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Sharp power-type Heronian and Lehmer means inequalities for the complete elliptic integrals

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Abstract. In the article, we prove that the inequalities

$$H_p(\mathscr{K}(r),\mathscr{E}(r)) > \frac{\pi}{2}, \quad L_q(\mathscr{K}(r),\mathscr{E}(r)) > \frac{\pi}{2}$$

hold for all $r \in (0,1)$ if and only if $p \geq -3/4$ and $q \geq -3/4$, where $H_p(a,b)$ and $L_q(a,b)$ are respectively the p-th power-type Heronian mean and q-th Lehmer mean of a and b, and $\mathscr{K}(r)$ and $\mathscr{E}(r)$ are respectively the complete elliptic integrals of the first and second kinds.

§1 Introduction

Let $p, q \in \mathbb{R}$ and $a, b > 0$. Then the p-th power-type Heronian mean $H_p(a,b)$ [1] and q-th Lehmer mean $L_q(a,b)$ [2] of a and b are defined by

$$H_p(a,b) = \begin{cases} \left[\frac{a^p+(ab)^{p/2}+b^p}{3}\right]^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0 \end{cases} \tag{1}$$

and

$$L_q(a,b) = \frac{a^{q+1}+b^{q+1}}{a^q+b^q}. \tag{2}$$

It is well-known that both $H_p(a,b)$ and $L_q(a,b)$ are continuous and strictly increasing with respect to $p, q \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. More properties for the p-th power-type Heronian and q-th Lehmer means can be found in the literature [3-8].

For $r \in (0,1)$, Legendre's complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ of the first and second

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kinds [9] are defined by

$$\begin{cases} \mathscr{K}(r) = \int_0^{\frac{\pi}{2}} \left(1 - r^2 \sin\theta\right)^{-1/2} d\theta, \\ \mathscr{K}'(r) = \mathscr{K}(r'), \\ \mathscr{K}(0^+) = \frac{\pi}{2}, \quad \mathscr{K}(1^-) = \infty \end{cases} \tag{3}$$

and

$$\begin{cases} \mathscr{E}(r) = \int_0^{\frac{\pi}{2}} \left(1 - r^2 \sin\theta\right)^{1/2} d\theta, \\ \mathscr{E}'(r) = \mathscr{E}(r'), \\ \mathscr{E}(0^+) = \frac{\pi}{2}, \quad \mathscr{E}(1^-) = 1, \end{cases} \tag{4}$$

where and in what follows we denote $r' = \sqrt{1-r^2}$. In the sequel, we denote $\mathscr{K}(r)$ and $\mathscr{E}(r)$ by $\mathscr{K}$ and $\mathscr{E}$ if no risk for confusion, respectively.

The complete elliptic integrals $\mathscr{K}$ and $\mathscr{E}$ play a very important role in many branches of mathematics such as classical analysis, number theory, geometric function theory, and conformal and quasi-conformal mappings [10-26]. Recently, the complete elliptic integrals $\mathscr{K}$ and $\mathscr{E}$ have attracted the attention of many researchers, in particular, many remarkable properties and inequalities for them can be found in the literature [27-34].

In 1990, Anderson, Vamanamurthy and Vuorinen [35, Theorem 3.31] proved that the inequality

$$\sqrt{\mathscr{K}(r)\mathscr{E}(r)} > \frac{\pi}{2} \tag{5}$$

holds for all $r \in (0, 1)$.

Later, Wang et al. [36] gave a generalization of inequality (5) and proved that the inequality

$$M_p(\mathscr{K}(r), \mathscr{E}(r)) > \frac{\pi}{2} \tag{6}$$

holds for all $r \in (0, 1)$ if and only if $p \geq -1/2$, where

$$M_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{1/p} \quad (p \neq 0), \quad M_0(a, b) = \sqrt{ab}$$

is the p-th power mean of two positive numbers a and b.

From (1) and (2) we clearly see that inequality (5) can be rewritten as

$$H_0(\mathscr{K}(r), \mathscr{E}(r)) = L_{-1/2}(\mathscr{K}(r), \mathscr{E}(r)) > \frac{\pi}{2}. \tag{7}$$

Inequalities (6) and (7) together with the monotonicity of $p \mapsto H_p(a, b)$ and $q \mapsto L_q(a, b)$ inspired us to ask what are the best possible parameters p and q such that the inequalities $H_p(\mathscr{K}(r), \mathscr{E}(r)) > \pi/2$ and $L_q(\mathscr{K}(r), \mathscr{E}(r)) > \pi/2$ hold for all $r \in (0, 1)$. The main purpose of this paper is to answer this question. Our main result is the following.

Theorem 1.1. *Let $p, q \in \mathbb{R}$. Then the inequalities*

$$H_p\left(\mathscr{K}(r), \mathscr{E}(r)\right) > \frac{\pi}{2} \tag{8}$$

and

$$L_q\left(\mathscr{K}(r), \mathscr{E}(r)\right) > \frac{\pi}{2} \tag{9}$$

hold for all $r \in (0, 1)$ if and only if $p \geq -3/4$ and $q \geq -3/4$.

§2 Lemmas

In order to prove our main result, we need several lemmas which we present in this section.

For $0 < r < 1$, the following derivative formulas were presented in [35, Appendix E, pp. 474-475]:

$$\frac{d\mathscr{K}}{dr} = \frac{\mathscr{E} - r'^2\mathscr{K}}{rr'^2}, \qquad \frac{d\mathscr{E}}{dr} = \frac{\mathscr{E} - \mathscr{K}}{r},$$
$$\frac{d(\mathscr{E} - r'^2\mathscr{K})}{dr} = r\mathscr{K}, \qquad \frac{d(\mathscr{K} - \mathscr{E})}{dr} = \frac{r\mathscr{E}}{r'^2}.$$

Lemma 2.1. *(See [35, Theorem 3.21(1), Exercise 3.43 (11) and (16)])*

(1) The function $r \mapsto (\mathscr{E} - r'^2\mathscr{K})/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, 1)$*;*

(2) The function $r \mapsto (\mathscr{K} - \mathscr{E})/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, \infty)$*;*

(3) The function $r \mapsto (\mathscr{E}^2 - r'^2\mathscr{K}^2)/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi^2/32, 1)$*.*

Lemma 2.2. *(See [36, Lemma 2.5]) For* $p \in \mathbb{R}$*, the function* $r \mapsto \left(\frac{\mathscr{K}}{\mathscr{E}}\right)^{p-1} \frac{\mathscr{E} - r'^2\mathscr{K}}{r'^2(\mathscr{K} - \mathscr{E})}$ *is strictly increasing on* $(0,1)$ *if and only if* $p \geq -1/2$*. In particular, the inequality*

$$\frac{\mathscr{E} - r'^2\mathscr{K}}{r'^2(\mathscr{K} - \mathscr{E})} > \left(\frac{\mathscr{K}}{\mathscr{E}}\right)^{3/2}$$

holds for all $r \in (0,1)$*.*

§3 Proof of Theorem 1.1

Proof of Theorem 1.1. We first prove that inequality (8) holds for all $r \in (0,1)$ if and only if $p \geq -3/4$. If $p = 0$, then inequality (8) reduces to inequality (5), which gives a validity of Theorem 1.1. Therefore, it suffices to show the inequality (8) for $p \neq 0$.

Let

$$F_p(r) = \frac{1}{p} \log \frac{\mathscr{K}(r)^p + \left[\mathscr{K}(r)\mathscr{E}(r)\right]^{p/2} + \mathscr{E}(r)^p}{3} - \log \frac{\pi}{2}. \tag{10}$$

Then differentiating (10) yields

$$\begin{aligned} F_p'(r) &= \frac{\mathscr{K}^{p-1}\frac{\mathscr{E}-r'^2\mathscr{K}}{rr'^2} + \frac{1}{2}(\mathscr{K}\mathscr{E})^{p/2-1}\left(\frac{\mathscr{E}-r'^2\mathscr{K}}{rr'^2}\mathscr{E} - \mathscr{K}\frac{\mathscr{K}-\mathscr{E}}{r}\right) - \mathscr{E}^{p-1}\frac{\mathscr{K}-\mathscr{E}}{r}}{\mathscr{K}^p + \left(\mathscr{K}\mathscr{E}\right)^{p/2} + \mathscr{E}^p} \\ &= \frac{\mathscr{E}^{p/2-1}r'^2(\mathscr{K} - \mathscr{E})(2\mathscr{E}^{p/2} + \mathscr{K}^{p/2})}{2rr'^2\left[\mathscr{K}^p + \left(\mathscr{K}\mathscr{E}\right)^{p/2} + \mathscr{E}^p\right]}\left[\widehat{F}_p(r) - 1\right], \end{aligned} \tag{11}$$

where

$$\widehat{F}_p(r) = \frac{\mathscr{K}^{p/2-1}(\mathscr{E} - r'^2\mathscr{K})(2\mathscr{K}^{p/2} + \mathscr{E}^{p/2})}{\mathscr{E}^{p/2-1}r'^2(\mathscr{K} - \mathscr{E})(2\mathscr{E}^{p/2} + \mathscr{K}^{p/2})}.$$

For $x \in (1, +\infty)$, we define

$$f_p(x) = \frac{x^{(p+1)/2}(2x^{p/2} + 1)}{x^{p/2} + 2}.$$

Then it follows from Lemma 2.2 that

$$\widehat{F}_p(r) > \left(\frac{\mathscr{K}}{\mathscr{E}}\right)^{(p+1)/2} \frac{2\mathscr{K}^{p/2}+\mathscr{E}^{p/2}}{\mathscr{K}^{p/2}+2\mathscr{E}^{p/2}} = f_p\left(\frac{\mathscr{K}}{\mathscr{E}}\right). \tag{12}$$

Taking the logarithmic derivative of $f_{-\frac{3}{4}}(x)$, one has

$$\frac{f'_{-\frac{3}{4}}(x)}{f_{-\frac{3}{4}}(x)} = \frac{(x^{3/8}-1)^2}{4x(2+5x^{3/8}+2x^{3/4})} > 0 \tag{13}$$

for $x \in (1, +\infty)$.

We clearly see from (13) that $f_{-\frac{3}{4}}(x) > f_{-\frac{3}{4}}(1) = 1$ for $x \in (1, +\infty)$. This in conjunction with (12) and $\mathscr{K}/\mathscr{E} > 1$ yields $\widehat{F}_{-\frac{3}{4}}(r) > f_{-\frac{3}{4}}(\mathscr{K}/\mathscr{E}) > 1$. Combining this with (11), we conclude that $F_{-\frac{3}{4}}(r)$ is strictly increasing on $(0,1)$ and so $F_{-\frac{3}{4}}(r) > F_{-\frac{3}{4}}(0) = 0$ for $r \in (0,1)$. It follows from (10) and the monotonicity of $H_p(a,b)$ with respect to p that

$$F_p(r) \geq F_{-\frac{3}{4}}(r) > 0 \tag{14}$$

for all $r \in (0,1)$ and $p \geq -3/4$.

Therefore, if $p \geq -3/4$, then inequality (8) holds for all $r \in (0,1)$ following from (10) and (14).

Next, we prove that $p = -3/4$ is the best possible parameter such that the inequality (8) holds for all $r \in (0,1)$. Several limit values as $r \to 0^+$ need to be used and presented as follows.

- It follows from Lemma 2.1 (1) and (2) that

$$\lim_{r\to 0^+} \frac{\log \mathscr{K}/\mathscr{E}}{r^2} = \frac{1}{2}\lim_{r\to 0^+}\left[\frac{\mathscr{E}-r'^2\mathscr{K}}{r^2r'^2\mathscr{K}} + \frac{\mathscr{K}-\mathscr{E}}{r^2\mathscr{E}}\right] = \frac{1}{2}. \tag{15}$$

- It follows from Lemma 2.1 that

$$\begin{aligned}\lim_{r\to 0^+} \frac{\log(\mathscr{E}-r'^2\mathscr{K})/[r'^2(\mathscr{K}-\mathscr{E})]}{r^2} &= \lim_{r\to 0^+}\left[\frac{\mathscr{K}}{2(\mathscr{E}-r'^2\mathscr{K})} + \frac{1}{r'^2} - \frac{\mathscr{E}}{2r'^2(\mathscr{K}-\mathscr{E})}\right] \\ &= -\frac{1}{2}\lim_{r\to 0^+}\left[\frac{\mathscr{E}^2-r'^2\mathscr{K}^2}{r^4}\cdot\frac{r^2}{\mathscr{K}-\mathscr{E}}\cdot\frac{r^2}{\mathscr{E}-r'^2\mathscr{K}}\right] + 1 = \frac{3}{4}.\end{aligned} \tag{16}$$

- It follows from Lemma 2.1 (1) and (2) that

$$\begin{aligned}\lim_{r\to 0^+} \frac{\log(2\mathscr{K}^{p/2}+\mathscr{E}^{p/2})/(\mathscr{K}^{p/2}+2\mathscr{E}^{p/2})}{r^2} &= \frac{3p}{4}\lim_{r\to 0^+}\frac{(\mathscr{E}\mathscr{K})^{p/2-1}(\mathscr{E}^2-2r'^2\mathscr{E}\mathscr{K}+r'^2\mathscr{K}^2)}{r^2r'^2(2\mathscr{K}^{p/2}+\mathscr{E}^{p/2})(\mathscr{K}^{p/2}+2\mathscr{E}^{p/2})} \\ &= \frac{p}{3\pi^2}\lim_{r\to 0^+}\left[\frac{\mathscr{E}(\mathscr{E}-r'^2\mathscr{K})}{r^2} + \frac{r'^2\mathscr{K}(\mathscr{K}-\mathscr{E})}{r^2}\right] = \frac{p}{12}.\end{aligned} \tag{17}$$

For $p < -3/4$, we clearly see from (15)-(17) that

$$\begin{aligned}\lim_{r\to 0^+}\frac{\log \widehat{F}_p(r)}{r^2} &= \left(\frac{p}{2}-1\right)\lim_{r\to 0^+}\frac{\log \mathscr{K}/\mathscr{E}}{r^2} + \lim_{r\to 0^+}\frac{\log(\mathscr{E}-r'^2\mathscr{K})/[r'^2(\mathscr{K}-\mathscr{E})]}{r^2} \\ &\quad + \lim_{r\to 0^+}\frac{\log(2\mathscr{K}^{p/2}+\mathscr{E}^{p/2})/(\mathscr{K}^{p/2}+2\mathscr{E}^{p/2})}{r^2} = \frac{1}{3}\left(p+\frac{3}{4}\right) < 0,\end{aligned}$$

which implies that there exists a small $\delta_1 \in (0,1)$ such that $\widehat{F}_p(r) < 1$ for $r \in (0, \delta_1)$. Combining this with (11) and $F_p(0) = 0$, we conclude that $F_p(r) < 0$ for $r \in (0, \delta_1)$ and $p < -3/4$. This in conjunction with (10) yields $H_p(\mathscr{K}(r), \mathscr{E}(r)) < \pi/2$ for $r \in (0, \delta_1)$ if $p < -3/4$.

Secondly, we prove the inequality (9).

Let

$$G_q(r) = \log \frac{\mathscr{K}(r)^{q+1} + \mathscr{E}(r)^{q+1}}{\mathscr{K}(r)^q + \mathscr{E}(r)^q} - \log \frac{\pi}{2}. \tag{18}$$

Differentiation of (18) gives rise to

$$\begin{aligned} G_q'(r) &= \frac{(q+1)\mathscr{K}^q \frac{\mathscr{E} - r'^2\mathscr{K}}{rr'^2} - (q+1)\mathscr{E}^q \frac{\mathscr{K}-\mathscr{E}}{r}}{\mathscr{K}^{q+1} + \mathscr{E}^{q+1}} - \frac{q\mathscr{K}^{q-1} \frac{\mathscr{E} - r'^2\mathscr{K}}{rr'^2} - q\mathscr{E}^{q-1} \frac{\mathscr{K}-\mathscr{E}}{r}}{\mathscr{K}^q + \mathscr{E}^q} \\ &= \frac{\mathscr{E}^{q-1}(\mathscr{K} - \mathscr{E})\left[\mathscr{E}^{q+1} + (q+1)\mathscr{E}\mathscr{K}^q - q\mathscr{K}^{q+1}\right]}{r(\mathscr{K}^{q+1} + \mathscr{E}^{q+1})(\mathscr{K}^q + \mathscr{E}^q)} \left[\widehat{G}_q(r) - 1\right], \end{aligned} \tag{19}$$

where

$$\widehat{G}_q(r) = \frac{\mathscr{K}^{q-1}(\mathscr{E} - r'^2\mathscr{K})\left[\mathscr{K}^{q+1} + (q+1)\mathscr{K}\mathscr{E}^q - q\mathscr{E}^{q+1}\right]}{\mathscr{E}^{q-1}r'^2(\mathscr{K} - \mathscr{E})\left[\mathscr{E}^{q+1} + (q+1)\mathscr{E}\mathscr{K}^q - q\mathscr{K}^{q+1}\right]}.$$

It follows from Lemma 2.2 that

$$\widehat{G}_q(r) > \left(\frac{\mathscr{K}}{\mathscr{E}}\right)^{q+1/2} \frac{\left[(\mathscr{K}/\mathscr{E})^{q+1} + (q+1)(\mathscr{K}/\mathscr{E}) - q\right]}{\left[1 + (q+1)(\mathscr{K}/\mathscr{E})^q - q(\mathscr{K}/\mathscr{E})^{q+1}\right]} = g_p\left(\frac{\mathscr{K}}{\mathscr{E}}\right), \tag{20}$$

where

$$g_q(x) = \frac{x^{q+1/2}[x^{q+1} + (q+1)x - q]}{1 + (q+1)x^q - qx^{q+1}}$$

for $x \in (1, +\infty)$.

The logarithmic derivative of $g_{-\frac{3}{4}}(x)$ yields

$$\frac{g'_{-\frac{3}{4}}(x)}{g_{-\frac{3}{4}}(x)} = \frac{3(x^{1/4} - 1)^2(1 + \sqrt{x})}{2x\left[3(x^{1/4} - 1)^4 + 4x^{1/4}(1 - x^{1/4} + \sqrt{x})\right]} > 0$$

for $x \in (1, +\infty)$. This in conjunction with (20) and $\mathscr{K}/\mathscr{E} > 1$ implies that $\widehat{G}_{-\frac{3}{4}}(r) > g_{-\frac{3}{4}}(\mathscr{K}/\mathscr{E}) > g_{-\frac{3}{4}}(1) = 1$. Combining this with (18) and (19), we clearly see that $G_{-\frac{3}{4}}(r) > G_{-\frac{3}{4}}(0) = 0$, that is,

$$L_{-\frac{3}{4}}(\mathscr{K}(r), \mathscr{E}(r)) > \frac{\pi}{2} \tag{21}$$

for $r \in (0, 1)$.

Therefore, the inequality (9) holds for all $r \in (0, 1)$ if $q \geq -3/4$ following from (21) together with the monotonicity of $L_q(a, b)$ with respect to q.

We now prove $p = -3/4$ is sharp for the inequality (9).

It follows from Lemma 2.1 (1) and (2) that

$$\begin{aligned} &\lim_{r\to 0^+} \frac{\log\left[\mathscr{K}^{q+1} + (q+1)\mathscr{K}\mathscr{E}^q - q\mathscr{E}^{q+1}\right]/\left[\mathscr{E}^{q+1} + (q+1)\mathscr{E}\mathscr{K}^q - q\mathscr{K}^{q+1}\right]}{r^2} \\ &= \frac{q+1}{2} \lim_{r\to 0^+} \frac{(\mathscr{E}^2 - 2r'^2\mathscr{E}\mathscr{K} + r'^2\mathscr{K}^2)\left[(\mathscr{E}^q + \mathscr{K}^q)^2 + q^2(\mathscr{E}\mathscr{K})^{q-1}(\mathscr{K} - \mathscr{E})^2\right]}{r^2r'^2\left[\mathscr{K}^{q+1} + (q+1)\mathscr{K}\mathscr{E}^q - q\mathscr{E}^{q+1}\right]\left[\mathscr{E}^{q+1} + (q+1)\mathscr{E}\mathscr{K}^q - q\mathscr{K}^{q+1}\right]} \\ &= \frac{2(q+1)}{\pi^2} \lim_{r\to 0^+} \left[\frac{\mathscr{E}(\mathscr{E} - r'^2\mathscr{K})}{r^2} + \frac{r'^2\mathscr{K}(\mathscr{K} - \mathscr{E})}{r^2}\right] = \frac{q+1}{2}. \end{aligned} \tag{22}$$

If $q < -3/4$, we clearly see from (15), (16) and (22) that

$$\begin{aligned}\lim_{r\to 0^+}\frac{\log \widehat{G}_q(r)}{r^2} &= (q-1)\lim_{r\to 0^+}\frac{\log \mathscr{K}/\mathscr{E}}{r^2}+\lim_{r\to 0^+}\frac{\log(\mathscr{E}-r'^2\mathscr{K})/[r'^2(\mathscr{K}-\mathscr{E})]}{r^2}\\ &+\lim_{r\to 0^+}\frac{\log\left[\mathscr{K}^{q+1}+(q+1)\mathscr{K}\mathscr{E}^q-q\mathscr{E}^{q+1}\right]/\left[\mathscr{E}^{q+1}+(q+1)\mathscr{E}\mathscr{K}^q-q\mathscr{K}^{q+1}\right]}{r^2}\\ &= q+\frac{3}{4}<0,\end{aligned}$$

which yields that there exists a small $\delta_2 \in (0,1)$ such that $\widehat{G}_q(r) < 1$ for $r \in (0, \delta_2)$. Combining this with (18) and (19), we conclude that $G_q(r) < G_q(0) = 0$ for $r \in (0, \delta_2)$ and $q < -3/4$. This in conjunction with (18) yields $L_p(\mathscr{K}(r), \mathscr{E}(r)) < \pi/2$ for $r \in (0, \delta_2)$ if $q < -3/4$.

Remark 3.1. For $p \neq 0$, it is easy to verify by (1) that

$$H_p(a,b)=\left[\frac{a^p+b^p}{2}-\frac{(a^{p/2}-b^{p/2})^2}{6}\right]^{1/p}\begin{cases}\le M_p(a,b), & p>0,\\ \ge M_p(a,b), & p<0.\end{cases}$$

This in conjunction with (6) yields $H_{-1/2}(\mathscr{K}(r), \mathscr{E}(r)) > \pi/2$ for all $r \in (0,1)$. Theorem 1.1 enables us to know that $p = -1/2$ is not the optimal value to make the inequality (8) valid.

On the other hand, Liu [5] studied the inequalities between the power and Lehmer means and established the inequality $M_{2\lambda+1} \le L_\lambda(x,y)$ for all $x, y > 0$ if $\lambda \in (-1,-1/2) \cup (0,+\infty)$ and the reverse inequality $M_{2\lambda+1} \ge L_\lambda(x,y)$ for all $x, y > 0$ if $\lambda \in (-\infty,-1) \cup (-1/2, 0)$.

Combining this with (6), we clearly see that

$$L_{-3/4}(\mathscr{K}(r),\mathscr{E}(r)) \ge M_{-1/2}(\mathscr{K}(r),\mathscr{E}(r)) > \frac{\pi}{2}$$

for all $r \in (0,1)$, which gives the sufficient condition of the second inequality in Theorem 1.1.

Remark 3.2. From (10) and (18), we clearly see that

$$\lim_{r\to 1^-} F_p(r) = \log\frac{2}{\pi}-\frac{\log 3}{p}, \tag{23}$$

if $p < -3/4$ and

$$\lim_{r\to 1^-} G_q(r)=\begin{cases}+\infty, & -1<q<-\frac{3}{4},\\ \log\frac{4}{\pi}, & q=-1,\\ \log\frac{2}{\pi}, & q<-1.\end{cases} \tag{24}$$

The necessary conditions for the reverse inequalities of (8) and (9) are $\lim_{r\to 1^-} F_p(r) \le 0$ and $\lim_{r\to 1^-} G_q(r) \le 0$, which are equivalent to $p \le -\log 3/\log(\pi/2)$ and $q < -1$ from (23) and (24). By numerical experiments, it enables us to present the following open problem.

Open Problem. Inequalities

$$H_p(\mathscr{K}(r),\mathscr{E}(r)) < \frac{\pi}{2} \quad \text{and} \quad L_q(\mathscr{K}(r),\mathscr{E}(r)) < \frac{\pi}{2}$$

hold for all $r \in (0,1)$ if and only if $p \le -\log 3/\log(\pi/2)$ and $q < -1$.

Declarations

Conflict of interest The authors declare no conflict of interest.

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