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q–differ-integral operator on p–valent functions associated with operator on Hilbert space

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Abstract. Making use of multivalent functions with negative coefficients of the type $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, which are analytic in the open unit disk and applying the q-derivative a q–differ-integral operator is considered. Furthermore by using the familiar Riesz-Dunford integral, a linear operator on Hilbert space H is introduced. A new subclass of p-valent functions related to an operator on H is defined. Coefficient estimate, distortion bound and extreme points are obtained. The convolution-preserving property is also investigated.

§1 Motivation and Introduction

The study of univalent and multivalent functions is one of the leading branches of geometric function theory. One of the fundamental problems in this theory is coefficient estimates of such functions. After solving this problem, we may find many interesting geometric properties. The operator on Hilbert space related to multivalent functions was studied by many researchers. Also, the q-derivative is introduced in 1909. In this paper, we define a new subclass of multivalent functions associated with q-derivative. Furthermore, operators on Hilbert space is considered and some geometric structures are investigated.
Let $\mathcal{N}$ be the class of analytic and multivalent functions defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is of the type:

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k. \tag{1}$$

In the theory of q–calculus, the q–shifted factorial is introduced for $w, q \in \mathbb{C}$, $n \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ as a product of n factors by:

$$(w, q)_n = \begin{cases} 1 & , \quad n = 0, \\ (1 - w)(1 - wq) \cdots (1 - wq^{n-1}) & , \quad n \in \mathbb{N}, \end{cases} \tag{2}$$

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and according to the basic analogue of the gamma function, we have

$$(q^w, q)_n = \frac{\Gamma_q(w+n)(1-q)^n}{\Gamma_q(w)}, \qquad (n > 0), \tag{3}$$

where the gamma function is defined by:

$$\Gamma_q(y) = \frac{(q,q)_\infty (1-q)^{1-y}}{(q^y, q)_\infty}, \qquad (0 < q < 1). \tag{4}$$

If $|q| < 1$, the q–shifted factorial $(w, q)_n$ is meaningful for $n = \infty$ as a convergent product:

$$(w, q)_\infty = \prod_{t=0}^{\infty} (1 - wq^t). \tag{5}$$

It is clear that $\Gamma_q(y) \to \Gamma(y)$ as $q \to 1$, where $\Gamma(y)$ is the ordinary Eular gamma function. Also we have:

$$\lim_{q \to 1^-} \frac{(q^w, q)_n}{(1-q)^n} = (w)_n, \tag{6}$$

where $(w)_n$ is the familiar Pochhammer symbol defined by:

$$(w)_n = w(w-1)\cdots(w+n-1). \tag{7}$$

The Jachson's q–derivative and q–integral of a function are respectively introduced by Gasper and Rahman [1]:

$$\mathcal{D}_{q,z} f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \qquad (z \neq 0, \quad z \neq 0), \tag{8}$$

and

$$\int_0^z f(t) d_q(t) = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k), \tag{9}$$

see also [2].

From (8), we get:

$$\mathcal{D}_q z^k = [k]_q z^{k-1}, \tag{10}$$

where

$$[k]_q = \frac{1-q^k}{1-q}, \tag{11}$$

$[k]_q$ is called q–analogue of k.

Now, we recall some definitions of fractional q–calculus operators, which were considered in [3].

Definition 1.1 (Fractional q–integral operator)**.** If $f(z)$ is analytic in a simply connected domain $\mathbb{D} \subset \mathbb{C}$ containing the origin, then the fractional q–integral of f of order γ ($\gamma > 0$), is defined by:

$$\mathcal{I}^{\gamma}_{q,z} f(z) = \mathcal{D}^{-\gamma}_{q,z} f(z) = \frac{1}{\Gamma_q(\gamma)} \int_0^z (z - qs)_{\gamma-1} f(s) d_q s, \tag{12}$$

where $(z - qs)_{\gamma-1}$ is single valued when $|\arg(\frac{-sq^\gamma}{z})| < \pi$, $|sq^\gamma| < 1$ and $|\arg(z)| < \pi$.

Definition 1.2 (Fractional q–derivative operator)**.** The fractional q–derivative operator of f

of order γ $(0 \leqslant \gamma < 1)$ is defined by:

$$\mathcal{D}_{q,z}^{\gamma} f(z) = \mathcal{D}_{q,z}\mathcal{I}_{q,z}^{1-\gamma} f(z) = \frac{1}{\Gamma_q(1-\gamma)}\mathcal{D}_{q,z}\int_0^z (z-qs)_{-\gamma} d_q s. \tag{13}$$

Definition 1.3 (Extended fractional q–derivative operator)**.** With the same hypothesis of definition 1.2, the extended fractional q–derivative of order γ, is defined by:

$$\mathcal{D}_{q,z}^{\gamma} f(z) = \mathcal{D}_{q,z}^{m}\mathcal{I}_{q,z}^{m-\gamma} f(z), \qquad (m-1 \leqslant \gamma < m, \quad m \in \mathbb{N}_0). \tag{14}$$

Now, we consider the linear operator, see [4]:

$$\begin{aligned} &\Omega_{q,p}^{\gamma} f(z) : \mathcal{N} \to \mathcal{N}, \\ &\Omega_{q,p}^{\gamma} f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)} a_k z^k. \end{aligned} \tag{15}$$

It is easy to show that $\Omega_{q,p}^{0} f(z) = f(z)$.

If $f(z) \in \mathcal{N}$, then the generalized Al. Oboudi [6] type differential operator $\mathcal{D}_{q,p,\lambda}^{\gamma,m}$ for $\gamma \geqslant 0$ and $m \in \mathbb{N}$ is defined by:

$$\begin{aligned} &\mathcal{D}_{q,p,\lambda}^{\gamma,0} f(z) = f(z), \\ &\mathcal{D}_{q,p,\lambda}^{\gamma,1} f(z) = (1-\lambda)\Omega_{q,p}^{\gamma} f(z) + \frac{\lambda z}{[p]_q}\mathcal{D}_q(\Omega_{q,p}^{\gamma} f(z)), \\ &\mathcal{D}_{q,p,\lambda}^{\gamma,m} f(z) = \mathcal{D}_{q,p,\lambda}^{\gamma,1}(\mathcal{D}_{q,p,\lambda}^{\gamma,m-1} f(z)), \end{aligned} \tag{16}$$

which was introduced by Selvakuman et al. [5].

We note that if $f(z)$ is given by (1), then by (16), we get:

$$\mathcal{D}_{q,p,\lambda}^{\gamma,m} f(z) = z^p - \sum_{k=p+1}^{\infty} \Big(\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big[1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big]\Big)^m a_k z^k. \tag{17}$$

By specializing the parameters in the operator $\mathcal{D}_{q,p,\lambda}^{\gamma,m}$, this operator reduces to some well-known differential operators. For example, see [6, 7] and [8].

Definition 1.4. The function $f \in \mathcal{N}$ is said to be a member of the class $\mathcal{ND}(\alpha, \beta, \theta)$ if it satisfies:

$$\left|\frac{z^{2-p}(\mathcal{D}_{q,p,\lambda}^{\gamma,m} f(z))'' + z^{1-p}(\mathcal{D}_{q,p,\lambda}^{\gamma,m} f(z))' - p^2}{2z^{-p}\mathcal{D}_{q,p,\lambda}^{\gamma,m} f(z) - \alpha(1+\beta)}\right| < \theta, \tag{18}$$

where α, β and θ belong to $[0, 1)$.

Let H be a Hilbert space on the $\mathbb{C}$ and T be a linear operator on H. Also $f(T)$ be the operator on H defined by Riesz-Dunford integral [9]:

$$f(T) = \frac{1}{2\pi i}\int_C f(z)(zI - T)^{-1} dz, \tag{19}$$

where C is a positively oriented simple closed rectifiable counter lying in $\mathbb{U}$, and containing the spectrum of T in its interior domain and I is the identity operator on H. See [10], also [11].

Definition 1.5. A function given by (1) is in the class $\mathcal{ND}(T, \alpha, \beta, \theta)$ if for all operator T with

$\|T\| < 1$ and $T \neq 0$ it satisfies the inequality:

$$\begin{aligned} L &= \|T^{2-p}[\mathcal{D}_{q,p,\lambda}^{\gamma,m} f(T)]'' + T^{1-p}[\mathcal{D}_{q,p,\lambda}^{\gamma,m} f(T)]' - p^2\| \\ &< \theta\|2T^{-p}[\mathcal{D}_{q,p,\lambda}^{\gamma,m} f(T)] - \alpha(1+\beta)\| = R. \end{aligned} \tag{20}$$

§2 Main Result

In this section, we obtain coefficient inequality and distortion bound for a function $f \in \mathcal{ND}(T,\alpha,\beta,\theta)$.

Theorem 2.1. *A function $f(z)$ given by* (1) *is in the class $\mathcal{ND}(T,\alpha,\beta,\theta)$ for all $T \neq 0$ if and only if:*

$$\sum_{k=p+1}^{\infty} \frac{(k^2+2\theta)}{\theta(2-\alpha(1+\beta))} \Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m a_k \leqslant 1. \tag{21}$$

Proof. Suppose that (21) is true. We have,

$$\begin{aligned} L - R &= \left\| -\sum_{k=p+1}^{\infty} k^2 \Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m a_k T^{k-p} \right\| \\ &\quad - \theta \left\| 2-\alpha(1+\beta) - \sum_{k=p+1}^{\infty} 2\Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m a_k T^{k-p} \right\| \\ &\leqslant \sum_{k=p+1}^{\infty} (k^2+2\theta)\Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m a_k - \theta(2-\alpha(1+\beta)) \\ &\leqslant 0. \end{aligned}$$

Hence, f is in the class $\mathcal{ND}(T,\alpha,\beta,\theta)$.

Conversely, suppose that $L < R$, so:

$$\begin{aligned} &\left\| -\sum_{k=p+1}^{\infty} \Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m a_k T^{k-p} \right\| \\ &< \theta \left\| 2-\alpha(1+\beta) - \sum_{k=p+1}^{\infty} 2\Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m a_k T^{k-p} \right\|. \end{aligned}$$

Setting $T = rI$ $(0 < r < 1)$ in the above inequality, we get:

$$\theta > \frac{\sum_{k=p+1}^{\infty} k^2 \Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m a_k r^{k-p}}{(2-\alpha(1+\beta)) - \sum_{k=p+1}^{\infty} 2\Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m a_k r^{k-p}}. \tag{22}$$

Upon clearing denominator in (22) and letting $r \to 1$, we obtain the required result. So the proof is complete. □

Remark 1. We note that the result (21) is sharp for the function $F(z)$ given by:

$$F(z)=z^p-\frac{\theta(2-\alpha(1+\beta))z^{p+1}}{((p+1)^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(p+2)}{\Gamma_q(p+1)\Gamma_q(p+2-\gamma)}\left(1-\lambda+\frac{[p+1]_q}{[p]_q}\lambda\right)\right]^m}. \tag{23}$$

Also, according to the Theorem 2.1, we get:

$$\sum_{k=p+1}^{\infty}a_k\leqslant\frac{\theta(2-\alpha(1+\beta))}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}. \tag{24}$$

Theorem 2.2. *If $f(z)$ of the form* (1) *be in the class $\mathcal{ND}(T,\alpha,\beta,\theta)$ with $\|T\|<1$ and $\|T\|\neq 0$, then:*

$$\begin{aligned}&\|T\|^p\left[1-\|T\|^{k-p}\frac{\theta(2-\alpha(1+\beta))}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}\right]\\&\leqslant\|f(T)\|\leqslant\|T\|^p\left[1+\|T\|^{k-p}\frac{\theta(2-\alpha(1+\beta))}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}\right].\end{aligned} \tag{25}$$

Proof. Since $f(z)=z^p-\sum\limits_{k=p+1}^{\infty}a_kz^k$, so by (24) we have:

$$\begin{aligned}\|f(T)\|&=\left\|T^p-\sum_{k=p+1}^{\infty}a_kT^k\right\|\\&\geqslant\|T^p\|-\|T\|^k\sum_{k=p+1}^{\infty}a_k\\&\geqslant\|T\|^p\left[1-\|T\|^{k-p}\frac{\theta(2-\alpha(1+\beta))}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}\right],\end{aligned}$$

and similarly:

$$\|f(T)\|\leqslant\|T\|^p\left[1+\|T\|^{k-p}\frac{\theta(2-\alpha(1+\beta))}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}\right].$$

Hence, the proof is complete. □

§3 Extreme Points

In this section, we introduce the extreme points of the class $\mathcal{ND}(T,\alpha,\beta,\theta)$ and show that this class is closed under convolution with respect to the parameters α and β.

Theorem 3.1. *Let $f_p(z)=z^p$ and*

$$f_k(z)=z^p+\frac{\theta(2-\alpha(1+\beta)}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}z^k,\qquad(k=p+1,p+2,\ldots). \tag{26}$$

Then, $f(z) \in \mathcal{ND}(T, \alpha, \beta, \theta)$ *if and only if it can be expressed by:*

$$f(z) = \sum_{k=p}^{\infty} x_k f_k(z),$$

where, $x_k \geqslant 0$ *and* $\sum\limits_{k=p}^{\infty} x_k = 1$.

Proof. Let $f(z) \in \mathcal{ND}(T, \alpha, \beta, \theta)$, then from (21), we have:

$$a_k \leq \sum_{k=p+1}^{\infty} a_k \leqslant \frac{\theta(2-\alpha(1+\beta)}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}, \qquad (k = p+1, p+2, \ldots).$$

Therefore by letting:

$$x_k = \frac{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}{\theta(2-\alpha(1+\beta))} a_k, \qquad (k = p+1, p+2, \ldots),$$

and $x_p = 1 - \sum\limits_{k=p+1}^{\infty} x_k$, we conclude the required result.

Conversely, let $f(z) = \sum\limits_{k=p}^{\infty} x_k f_k(z)$, then:

$$\begin{aligned} f(z) &= x_p z^p + \sum_{k=p+1}^{\infty} x_k \left\{ z^p + \frac{\theta(2-\alpha(1+\beta))z^k}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m} \right\} \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\theta(2-\alpha(1+\beta))x_k}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m} z^k \\ &= z^p + \sum_{k=p+1}^{\infty} w_k z^k, \end{aligned}$$

where

$$w_k = \frac{\theta(2-\alpha(1+\beta))x_k}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}.$$

But

$$\begin{aligned} \sum_{k=p+1}^{\infty} w_k &= \frac{k^2+2\theta}{\theta(2-\alpha(1+\beta))}\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m \\ &= \sum_{k=p+1}^{\infty} x_k = 1 - x_p \leqslant 1. \end{aligned}$$

So, by Theorem 2.1, we get $f(z) \in \mathcal{ND}(T, \alpha, \beta, \theta)$. □

Remark 2. The extreme points of the class $\mathcal{ND}(T, \alpha, \beta, \theta)$ are the functions $f_p(z) = z^p$ and $f_k(z)$ $(k \geqslant p+1)$ defined by (26).

Theorem 3.2. *Let* $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k$ *and* $g(z) = z^p - \sum\limits_{k=p+1}^{\infty} b_k z^k$ *be in the class*

$\mathcal{ND}(T,\alpha,\beta,\theta)$, then $(f*g)(z)$ defined by:

$$(f*g)(z)=z^p-\sum_{k=p+1}^{\infty}a_kb_kz^k,$$

belongs to $\mathcal{ND}(T,\alpha,\beta^*,\theta)$, where:

$$\beta^*\leqslant\frac{2}{\alpha}-\left\{1+\frac{\theta(2-\alpha(1+\beta))^2}{\alpha(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}\right\}.$$

Proof. By Theorem 2.1, it is sufficient to show that:

$$\sum_{k=p+1}^{\infty}\frac{(k^2+2\theta)}{\theta(2-\alpha(1+\beta^*))}\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m a_kb_k\leqslant 1.$$

By using Cauchy-Schwarz inequality from (21), we obtain:

$$\sum_{k=p+1}^{\infty}\frac{(k^2+2\theta)}{\theta(2-\alpha(1+\beta))}\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m \sqrt{a_kb_k}\leqslant 1.$$

Hence, we find the largest β^* such that:

$$\sum_{k=p+1}^{\infty}\frac{(k^2+2\theta)}{\theta(2-\alpha(1+\beta^*))}\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m a_kb_k$$
$$\leqslant\sum_{k=p+1}^{\infty}\frac{(k^2+2\theta)}{\theta(2-\alpha(1+\beta))}\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m \sqrt{a_kb_k}\leqslant 1,$$

or equivalently

$$\sqrt{a_kb_k}\leqslant\frac{2-\alpha(1+\beta^*)}{2-\alpha(1+\beta)},\qquad (k\geqslant p+1).$$

This inequality holds if

$$\frac{\theta(2-\alpha(1+\beta))}{(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}\leqslant\frac{2-\alpha(1+\beta^*)}{2-\alpha(1+\beta)},$$

or equivalently

$$\beta^*\leqslant\frac{2}{\alpha}-\left\{1+\frac{\theta(2-\alpha(1+\beta))^2}{\alpha(k^2+2\theta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}\right\}.$$

□

Theorem 3.3. *With the same assumptions of Theorem 3.2, $(f*g)(z)$ belongs to $\mathcal{ND}(T,\alpha^*,\beta,\theta)$, where:*

$$\alpha^*\leqslant\frac{2}{1+\beta}-\frac{\theta(2-\alpha(1+\beta))^2}{(k^2+2\theta)(1+\beta)\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m}.$$

Proof. It is sufficient to show that:

$$\sum_{k=p+1}^{\infty}\frac{(k^2+2\theta)}{\theta(2-\alpha^*(1+\beta))}\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m a_kb_k\leqslant 1.$$

By using Cauchy-Schwarz inequality from (21) we get:

$$\sum_{k=p+1}^{\infty}\frac{(k^2+2\theta)}{\theta(2-\alpha(1+\beta))}\left[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\left(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right)\right]^m \sqrt{a_kb_k}\leqslant 1.$$

Hence, we find the largest α^* such that:

$$\sum_{k=p+1}^{\infty}\frac{(k^2+2\theta)}{\theta(2-\alpha^*(1+\beta))}\Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m a_k b_k$$

$$\leqslant \sum_{k=p+1}^{\infty}\frac{(k^2+2\theta)}{\theta(2-\alpha(1+\beta))}\Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m \sqrt{a_k b_k}\leqslant 1,$$

or equivalently

$$\sqrt{a_k b_k}\leqslant \frac{2-\alpha^*(1+\beta)}{2-\alpha(1+\beta)},\qquad (k\geqslant p+1).$$

This inequality holds if

$$\frac{\theta(2-\alpha(1+\beta))}{(k^2+2\theta)\Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m}\leqslant \frac{2-\alpha^*(1+\beta)}{2-\alpha(1+\beta)},$$

or equivalently

$$\alpha^*\leqslant \frac{2}{1+\beta}-\frac{\theta(2-\alpha(1+\beta))^2}{(k^2+2\theta)(1+\beta)\Big[\frac{\Gamma_q(p+1-\gamma)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\gamma)}\Big(1-\lambda+\frac{[k]_q}{[p]_q}\lambda\Big)\Big]^m}.$$

So the proof is complete. □

§4 Conclusion

Studying the theory of analytic functions has been a area of concern for many authors. A more specific field is the study of operator on Hilbert space in complex plane. Literature review indicates lots of researches on the classes of p-valent analytic functions. The interplay of geometric structures are very important aspect in complex analysis.This work will lead possibility to define new subclasses of p-valent functions by using q-analogue of the well-known operators or even conclude some geometric properties such as radii of starlikeness, convexity , close-to-convexity and so on. Also neighborhood and partial sum properties will be investigate. This idea could be topics for next researches.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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