$

We calculate the p-th quantile $\xi_{i,p}^l$ using the nonparametric distribution function $\hat{F}_i^l$, which is estimated based on the competing risks data with progressive censoring at S_i. Bordes [2] introduced the product limit estimation method of nonparametric reliability estimation for the progressive censoring observation. We improve this method for progressive censoring data in the presence of competing risks $\underline{t}_i=(t_{ij},\delta_{ij},\mathcal{R}_{ij})$. The reliability function $R_i^l(t)$ is estimated by

$$\hat{R}_i^l(t)=\prod_{\{1\le j\le r_i,t_{ij}\le t\}}\left[1-\frac{I(l=1,\delta_{ij}=1)+I(l=2,\delta_{ij}=0)}{\alpha_{ij}}\right], \quad (13)$$

where $\alpha_{ij}=\sum_{j_1=j}^{r_i}\mathcal{R}_{ij_1}+m-j+1$ and $I(\cdot)$ is an indicator function. Then we get the approximate estimates $\hat{\xi}_{i,p}^l$ using $\hat{R}_i^l(t)$. To summarize the time transformation procedures, a diagram is presented in Figure 1.

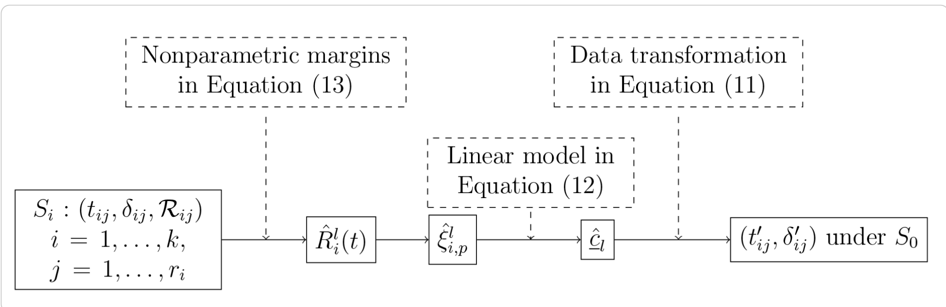

Figure 1. Transformation diagram of failure times.

We obtain the reliability estimate $\hat{R}_i^l(t)$ of competing risk under S_0 using Equation (13) for the transformed competing risks data with progressive censoring. Afterwards, the overall reliability function $R_0(t)$ for the dependent competing risks model is estimated using a kernel copula estimation $\hat{C}_0(u,v)$. The kernel copula density estimation $\hat{c}_0(u,v)$ of $\hat{C}_0(u,v)$ has been introduced by Chen and Fan [6], Nalger [23] and Geenens et al. [9]. In this method, the uniform data (u,v) need to be transformed using the inverse function of standard normal cumulative distribution function. The copula density is expressed as

$$\hat{c}_0(u,v) = \frac{\hat{f}(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))} \tag{14}$$

where $\hat{f}(\cdot,\cdot)$ is a standard kernel estimator on the transformed data $(\Phi^{-1}(u), \Phi^{-1}(v))$. The reliability estimate under S_0 is of the form

$$\hat{R}_0(t) = \hat{C}_0(u,v)|_{u=\hat{R}_0^1(t), v=\hat{R}_0^2(t)} = \int_0^{\hat{R}_0^1(t)} \int_0^{\hat{R}_0^2(t)} \hat{c}_0(u,v)dudv. \tag{15}$$

In Equation (15) the nonparametric kernel copula and nonparametric margins are used to express the nonparametric reliability estimator. It is remarked that the semiparmetric estimates of reliability $\tilde{R}_0(t)$ can be obtained when the nonparametric margins $\hat{R}_0^l(t)$ in Equations (14)-(15) is replaced by the parametric margins $u = R_0^1(t;\hat{\theta}_{01}), v = R_0^2(t;\hat{\theta}_{02})$.

§5 Simulation study

We simulate the accelerated model in the two cases that the dependence of bivariate and trivariate exponential competing risks is measured by Clayton family. The comparison of estimation methods is presented in this section.

5.1 Bivariate dependent competing risks

Consider $k = 3$ stress levels as $S_1 = 240, S_2 = 280, S_3 = 300$, respectively, and the use stress level $S_0 = 220$. Specify the accelerated function $\log\theta_{il} = a_l + b_l \log S_i$ for $i = 1,2,3, l = 1,2$. The coefficients in the acceleration function are set as $a_1 = -22.8518, b_1 = 4.1717, a_2 = -8.9373, b_2 = 1.7551$.

Under each stress S_i in CSALT, there are n_i tested products, and $\mathcal{R}_{ij}$ products are withdrawn when the j-th failure takes place. The test under S_i is terminated until r_i failures are observed for $j = 1,2,\ldots,r_i, i = 1,2,3$. To compare the numerical results, we consider two progressive schemes given in Table 2 in the constant-stress accelerated life test.

Table 2. Removal schemes under different sample sizes.

Scheme	Sample size	Stress	Removals
Scheme 1	$(n_1,n_2,n_3) = (60\ 60\ 80)$ $(r_1,r_2,r_3) = (30\ 30\ 40)$	S_1	$((2,\ldots,2)_{1\times15},(0,\ldots,0)_{1\times15})$
		S_2	$((2,\ldots,2)_{1\times15},(0,\ldots,0)_{1\times15})$
		S_3	$((2,\ldots,2)_{1\times20},(0,\ldots,0)_{1\times20})$
Scheme 2	$(n_1,n_2,n_3) = (100\ 100\ 100)$ $(r_1,r_2,r_3) = (50\ 50\ 50)$	S_1 S_2 S_3	$((2,\ldots,2)_{1\times25},(0,\ldots,0)_{1\times25})$

Denote $c_u(v) = \frac{\partial C(u,v)}{\partial u}$. The competing risks data is simulated by means of the below algorithm using the conditional approach for an Archimedean copula $C(u,v)$.

(1) Generate two independent progressive uniform (0,1) samples u_{ij} and w_{ij} using $(\mathcal{R}_{i1}, \mathcal{R}_{i2}, \ldots, \mathcal{R}_{ir_i})$ for $j = 1, 2, \ldots, r_i, i = 1, 2, 3$ using the algorithm in [3].
(2) Set $v = c_u^{-1}(w)$, where $c_u^{-1}(w)$ is the quasi-inverse of $c_u(w)$. When the Archimedean copula is Clayton family,
$$v_{ij} = c_{u_{ij}}^{-1}(w_{ij}) = \left[{w_{ij}}^{-\theta_c/(\theta_c+1)} {u_{ij}}^{-\theta_c} - {u_{ij}}^{-\theta_c} + 1 \right]^{-1/\theta_c}.$$
(3) Set $x_{ij1} = -\log(1 - u_{ij})/\lambda_{i1}$ and $x_{ij2} = -\log(1 - v_{ij})/\lambda_{i2}$.
(4) Obtain $(t_{ij}, \delta_{ij}) = (\min(x_{ij1}, x_{ij2}), I(x_{ij1} < x_{ij2}))$ for $j = 1, 2, \ldots, r_i, i = 1, 2, 3$.

Based on the simulated competing risks data, we obtain MLEs by solving the log-likelihood equations in Equation (8), the mean square errors (MSEs), the absolute errors (AEs), the relative errors (REs). The interval lengths (ILs) and the coverage probabilities (CPs) of the 95% confidence intervals of parameters, namly asymptotic confidence intervals (ACI), percentile bootstrap (Boot-p) confidence intervals and bootstrap-t (Boot-t) confidence intervals, are presented. As larger ILs impliy higher CPs, it is not expected to separately compare ILs and CPs. An alternative method using interval scores (IS) is introduced in [11] to improve the separate comparision of confidence intervals. This score is often used for quantile and interval prediction. As this score evaluates ILs and CPs simultaneously, it is also a good choice to show the performance of intervals. Then we apply it to compare the performance (Perfm) of interval estimation for model parameters. Let l_j and u_j for $j = 1, 2, \ldots, N$ be the lower and upper intervals in the $(1 - \alpha)100\%$ confidence intervals of certain accelerated distribution parameter θ at levels $\alpha/2$ and $1 - \alpha/2$ for each simulated data. θ_0 is the real value of θ. The interval score of θ is defined as
$$\mathrm{IS}(\alpha) = \frac{1}{N} \sum_{j=1}^{N} \left[(u_j - l_j) + \frac{2}{\alpha}(l_j - \theta_0) I(\theta_0 < l_j) + \frac{2}{\alpha}(\theta_0 - u_j) I(\theta_0 > u_j) \right],$$
where N is the simulation number. In this study, we set $N = 500$. These results are shown in Tables (3)-(5) in the case that the Kendall's tau is $\tau = 0.5$ for bivariate Clayton family.

Table 3. Parameter estimates for accelerated bivariate competing risks distributions under Scheme 1.

Method	Perfm	λ_{11}	λ_{12}	λ_{21}	λ_{22}	λ_{31}	λ_{32}
MLE	MLEs	0.9574	1.8378	2.1122	2.4444	2.6085	2.9147
	MSEs	0.0018	0.0263	0.0126	0.0031	0.0118	0.0073
	AEs	0.0426	0.1622	0.1122	0.0556	0.1085	0.0853
	REs	0.0426	0.0811	0.0561	0.0222	0.0434	0.0284
ACI	ILs	1.3210	1.4278	2.0943	2.1085	2.1891	2.2025
	CPs	0.9340	0.8700	0.9780	0.9500	0.9940	0.9540
	IS	1.6930	2.3858	2.2549	2.3862	2.2731	2.5623
Boot-p	ILs	1.0991	1.3840	1.8602	1.9840	1.9136	2.0313
	CPs	0.8400	0.8640	0.9360	0.9480	0.9400	0.9460
	IS	2.2557	2.4300	2.4315	2.3078	2.4373	2.4514
Boot-t	ILs	2.5014	3.1648	3.8361	4.3162	3.9071	4.3183
	CPs	0.9900	0.9980	1.0000	1.0000	1.0000	1.0000
	IS	2.6039	3.1648	3.8361	4.3162	3.9071	4.3183

Table 4. Parameter estimates for accelerated bivariate competing risks distributions under Scheme 2.

Method	Perfm	λ_{11}	λ_{12}	λ_{21}	λ_{22}	λ_{31}	λ_{32}
MLE	MLEs	0.9711	1.8296	2.1007	2.4013	2.5862	2.8741
	MSEs	0.0008	0.0290	0.0101	0.0097	0.0074	0.0159
	AEs	0.0289	0.1704	0.1007	0.0987	0.0862	0.1259
	REs	0.0289	0.0852	0.0503	0.0395	0.0345	0.0420
ACI	ILs	1.0378	1.1004	1.6045	1.6054	1.9367	1.9447
	CPs	0.9540	0.8480	0.9880	0.9540	0.9740	0.9540
	IS	1.1707	1.9027	1.7115	1.7856	2.1046	2.4183
Boot-p	ILs	0.8714	1.0318	1.3792	1.4665	1.6776	1.7765
	CPs	0.8440	0.7660	0.9240	0.9360	0.9340	0.9380
	IS	1.8412	2.4778	1.7553	1.7516	2.1150	2.3325
Boot-t	ILs	1.9186	2.3333	2.7973	3.1287	3.4068	3.7604
	CPs	1.0000	0.9980	1.0000	1.0000	1.0000	0.9980
	IS	1.9186	2.3338	2.7973	3.1287	3.4068	3.7627

Table 5. Parameter estimates in the bivariate case under S_0.

Scheme	Perfm	a_1	b_1	a_2	b_2	λ_{01}	λ_{02}
Scheme 1	MLEs	-25.2326	4.5995	-10.5270	2.0306	0.6542	1.5301
	MSEs	5.6684	0.1831	2.5274	0.0759	0.0025	0.0281
	AEs	2.3808	0.4278	1.5898	0.2755	0.0497	0.1676
	REs	0.1042	0.1026	0.1779	0.1570	0.0706	0.0987
Scheme 2	MLE	-24.6342	4.4928	-10.2578	1.9804	0.6693	1.5279
	MSEs	3.1772	0.1031	1.7439	0.0508	0.0012	0.0288
	AEs	1.7825	0.3211	1.3206	0.2253	0.0346	0.1697
	REs	0.0780	0.0770	0.1478	0.1284	0.0492	0.1000

From Tables 3 and 4, we see that the MSEs, AEs and REs of parameter estimators from the first competing failure cause decrease when the sample size n_i at each accelerated stress level increases. This imply that more failure information on the first cause occurs than the other cause. Comparing the interval scores, it shows that ACI performs better than the bootstrap approach. From the estimates under the normal stress level in Table 5, it indicates that likelihood based estimation method on the accelerated dependent competing risks with progressive censoring scheme has a good performance.

For the nonparametric estimation, we set $M = 1$ and $H_1(S) = \log(S) - \log(S_0)$ in Equation (12). Two nonparamteric reliability estimates using nonparametric copula are obtained when the marginal distributions of failure causes are fitted in parametric and nonparametric methods, which are presented in Figure 2 along with parametric methods. The parametric reliability under the assumption of independence is also displayed.

From Figure 2, we observe that the reliability estimates using parametric margins in the dependent competing risks perfom better than the nonparametric copula and margins and the independent parametric method. The error of the method using parametric copula and margins has a small change with increasing t. The nonparamtric copula based reliability estimate with

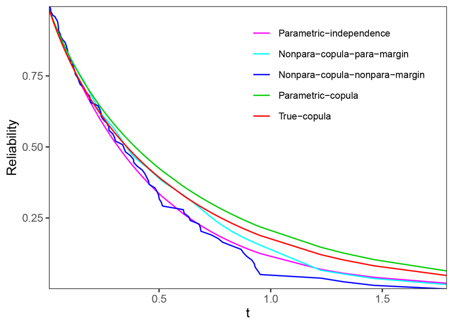

Figure 2. Comparision of reliability estimation.

parametric margins is close to the true model when t is small. However, the error becomes larger as t increases.

5.2 Trivariate dependent competing risks

In the trivariate competing risks model, the setting of accelerated life testing and the first two competing risks remain the same with the bivariate case. The coefficients of the third competing risk are set as $a_3 = -15.5174$ and $b_3 = 2.8984$. The random samples of three-dimensional variables for Clayton copula are generated using the Marchall-Olkin approach [19] instead of the conditional sampling approach in Step (2) of the bivariate simulation algorithm. Here we consider the removal Scheme 1. The likelihood based parameter estimates are presented in Tables 6 and 7.

Table 6. Parameter estimates for accelerated trivariate competing risks distributions.

Method	Perfm	λ_{11}	λ_{12}	λ_{13}	λ_{21}	λ_{22}	λ_{23}	λ_{31}	λ_{32}	λ_{33}
MLE	MLEs	0.6240	2.1008	1.5252	1.8574	2.4912	1.8464	2.2895	2.9444	2.9478
	MSEs	0.1414	0.0102	0.0006	0.0203	0.0001	0.0236	0.0443	0.0031	0.0027
	AEs	0.3760	0.1008	0.0252	0.1426	0.0088	0.1536	0.2105	0.0556	0.0522
	REs	0.3760	0.0504	0.0168	0.0713	0.0035	0.0768	0.0842	0.0185	0.0174
ACI	ILs	1.3355	1.8203	1.77l16	2.3265	2.4016	2.3140	2.6106	2.6945	2.6890
	CPs	0.6640	0.9580	0.9440	0.9320	0.9440	0.9120	0.9240	0.9420	0.9420
	IS	4.7063	2.1024	2.2260	3.0428	2.8598	3.3238	3.5071	3.1426	3.0793
Boot-p	ILs	0.6927	1.9051	1.6725	2.0589	2.3773	2.0717	2.2316	2.4826	2.4859
	CPs	0.6050	0.9560	0.9000	0.8220	0.9400	0.8100	0.7600	0.9200	0.9180
	IS	9.9845	2.1383	2.6994	5.0832	2.9645	5.3893	7.9495	3.7096	3.5736
Boot-t	ILs	2.0576	4.1773	3.7208	4.9489	5.1207	4.9384	5.5558	5.3139	5.3611
	CPs	0.7580	1.0000	0.9980	0.9900	1.0000	0.9960	0.9920	1.0000	1.0000
	IS	4.9429	4.1773	3.7921	5.0809	5.1207	5.1237	5.7065	5.3139	5.3611

Table 7. Parameter estimates in the trivariate case under S_0.

Perfm	a_1	b_1	a_2	b_2	a_3	b_3	λ_{01}	λ_{02}	λ_{03}
MLEs	-33.5114	6.0354	-7.1876	1.4446	-14.2396	2.6657	0.3833	1.8299	1.1483
MSEs	113.6275	3.4732	3.0612	0.0964	1.6328	0.0542	0.1028	0.0175	0.0007
AEs	10.6596	1.8637	1.7496	0.3105	1.2778	0.2327	0.3206	0.1323	0.0256
REs	0.4665	0.4467	0.1958	0.1769	0.0823	0.0803	0.4554	0.0779	0.0228

From Tables 6 and 7, it can be seen that MLEs of parameters at the accelerated stress levels perform good. The ILs and IS of ACI are smaller than the bootstrap method. Comparing Tables 3 and 6, we find that ILs and IS in three-dimensional case are larger than the bivariate case for the same parameters. The estimates of the coefficients in the acceleration functions have larger MSEs, AEs and REs from Tables 5 and 7 under Scheme 1. This indicates the accuracy of likelihood based estimation is reduced with the increasing number of failure causes.

§6 Real data analysis

For illustration of the parametric and nonparametric method, we analyze a competing risks data of Class-H insulation systems for electric motors in a constant accelerated life test. The setting of the stress levels are at 4 accelerated temperatures $(S_1, S_2, S_3, S_4) = (453\text{K}, 463\text{K}, 493\text{K}, 513\text{K})$ and the normal stress level $S_0 = 423\text{K}$. The details of this dataset refer to Klein and Basu [16] and Nelson [24]. We choose two failure causes, namely Turn and Phase causes in the bivariate competing risks, and the failure times are used by dividing 10^4 in case of the occurring of likelihood tending to zero. The failure times of each competing risk are considered to be from exponential distribution $\text{Exp}(\lambda_{il}), i = 1, 2, 3, 4, l = 1, 2$. The acceleration function is expressed as $\log \lambda_{il} = a_l + b_l \log(S_i)$.

The parametric inference in Section 3 is based on a copula with the specified copula parameter θ_c. Here we consider three copulas, namely Clayton, Gumbel and Frank, to capture the dependence between the two competing risks. The three copulas show three tail behaviors. Clayton copula can capture the lower tail dependence, and Gumbel copula shows the opposite behavior. For Frank copula, it is a symmetric copula. We use the Akaike information criterion (AIC) ([8], [26]) to select the copula. According to the relationship between Kendall's τ and copula parameter, we discretize the τ interval $[0, 1]$ with the length 0.005 to get the possible copula parameters. For each θ_c corresponding to the discrete τ, AIC is calculated and compared to select the copula and θ_c. For this dataset, Frank copula with $\hat{\theta}_c = 0.2883$ is determined corresponding to $\hat{\tau} = 0.1260$.

The simulation study shows that the asymptotic confidence intervals of dependent model parameters are better than the bootstrap method. Thus we present the parameter estimates under S_0 and asymptotic confidence intervals in Tables 8 and 9. The overall reliability estimation under S_0 is shown in Figure 3.

Table 8. Parameter estimates under accelerated stress levels.

Perfm	λ_{11}	λ_{12}	λ_{21}	λ_{22}
MLE	0.8796	1.4431	3.0456	4.9805
ACI-Lower	0.4346	0.8778	1.5071	3.0326
ACI-Upper	1.3245	2.0085	4.5842	6.9285
ACI-IL	0.8899	1.1307	3.0771	3.8958
	λ_{31}	λ_{32}	λ_{41}	λ_{42}
MLE	3.4257	5.5874	5.5312	11.1010
ACI-Lower	1.6981	3.4064	2.5315	6.9262
ACI-Upper	5.1533	7.7684	8.5308	15.2758
ACI-ILs	3.4552	4.3620	5.9993	8.3496

Table 9. Parameter estimates under S_0.

Parameters	Estimates
a_1	-71.3253
b_1	11.7123
a_2	-79.2305
b_2	13.0808
λ_{01}	0.6086
λ_{02}	0.8814

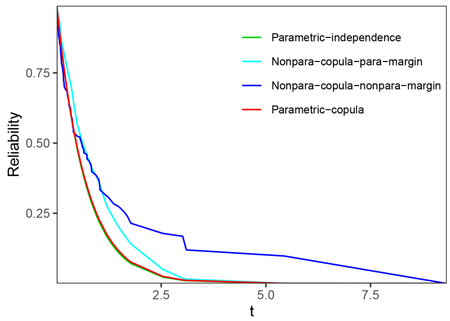

Figure 3. Reliability estimation under S_0 in the Data analysis.

From Figure 3, we see that the reliability function using frank copula model is close to the independence copula. It can be interpreted by the determined small $\tau = 0.126$ using AIC.

§7 Conclusion

In this paper, the explicit functional forms of the likelihood function and likelihood equations are presented for specified Archimedean copula-based dependent competing risks model in constant-stress accelerated life test. The progressive censoring competing risks data is analyzed using the parametric estimation method, and a non-parametric method is introduced to obtain the nonparametric reliability at the use condition. The simulation study and data analysis indicate that the estimation methods have a good performance, and the asymptotic confidence interval is better than bootstrap confidence intervals. The non-parametric reliability method can be extended to multivariate competing risks model in future work.

Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]School of Mathematics and Statistics, Xidian University, Xi'an 710126, China.
Email: cfzhang917@xidian.edu.cn

[2]School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, China.
Email: ymshi@nwpu.edu.cn

[3]School of Mathematics, Yunnan Normal University, Kunming 650500, China.
Email: liang610112@163.com