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Complete moment convergence for ND random variables under the sub-linear expectations

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Abstract. In this article, we establish a general result on complete moment convergence for arrays of rowwise negatively dependent(ND) random variables under the sub-linear expectations. As applications, we can obtain a series of results on complete moment convergence for ND random variables under the sub-linear expectations.

§1 Introduction and notation

In the classical probability theory, the additivity of the probabilities and the expectations is assumed. But in practice, such additivity assumption is not feasible in many areas of applications because the uncertainty phenomena can not be modeled using additive probabilities or additive expectations. Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus [7,9,12,14,15]. Peng [15,16]) introduced the general framework of the sub-linear expectation in a general function space by relaxing the linear property of the classical expectation to the sub-additivity and positive homogeneity (cf. Definition 1.1 below). Under Peng's sub-linear expectation framework, many limit theorems have been established recently, including the central limit theorem and weak law of large numbers [16,17,18], strong law of large numbers [3,10,4,22], the law of the iterated logarithm [5,24,22], the moment inequalities for the maximum partial sums and the Kolomogov strong law of large numbers [25], and so on. Investigating the limit theorems in sub-linear expectation space is of great significance in the theory and application. Because sub-linear expectation and capacity are not additive, the study of the limit theorems under sub-linear expectation becomes much more complex and challenging.

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Throughout this paper, C denotes a positive constant which may differ from one place to another. Denote $a_+ = \max\{0, a\}$.

We use the framework and notations of [16]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l.Lip}(\mathbb{R}^n)$, where $C_{l.Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \le C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \ \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

for some $C > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". If X is an element of set $\mathcal{H}$, then we denote $X \in \mathcal{H}$.

Definition 1.1 A sub-linear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\widehat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have
(a) Monotonicity: If $X \ge Y$ then $\widehat{\mathbb{E}}[X] \ge \widehat{\mathbb{E}}[Y]$;
(b) Constant preserving: $\widehat{\mathbb{E}}[c] = c$;
(c) Sub-additivity: $\widehat{\mathbb{E}}[X + Y] \le \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
(d) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X], \lambda > 0$.
Here $\bar{\mathbb{R}} = [-\infty, +\infty]$. The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X], \ \ \forall X \in \mathcal{H}.$$

From the definition, we can easily get that $\widehat{\mathcal{E}}[X] \le \widehat{\mathbb{E}}[X]$, $\widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c$, $\widehat{\mathbb{E}}[X - Y] \ge \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$ and $|\widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]| \le \widehat{\mathbb{E}}[|X - Y|]$. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathcal{E}}[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

Definition 1.2 [16]

(i)(Independence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, a random vector $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$, $X_i \in \mathcal{H}$ under $\widehat{\mathbb{E}}$ if for each test function $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]|_{\boldsymbol{x}=\boldsymbol{X}}]$, whenever $\bar{\varphi}(\boldsymbol{x}) := \widehat{\mathbb{E}}[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|] < \infty$ for all $\boldsymbol{x}$ and $\widehat{\mathbb{E}}[|\bar{\varphi}(\boldsymbol{X})|] < \infty$.

Definition 1.3 [24]

(i) (Negative dependence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, a random vector $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_n), Y_i \in \mathcal{H}$ is said to be negatively dependent (ND) to another random vector $\boldsymbol{X} = (X_1, X_2, \cdots, X_n), X_i \in \mathcal{H}$ under $\widehat{\mathbb{E}}$ if for each pair of test functions $\varphi_1 \in C_{l.Lip}(\mathbb{R}^m)$ and $\varphi_2 \in C_{l.Lip}(\mathbb{R}^n)$ we have $\widehat{\mathbb{E}}[\varphi_1(\boldsymbol{X})\varphi_2(\boldsymbol{Y}) \le \widehat{\mathbb{E}}[\varphi_1(\boldsymbol{X})]\widehat{\mathbb{E}}[\varphi_2(\boldsymbol{Y})]$, whenever either φ_1, φ_2 are coordinatewise nondecreasing or φ_1, φ_2 are coordinatewise non-increasing with $\varphi_1(\boldsymbol{X}) \ge 0$, $\widehat{\mathbb{E}}[\varphi_2(\boldsymbol{Y})] \ge 0$, $\widehat{\mathbb{E}}[|\varphi_1(\boldsymbol{X})\varphi_2(\boldsymbol{Y})|] < \infty$, $\widehat{\mathbb{E}}[|\varphi_1(\boldsymbol{X})|] < \infty$, $\widehat{\mathbb{E}}[|\varphi_2(\boldsymbol{Y})|] < \infty$.

(ii) (ND random variables) Let $\{X_n; n \ge 1\}$ be a sequence of random variables in the sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. $X_1, X_2, \cdots$ are said to be negatively dependent if X_{i+1} is negatively dependent to $(X_1, X_2, \cdots, X_i)$ for each $i \ge 1$.

It is obvious that, if $\{X_n; n \ge 1\}$ is a sequence of independent random variables and $f_1(x), f_2(x), \cdots \in C_{l.Lip}(\mathbb{R})$ then $\{f_n(X_n); n \ge 1\}$ is also a sequence of independent random variables; if $\{X_n; n \ge 1\}$ is a sequence of negatively dependent random variables and $f_1(x), f_2(x), \cdots$

$\cdot \in C_{l.Lip}(\mathbb{R})$ are non-decreasing (resp. non-increasing) functions, then $\{f_n(X_n); n \geq 1\}$ is also a sequence of negatively dependent random variables.

Definition 1.4 An array of random variables $\{X_{ni}; 1 \leq i \leq k_n,\ n \geq 1\}$ in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called rowwise negatively dependent random variables if for every given $n \geq 1$, $\{X_{ni}; 1 \leq i \leq k_n\}$ is a sequence of negatively dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, where and in the sequel $\{k_n,\ n \geq 1\}$ is a sequence of positive integers and $k_n \to \infty$, when $n \to \infty$.

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0,1]$ is called a capacity if

$$V(\phi) = 0,\ V(\Omega) = 1,\ \text{ and } V(A) \leq V(B)\ \ \forall A \subset B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sub-linear space, and $\widehat{\mathcal{E}}$ be the conjugate expectation of $\widehat{\mathbb{E}}$. We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\},\ \mathcal{V}(A) := 1 - \mathbb{V}(A^c),\ \forall A \in \mathcal{F},$$

where A^c is the complement set of A. It is obvious that $\mathbb{V}$ is sub-additive and

$$\mathbb{V}(A) := \widehat{\mathbb{E}}[I_A],\ \mathcal{V}(A) := \widehat{\mathcal{E}}[I_A],\ \text{if } I_A \in \mathcal{H},$$

$$\widehat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g],\ \widehat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \widehat{\mathcal{E}}[g],\ \text{if } f \leq I_A \leq g, f, g \in \mathcal{H}.$$

This implies Markov inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \widehat{\mathbb{E}}[|X|^p]/x^p,\ \ \forall x > 0, p > 0$$

from $I\{|X| \geq x\} \leq |X|^p/x^p \in \mathcal{H}$. By Lemma 4.1 in Zhang (2016a), we have Hölder inequality inequality: $\forall X, Y \in \mathcal{H}, p, q > 1$, satisfying $p^{-1} + q^{-1} = 1$,

$$\widehat{\mathbb{E}}[|XY|] \leq (\widehat{\mathbb{E}}[|X|^p])^{\frac{1}{p}} (\widehat{\mathbb{E}}[|Y|^q])^{\frac{1}{q}},$$

particularly, Jensen inequality:

$$(\widehat{\mathbb{E}}[|X|^r])^{\frac{1}{r}} \leq (\widehat{\mathbb{E}}[|X|^s])^{\frac{1}{s}},\ \text{ for } 0 < r \leq s.$$

We define the Choquet integrals/expecations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) := \int_0^\infty V(X \geq x)dx + \int_{-\infty}^0 (V(X \geq x) - 1)dx$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively. If $\lim_{c\to\infty} \widehat{\mathbb{E}}[(|X| - c)_+] = 0$, then $\widehat{\mathbb{E}}[|X|]$ $\leq C_{\mathbb{V}}(|X|)$ [24].

Complete convergence and complete moment convergence theorems are important limit theorems in probability theory. Many of related results of complete convergence have been obtained in the probability space. We refer the reader to [1,13,11,8,21], and so on. For the results of complete moment convergence in the probability space, we refer the reader to [6,2,20,19,23], and so on.

Wu et al. [23] established the following complete moment convergence theorem for END random variables in classical probability space.

Theorem A Let $q \geq 1$, $\{X_{ni}; 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise END random variables and $\{c_n\}$ be a sequence of positive real numbers. Assume that the following conditions hold:

(a) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} E|X_{ni}|I(|X_{ni}| > \varepsilon) < \infty$ for any $\varepsilon > 0$;
(b) for some $\delta > 0$, there exists $\eta > 1$ such that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \le \delta) \right)^{\eta} < \infty;$$

(c) $\sum_{i=1}^{k_n} E|X_{ni}|I(|X_{ni}| > \frac{\delta}{16\eta}) \to 0,$ as $n \to \infty$.
Then for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n E \left\{ \left| \sum_{i=1}^{k_n} (X_{ni} - EX_{ni} I(|X_{ni}| \le \delta)) \right| - \varepsilon \right\}_+ < \infty.$$

The main purpose of this article is to investigate more general complete moment convergence for arrays of rowwise ND random variables under the sub-linear expectations.

§2 Main results

In this paper, we define $g(x)$ in the following: For $0 < \mu < 1$, let $g(x) \in C_{l.Lip}(\mathbb{R})$ such that $0 \le g(x) \le 1$ for all x and $g(x) = 1$ if $|x| \le \mu$, $g(x) = 0$ if $|x| > 1$ and when $x > 0$, $g(x) \downarrow$. Then

$$I(|x| \le \mu) \le g(x) \le I(|x| \le 1),\ I(|x| > 1) \le 1 - g(x) \le I(|x| > \mu). \tag{2.1}$$

Theorem 2.1. *Let $\{X_{ni}; 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise ND random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ and $\{c_n\}$ be a sequence of positive real numbers, $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function with $f(0) = 0$ and $\eta \ge 1$ be a constant. There exits some constant $\delta > 0$ and μ is the same as in (2.1). Assume that the following conditions hold:*
(a) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \widehat{\mathbb{E}} \left[f \left(8\eta |X_{ni}| \left(1 - g\left(\frac{X_{ni}}{\varepsilon} \right) \right) \right) \right]$
$\le \sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} C_{\mathbb{V}} \left[f \left(8\eta |X_{ni}| \left(1 - g\left(\frac{X_{ni}}{\varepsilon} \right) \right) \right) \right] < \infty,$ *for any $\varepsilon > 0$;*
(b) $\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} \widehat{\mathbb{E}} \left[X_{ni}^2 g\left(\frac{X_{ni}}{\delta} \right) \right] \right)^{\eta} < \infty;$
(c) $\sum_{i=1}^{k_n} \widehat{\mathbb{E}} \left[|X_{ni}| \left(1 - g\left(\frac{16\eta X_{ni}}{\mu\delta} \right) \right) \right] \to 0,$ *as $n \to \infty$;*
(d) let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be the inverse function for $f(t)$, which is $h(f(t)) = t, t \ge 0$ and $s(t) = \max_{\mu\delta \le x \le h(t)/\mu} \frac{x}{f(x)}$. Suppose that the constants η, δ and the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the condition

$$\int_{f(\delta)}^{\infty} h^{-\eta}(t) s(t)\, dt < \infty.$$

Then for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n C_{\mathbb{V}} \left[f \left(\left\{ \sum_{i=1}^{k_n} \left(X_{ni} - \widehat{\mathbb{E}} \left[X_{ni} g\left(\frac{X_{ni}}{\delta} \right) \right] \right) - \varepsilon \right\}_+ \right) \right] < \infty. \tag{2.2}$$

Let $f(t) = t^q,\ t \ge 0, q > 0$ in Theorem 2.1, we can obtain the following interesting result:

Corollary 2.1. *Let $q > 0$, $\{X_{ni}; 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise ND random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ and $\{c_n\}$ be a sequence of positive real numbers. Assume that the following conditions hold:*
(a) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \widehat{\mathbb{E}} \left[|X_{ni}|^q \left(1 - g\left(\frac{X_{ni}}{\varepsilon} \right) \right) \right]$
$\le \sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} C_{\mathbb{V}} \left[|X_{ni}|^q \left(1 - g\left(\frac{X_{ni}}{\varepsilon} \right) \right) \right] < \infty$ *for any $\varepsilon > 0$;*

(b) for some $\delta > 0$*, there exists* $\eta > \max(1, q)$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} \widehat{\mathbb{E}} \left[X_{ni}^2 g\left(\frac{X_{ni}}{\delta} \right) \right] \right)^{\eta} < \infty;$$

(c) $\sum_{i=1}^{k_n} \widehat{\mathbb{E}} \left[|X_{ni}| \left(1 - g\left(\frac{16\eta X_{ni}}{\mu\delta} \right) \right) \right] \to 0$*, as* $n \to \infty$*.*

Then for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n C_{\mathbb{V}} \left[\left\{ \sum_{i=1}^{k_n} \left(X_{ni} - \widehat{\mathbb{E}} \left[X_{ni} g\left(\frac{X_{ni}}{\delta} \right) \right] \right) - \varepsilon \right\}_+^q \right] < \infty.$$

Remark 2.1 We can obtain the corresponding result of Theorem A when $f(t) = t$ $(t \ge 0)$ in our Theorem 2.1 under the sub-linear expectations. We establish a more general complete moment convergence theorem for arrays of rowwise ND random variables under the sub-linear expectations.

Remark 2.2 Letting $f(t)$ be some special functions, we can obtain a series of results on complete moment convergence for ND random variables.

§3 Proofs of main results

In order to prove our results, we need the following lemmas.

Lemma 3.1. *Let* $\{X_{ni}; 1 \le i \le k_n, n \ge 1\}$ *be an array of rowwise negatively dependent random variables in* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *with* $\widehat{\mathbb{E}}[X_{ni}] \le 0$*, for* $1 \le i \le k_n, n \ge 1$*. Let* $B_n' = \sum_{i=1}^{k_n} \widehat{\mathbb{E}}[X_{ni}^2]$*. Then for any given* n *and for all* $x > 0, y > 0$

$$\mathbb{V}\left(\sum_{i=1}^{k_n} X_{ni} > x \right) \le \left(\max_{1 \le i \le k_n} |X_{ni}| > y \right) + \exp\left\{ \frac{x}{y} - \frac{x}{y} \ln\left(1 + \frac{xy}{B_n'} \right) \right\}.$$

Proof Similar to the proof of Theorem 3.1 of [24], we can have

$$\mathbb{V}\left(\sum_{i=1}^{k_n} X_{ni} > x \right) \le \left(\max_{1 \le i \le k_n} |X_{ni}| > y \right) + \exp\left\{ \frac{x}{y} - \frac{x}{y} \left(\frac{B_n'}{xy} + 1 \right) \ln\left(1 + \frac{xy}{B_n'} \right) \right\}.$$

Hence we have

$$\mathbb{V}\left(\sum_{i=1}^{k_n} X_{ni} > x \right) \le \left(\max_{1 \le i \le k_n} |X_{ni}| > y \right) + \exp\left\{ \frac{x}{y} - \frac{x}{y} \ln\left(1 + \frac{xy}{B_n'} \right) \right\}.$$

Lemma 3.2. *Let* $\{X_{ni}; 1 \le i \le k_n, n \ge 1\}$ *be an array of rowwise ND random variables in* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *and* $\{c_n\}$ *be a sequence of positive real numbers. Suppose that for every* $\varepsilon > 0$ *and some* $\delta > 0$*:*

(i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \mathbb{V}(|X_{ni}| > \varepsilon) < \infty$*;*

(ii) there exists $j \ge 1$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} \widehat{\mathbb{E}} \left[\left(X_{ni} g\left(\frac{X_{ni}}{\delta} \right) - \widehat{\mathbb{E}} \left[X_{ni} g\left(\frac{X_{ni}}{\delta} \right) \right] \right)^2 \right] \right)^j < \infty.$$

Then

$$\sum_{n=1}^{\infty} c_n \mathbb{V}\left(\sum_{i=1}^{k_n} \left(X_{ni} - \widehat{\mathbb{E}} \left[X_{ni} g\left(\frac{X_{ni}}{\delta} \right) \right] \right) > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0.$$

Proof Denote $S_n^{'} = \sum_{i=1}^{k_n}\left(X_{ni} - \widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right)$. Note that

$$\mathbb{V}(S_n^{'} > \varepsilon) \le \mathbb{V}\left(S_n^{'} > \varepsilon, \bigcup_{i=1}^{k_n}(|X_{ni}| > \mu\delta)\right) + \mathbb{V}\left(S_n^{'} > \varepsilon, \bigcap_{i=1}^{k_n}(|X_{ni}| \le \mu\delta)\right)$$
$$\le \sum_{i=1}^{k_n}\mathbb{V}(|X_{ni}| > \mu\delta) + \mathbb{V}\left(\sum_{i=1}^{k_n}\left(X_{ni}g\left(\frac{X_{ni}}{\delta}\right) - \widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right) > \varepsilon\right)$$

Hence, by condition (i), it is sufficient to prove that

$$\sum_{n=1}^{\infty} c_n\mathbb{V}\left(\sum_{i=1}^{k_n}\left(X_{ni}g\left(\frac{X_{ni}}{\delta}\right) - \widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right) > \varepsilon\right) < \infty.$$

For any $a > 0$ and set

$$d = \min\left\{1, \frac{a}{6\delta}\right\}, N_1 = \left\{n : \sum_{i=1}^{k_n}\mathbb{V}\left(|X_{ni}| > \min\left\{\mu\delta, \mu\frac{a}{6}\right\}\right) > d\right\}, \text{ and } N_2 = \mathbb{N}/N_1.$$

Note that

$$\sum_{n\in N_1} c_n\mathbb{V}\left(\sum_{i=1}^{k_n}\left(X_{ni}g\left(\frac{X_{ni}}{\delta}\right) - \widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right) > \varepsilon\right)$$
$$\le \sum_{n\in N_1} c_n \le \frac{1}{d}\sum_{n=1}^{\infty} c_n\sum_{i=1}^{k_n}\mathbb{V}\left(|X_{ni}| > \min\left\{\mu\delta, \mu\frac{a}{6}\right\}\right) < \infty.$$

Hence, it remains to prove that $\sum_{n\in N_2} c_n\mathbb{V}\left(\sum_{i=1}^{k_n}\left(X_{ni}g\left(\frac{X_{ni}}{\delta}\right) - \widehat{\mathbb{E}}[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)]\right) > \varepsilon\right) < \infty$.

By Lemma 3.1 we get

$$\sum_{n\in N_2} c_n\mathbb{V}\left(\sum_{i=1}^{k_n}\left(X_{ni}g\left(\frac{X_{ni}}{\delta}\right) - \widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right) > \varepsilon\right)$$
$$\le \sum_{n\in N_2} c_n\mathbb{V}\left(\max_{1\le i\le k_n}\left|X_{ni}g\left(\frac{X_{ni}}{\delta}\right) - \widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right| > a\right)$$
$$+ \sum_{n\in N_2} c_n \exp\left\{\frac{\varepsilon}{a} - \frac{\varepsilon}{a}\ln(1 + \frac{\varepsilon a}{B_n})\right\},$$

where $B_n = \sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[\left(X_{ni}g\left(\frac{X_{ni}}{\delta}\right) - \widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right)^2\right]$.

If $a/6 < \delta$, for $n \in N_2$, by the fact $\left|g\left(\frac{X_{ni}}{\delta}\right) - g\left(\frac{X_{ni}}{a/6}\right)\right| \le I\left(\mu\frac{a}{6} < |X_{ni}| \le \delta\right)$, we have,

$$\max_{1\le i\le k_n}\left|\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right|$$
$$\le \max_{1\le i\le k_n}\left|\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{a/6}\right)\right]\right| + \max_{1\le i\le k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left|g\left(\frac{X_{ni}}{\delta}\right) - g\left(\frac{X_{ni}}{a/6}\right)\right|\right]$$
$$\le a/6 + \delta\sum_{i=1}^{k_n}\mathbb{V}\left(|X_{ni}| > \mu\frac{a}{6}\right)$$
$$\le a/6 + \delta\sum_{i=1}^{k_n}\mathbb{V}\left(|X_{ni}| > \min\left\{\delta, \mu\frac{a}{6}\right\}\right) \le a/6 + \delta d \le a/3.$$

Therefore,

$$\sum_{n\in N_2} c_n \mathbb{V}\left(\max_{1\le i\le k_n}\left|X_{ni}g\left(\frac{X_{ni}}{\delta}\right)-\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right|>a\right)$$
$$\le \sum_{n=1}^{\infty} c_n \mathbb{V}\left(\max_{1\le i\le k_n}\left|X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right|>\frac{2}{3}a\right)\le \sum_{n=1}^{\infty} c_n\sum_{i=1}^{k_n}\mathbb{V}(|X_{ni}|>\frac{2}{3}a)<\infty.$$

If $a/6\ge\delta$, we can also easily get $\sum_{n\in N_2} c_n\mathbb{V}\left(\max_{1\le i\le k_n}\left|X_{ni}g\left(\frac{X_{ni}}{\delta}\right)-\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right|>a\right)$ $<\infty$.

Letting $a=\frac{\varepsilon}{j}$, we have

$$\sum_{n\in N_2} c_n\exp\left\{\frac{\varepsilon}{a}-\frac{\varepsilon}{a}\ln\left(1+\frac{\varepsilon a}{B_n}\right)\right\}\le \exp\left\{\frac{\varepsilon}{a}\right\}\sum_{n\in N_2} c_n\left(1+\frac{\varepsilon a}{B_n}\right)^{-\frac{\varepsilon}{a}}$$
$$\le \exp\left\{\frac{\varepsilon}{2a}\right\}\sum_{n\in N_2} c_n\left(\frac{B_n}{\varepsilon a}\right)^{j}\le C\sum_{n=1}^{\infty}c_nB_n^j$$
$$=C\sum_{n=1}^{\infty}c_n\left(\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[\left(X_{ni}g\left(\frac{X_{ni}}{\delta}\right)-\widehat{\mathbb{E}}[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)]\right)^2\right]\right)^j<\infty.$$

We complete the proof of Lemma 3.2.

Proof. **Proof of Theorem 2.1.** First of all, from condition (a), we have

$$(a')\ \sum_{n=1}^{\infty}c_n\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[f\left(|X_{ni}|\left(1-g\left(\frac{X_{ni}}{\varepsilon}\right)\right)\right)\right]$$
$$\le\sum_{n=1}^{\infty}c_n\sum_{i=1}^{k_n}C_{\mathbb{V}}\left[f\left(|X_{ni}|\left(1-g\left(\frac{X_{ni}}{\varepsilon}\right)\right)\right)\right]<\infty \text{ for any } \varepsilon>0.$$

Next, by Markov's inequality and condition (a'), we have that

$$\sum_{n=1}^{\infty}c_n\sum_{i=1}^{k_n}\mathbb{V}(|X_{ni}|>\varepsilon)\le C\sum_{n=1}^{\infty}c_n\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[f\left(|X_{ni}|\left(1-g\left(\frac{X_{ni}}{\varepsilon}\right)\right)\right)\right]<\infty,$$

which implies that condition (i) of Lemma 3.2 holds.

For $n\ge1$, denote $S_n=\sum_{i=1}^{k_n}\left(X_{ni}-\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right)$.

We can easily get that

$$\begin{aligned}\sum_{n=1}^{\infty}c_nC_{\mathbb{V}}\left[f\left(\{S_n-\varepsilon\}_+\right)\right]&=\sum_{n=1}^{\infty}c_n\int_0^{\infty}\mathbb{V}(S_n-\varepsilon>h(t))\,dt\\&=\sum_{n=1}^{\infty}c_n\int_0^{f(\delta)}\mathbb{V}(S_n>\varepsilon+h(t))\,dt+\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}\mathbb{V}(S_n>\varepsilon+h(t))\,dt\\&\le f(\delta)\sum_{n=1}^{\infty}c_n\mathbb{V}(S_n>\varepsilon)+\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}\mathbb{V}(S_n>h(t))\,dt\\&\hat{=}I_1+I_2.\end{aligned}\tag{3.1}$$

By Lemma 3.2, we have that $I_1<\infty$. It is enough to show that $I_2<\infty$.

For $t \geq f(\delta)$, it is easy to get that

$$\mathbb{V}(S_n > h(t)) \leq \sum_{i=1}^{k_n} \mathbb{V}(|X_{ni}| > \mu h(t)) + \mathbb{V}\left(\sum_{i=1}^{k_n}\left(X_{ni} g\left(\frac{X_{ni}}{h(t)}\right) - \widehat{\mathbb{E}}\left[X_{ni} g\left(\frac{X_{ni}}{\delta}\right)\right]\right) > h(t)\right),$$

which implies that

$$\begin{aligned} I_2 \leq & \sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \int_{f(\delta)}^{\infty} \mathbb{V}(|X_{ni}| > \mu h(t)) dt \\ & + \sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \mathbb{V}\left(\sum_{i=1}^{k_n}\left(X_{ni} g\left(\frac{X_{ni}}{h(t)}\right) - \widehat{\mathbb{E}}\left[X_{ni} g\left(\frac{X_{ni}}{\delta}\right)\right]\right) > h(t)\right) dt \\ \hat{=} & I_3 + I_4. \end{aligned} \tag{3.2}$$

By Markov's inequality and condition $(a^{'})$, we have that

$$\begin{aligned} I_3 & \leq \sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \int_{f(\delta)}^{\infty} \mathbb{V}\left(f(|X_{ni}/\mu|) > t\right) dt \\ & \leq C \sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} C_{\mathbb{V}}\left[f\left(|X_{ni}|\left(1 - g\left(\frac{X_{ni}}{\delta}\right)\right)\right)\right] < \infty. \end{aligned} \tag{3.3}$$

For fixed $n \geq 1, 1 \leq i \leq k_n$ and $t \geq f(\delta)$, denote

$$Y_{ni} = h(t) I(X_{ni} > h(t)) + X_{ni} I(|X_{ni}| \leq h(t)) - h(t) I(X_{ni} < -h(t)) \quad \text{and}$$
$$Y_{ni}^{'} = h(t) I(X_{ni} > h(t)) - h(t) I(X_{ni} < -h(t)).$$

By the definition of Y_{ni}, $\{Y_{ni} - EY_{ni}, 1 \leq i \leq k_n,\ n \geq 1\}$ is an array of rowwise ND random variables under the sub-linear expectations.

By the definition of Y_{ni}, C_r inequality and (2.1), we have

$$\widehat{\mathbb{E}}[Y_{ni}] \leq \widehat{\mathbb{E}}\left[X_{ni} g\left(\frac{\mu X_{ni}}{h(t)}\right)\right] + h(t)\mathbb{V}(|X_{ni}| > \mu h(t))$$

Hence, for $t \geq f(\delta)$, we have

$$\begin{aligned} & \mathbb{V}\left(\sum_{i=1}^{k_n}\left(X_{ni} g\left(\frac{X_{ni}}{h(t)}\right) - \widehat{\mathbb{E}}\left[X_{ni} g\left(\frac{X_{ni}}{\delta}\right)\right]\right) > h(t)\right) \\ & \leq \mathbb{V}\left(\sum_{i=1}^{k_n}\left(Y_{ni} - Y_{ni}^{'} - \widehat{\mathbb{E}}[Y_{ni}] + \widehat{\mathbb{E}}[Y_{ni}] - \widehat{\mathbb{E}}\left[X_{ni} g\left(\frac{X_{ni}}{\delta}\right)\right]\right) > h(t)\right) \\ & \leq \mathbb{V}\Bigg(\sum_{i=1}^{k_n}\Bigg((Y_{ni} - \widehat{\mathbb{E}}[Y_{ni}]) + |Y_{ni}^{'}| + \Bigg|\widehat{\mathbb{E}}\left[X_{ni} g\left(\frac{\mu X_{ni}}{h(t)}\right)\right] \\ & - \widehat{\mathbb{E}}\left[X_{ni} g\left(\frac{X_{ni}}{\delta}\right)\right]\Bigg| + h(t)\mathbb{V}(|X_{ni}| > \mu h(t))\Bigg) > h(t)\Bigg) \end{aligned} \tag{3.4}$$

By $g(x) \downarrow$, $x > 0$, condition (c) and the fact $\left|g\left(\frac{\mu X_{ni}}{h(t)}\right) - g\left(\frac{X_{ni}}{\delta}\right)\right| \leq I(\mu\delta < |X_{ni}| \leq h(t)/\mu)$, we have

$$\max_{t\geq f(\delta)}\frac{1}{h(t)}\sum_{i=1}^{k_n}\left|\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{\mu X_{ni}}{h(t)}\right)\right]-\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right|$$
$$\leq\max_{t\geq f(\delta)}\frac{1}{h(t)}\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left|g\left(\frac{\mu X_{ni}}{h(t)}\right)-g\left(\frac{X_{ni}}{\delta}\right)\right|\right]$$
$$\leq\delta^{-1}\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left(1-g\left(\frac{X_{ni}}{\mu\delta}\right)\right)\right]$$
$$\leq\delta^{-1}\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left(1-g\left(\frac{16\eta X_{ni}}{\mu\delta}\right)\right)\right]\to 0,$$

as $n\to\infty$.

Hence, for all n large enough,

$$\sum_{i=1}^{k_n}\left|\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{\mu X_{ni}}{h(t)}\right)\right]-\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right|<\frac{h(t)}{4},\ \ t\geq f(\delta). \tag{3.5}$$

By Markov's inequality and condition (c), we have

$$\max_{t\geq f(\delta)}\frac{1}{h(t)}\sum_{i=1}^{k_n}h(t)\mathbb{V}(|X_{ni}|>\mu h(t))$$
$$=\max_{t\geq f(\delta)}\sum_{i=1}^{k_n}\mathbb{V}(|X_{ni}|>\mu h(t))\leq\sum_{i=1}^{k_n}\mathbb{V}(|X_{ni}|>\mu\delta) \tag{3.6}$$
$$\leq(\mu\delta)^{-1}\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left(1-g\left(\frac{X_{ni}}{\mu\delta}\right)\right)\right]\to 0,\ \ \text{as}\ \ n\to\infty.$$

Hence, for all n large enough,

$$\sum_{i=1}^{k_n}h(t)\mathbb{V}(|X_{ni}|>\mu h(t)))<\frac{h(t)}{4},\ \ t\geq f(\delta). \tag{3.7}$$

Combining (3.4), (3.5) and (3.7), for n large enough, we have

$$\mathbb{V}\left(\sum_{i=1}^{k_n}\left(X_{ni}g\left(\frac{X_{ni}}{h(t)}\right)-\widehat{\mathbb{E}}\left[X_{ni}g\left(\frac{X_{ni}}{\delta}\right)\right]\right)>h(t)\right)$$
$$\leq\mathbb{V}\left(\sum_{i=1}^{k_n}(Y_{ni}-\widehat{\mathbb{E}}[Y_{ni}])>\frac{h(t)}{4}\right)+\mathbb{V}\left(\sum_{i=1}^{k_n}|Y'_{ni}|>\frac{h(t)}{4}\right)$$

Therefore

$$I_4\leq C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}\mathbb{V}\left(\sum_{i=1}^{k_n}(Y_{ni}-\widehat{\mathbb{E}}[Y_{ni}])>\frac{h(t)}{4}\right)\,dt$$
$$+C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}\mathbb{V}\left(\sum_{i=1}^{k_n}|Y'_{ni}|>\frac{h(t)}{4}\right)\,dt\hat{=}I_5+I_6. \tag{3.8}$$

By the fact $|Y'_{ni}|\leq h(t)I(|X_{ni}|>h(t))\leq h(t)\left(1-g\left(\frac{X_{ni}}{h(t)}\right)\right)$, Markov's inequality, (2.1) and condition $(a^{'})$, we can obtain

$$\begin{aligned} I_6 &= C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \mathbb{V}\left(\sum_{i=1}^{k_n} |Y_{ni}^{'}| > \frac{h(t)}{4}\right) dt \\ &\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \mathbb{V}\left(\sum_{i=1}^{k_n} h(t)\left(1-g\left(\frac{X_{ni}}{h(t)}\right)\right) > \frac{h(t)}{4}\right) dt \\ &\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \widehat{\mathbb{E}}\left[\sum_{i=1}^{k_n}\left(1-g\left(\frac{X_{ni}}{h(t)}\right)\right)\right] dt \\ &\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \sum_{i=1}^{k_n} \mathbb{V}(|X_{ni}| > \mu h(t)))\, dt \\ &\le C\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} C_{\mathbb{V}}\left[f\left(|X_{ni}|\left(1-g\left(\frac{X_{ni}}{\mu\delta}\right)\right)\right)\right] < \infty. \end{aligned} \tag{3.9}$$

Hence to prove $I_4 < \infty$, it is enough to show that $I_5 < \infty$. Denote $B_n = \sum_{i=1}^{k_n} \widehat{\mathbb{E}}(Y_{ni} - \widehat{\mathbb{E}}[Y_{ni}])^2$. Applying Lemma 3.1 with $x = h(t)/4$ and $y = h(t)/(4\eta)$, we have

$$\begin{aligned} I_5 &\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \mathbb{V}\left(\max_{1\le i\le k_n} |Y_{ni} - \widehat{\mathbb{E}}[Y_{ni}]| > \frac{h(t)}{4\eta}\right) dt \\ &+ C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \exp\left\{\eta - \eta\ln\left(1+\frac{h^2(t)}{16\eta B_n}\right)\right\} dt \,\hat{=} I_7 + I_8. \end{aligned} \tag{3.10}$$

By Markov's inequality and condition (c), we can get that for all n large enough,

$$\sum_{i=1}^{k_n} \mathbb{V}\left(|X_{ni}| > \frac{\mu\delta}{16\eta}\right) \le C\sum_{i=1}^{k_n} \widehat{\mathbb{E}}\left[|X_{ni}|\left(1-g\left(\frac{16\eta X_{ni}}{\mu\delta}\right)\right)\right] < \frac{\mu}{(1+\mu)16\eta}. \tag{3.11}$$

Thus by the fact $\left|g\left(\frac{\mu X_{ni}}{h(t)}\right) - g\left(\frac{16\eta X_{ni}}{\delta}\right)\right| \le I(\frac{\mu\delta}{16\eta} < |X_{ni}| \le h(t)/\mu)$ and (3.11), we have

$$\begin{aligned} &\max_{t\ge f(\delta)} \max_{1\le i\le k_n} \frac{1}{h(t)} |\widehat{\mathbb{E}}[Y_{ni}]| \le \max_{t\ge f(\delta)} \max_{1\le i\le k_n} \frac{1}{h(t)} \widehat{\mathbb{E}}[|Y_{ni}|] \\ &\le \max_{t\ge f(\delta)} \max_{1\le i\le k_n} \frac{1}{h(t)}\left(\widehat{\mathbb{E}}\left[|X_{ni}| g\left(\frac{\mu X_{ni}}{h(t)}\right)\right] + h(t)\mathbb{V}(|X_{ni}| > \mu h(t))\right) \\ &\le \max_{t\ge f(\delta)} \max_{1\le i\le k_n} \frac{1}{h(t)}\left(\widehat{\mathbb{E}}\left[|X_{ni}| g\left(\frac{16\eta X_{ni}}{\delta}\right)\right] + \widehat{\mathbb{E}}\left[|X_{ni}|\left|g\left(\frac{\mu X_{ni}}{h(t)}\right) - g\left(\frac{16\eta X_{ni}}{\delta}\right)\right|\right]\right) \\ &+ \max_{t\ge f(\delta)} \max_{1\le i\le k_n} \frac{1}{h(t)} \left(h(t)\mathbb{V}(|X_{ni}| > \mu h(t))\right) \\ &\le \frac{1}{\delta}\frac{\delta}{16\eta} + \max_{t\ge f(\delta)} \frac{1}{h(t)} \sum_{i=1}^{k_n} \widehat{\mathbb{E}}\left[|X_{ni}|\left(g\left|\frac{\mu X_{ni}}{h(t)}\right| - g\left(\frac{16\eta X_{ni}}{\delta}\right)\right)\right] + \sum_{i=1}^{k_n} \mathbb{V}(|X_{ni}| > \mu\delta) \\ &\le \frac{1}{16\eta} + \max_{t\ge f(\delta)} \frac{1}{h(t)} \cdot \frac{h(t)}{\mu} \cdot \sum_{i=1}^{k_n} \mathbb{V}\left(|X_{ni}| > \frac{\mu\delta}{16\eta}\right) + \sum_{i=1}^{k_n} \mathbb{V}(|X_{ni}| > \mu\delta) \\ &\le \frac{1}{16\eta} + (1/\mu + 1)\sum_{i=1}^{k_n} \mathbb{V}(|X_{ni}| > \frac{\mu\delta}{16\eta}) \end{aligned}$$

$$< \frac{1}{16\eta} + \frac{1}{16\eta} = \frac{1}{8\eta}.$$

Therefore by the fact $|Y_{ni}| \le |X_{ni}|$ and condition (a), we have

$$\begin{aligned} I_7 &\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \mathbb{V}\left(\max_{1\le i\le k_n} |Y_{ni}| > \frac{h(t)}{8\eta}\right) dt \le C\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n}\int_{f(\delta)}^{\infty} \mathbb{V}\left(|X_{ni}| > \frac{h(t)}{8\eta}\right) dt \\ &\le C\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} C_{\mathbb{V}}\left[f\left(8\eta|X_{ni}|\left(1-g\left(\frac{8\eta X_{ni}}{\delta}\right)\right)\right)\right] < \infty. \end{aligned} \tag{3.12}$$

By C_r-inequality, we have

$$\begin{aligned} I_8 &= C\sum_{n=1}^{\infty} c_n e^{\eta} \int_{f(\delta)}^{\infty} \left(1 + \frac{h^2(t)}{16\eta B_n}\right)^{-\eta} dt \le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} h^{-2\eta}(t) B_n^{\eta}\, dt \\ &= C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} h^{-2\eta}(t)\left(\sum_{i=1}^{k_n} \widehat{\mathbb{E}}(Y_{ni} - \widehat{\mathbb{E}}[Y_{ni}])^2\right)^{\eta} dt \\ &\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} h^{-2\eta}(t)\left(\sum_{i=1}^{k_n} \widehat{\mathbb{E}}[Y_{ni}^2]\right)^{\eta} dt. \end{aligned}$$

By the definition of Y_{ni}, C_r inequality and (2.1), we have

$$\begin{aligned} \widehat{\mathbb{E}}[Y_{ni}^2] &\le C\widehat{\mathbb{E}}\left[X_{ni}^2 g\left(\frac{\mu X_{ni}}{h(t)}\right)\right] + Ch^2(t)\mathbb{V}(|X_{ni}| > \mu h(t)) \\ &\le C\widehat{\mathbb{E}}\left[X_{ni}^2 g\left(\frac{X_{ni}}{\delta}\right)\right] + C\widehat{\mathbb{E}}\left[X_{ni}^2\left|g\left(\frac{\mu X_{ni}}{h(t)}\right) - g\left(\frac{X_{ni}}{\delta}\right)\right|\right] + Ch^2(t)\mathbb{V}(|X_{ni}| > \mu h(t)). \end{aligned}$$

Hence

$$\begin{aligned} I_8 &\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} h^{-2\eta}(t)\left(\sum_{i=1}^{k_n} \widehat{\mathbb{E}}[Y_{ni}^2]\right)^{\eta} dt \\ &\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} h^{-2\eta}(t)\left(\sum_{i=1}^{k_n} \widehat{\mathbb{E}}\left[X_{ni}^2 g\left(\frac{X_{ni}}{\delta}\right)\right]\right)^{\eta} dt \\ &+ C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} h^{-2\eta}(t)\left(\sum_{i=1}^{k_n} \widehat{\mathbb{E}}\left[X_{ni}^2\left|g\left(\frac{\mu X_{ni}}{h(t)}\right) - g\left(\frac{X_{ni}}{\delta}\right)\right|\right]\right)^{\eta} dt \\ &+ C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \left(\sum_{i=1}^{k_n} \mathbb{V}(|X_{ni}| > \mu h(t))\right)^{\eta} dt \hat{=} I_9 + I_{10} + I_{11}. \end{aligned} \tag{3.13}$$

By the definition of $s(t) = \max_{\mu\delta \le x \le h(t)/\mu} \frac{x}{f(x)}$, we have $s(t) \ge \frac{\mu\delta}{f(\mu\delta)}$. Then $\delta \le \frac{f(\mu\delta)}{\mu} s(t)$. For $t > f(\delta)$, we have

$$\begin{aligned} h^{-2\eta}(t) &\le h^{-\eta}(t)\delta^{-\eta} \le h^{-\eta}(t)\delta^{-\eta-1}\frac{f(\mu\delta)}{\mu}s(t) \\ &= \delta^{-\eta-1}\frac{f(\mu\delta)}{\mu}h^{-\eta}(t)s(t) = Ch^{-\eta}(t)s(t). \end{aligned}$$

Hence by conditions (b) and (d), we have

$$I_9 = C\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} \widehat{\mathbb{E}}\left[X_{ni}^2 g\left(\frac{X_{ni}}{\delta}\right)\right]\right)^{\eta} \int_{f(\delta)}^{\infty} h^{-\eta}(t)s(t)\, dt < \infty. \tag{3.14}$$

Note that

$$\left|g\left(\frac{\mu X_{ni}}{h(t)}\right)-g\left(\frac{X_{ni}}{\delta}\right)\right|\le I(\mu\delta<|X_{ni}|\le h(t)/\mu),\quad t\ge f(\delta)$$

then we have

$$I_{10}\le C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}h^{-\eta}(t)\left(\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left|g\left(\frac{\mu X_{ni}}{h(t)}\right)-g\left(\frac{X_{ni}}{\delta}\right)\right|\right]\right)^{\eta}dt.$$

By condition (c), for all n large enough, we have

$$\begin{aligned}\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left|g\left(\frac{\mu X_{ni}}{h(t)}\right)-g\left(\frac{X_{ni}}{\delta}\right)\right|\right]&\le\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left(1-g\left(\frac{X_{ni}}{\mu\delta}\right)\right)\right]\\&\le\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left(1-g\left(\frac{16\eta X_{ni}}{\mu\delta}\right)\right)\right]<1.\end{aligned}$$

Hence by the fact $\eta\ge1$, conditions $(a^{'})$ and (d), we get

$$\begin{aligned}I_{10}&\le C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}h^{-\eta}(t)\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[|X_{ni}|\left|g\left(\frac{\mu X_{ni}}{h(t)}\right)-g\left(\frac{X_{ni}}{\delta}\right)\right|\right]dt\\&\le C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}h^{-\eta}(t)\\&\left(\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[\frac{|X_{ni}|\left|g\left(\frac{\mu X_{ni}}{h(t)}\right)-g\left(\frac{X_{ni}}{\delta}\right)\right|}{f\left(|X_{ni}|\left|g\left(\frac{\mu X_{ni}}{h(t)}\right)-g\left(\frac{X_{ni}}{\delta}\right)\right|\right)}f\left(|X_{ni}|\left|g\left(\frac{\mu X_{ni}}{h(t)}\right)-g\left(\frac{X_{ni}}{\delta}\right)\right|\right)\right]\right)dt\\&\le C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}h^{-\eta}(t)s(t)\left(\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[f\left(|X_{ni}|\left|g\left(\frac{\mu X_{ni}}{h(t)}\right)-g\left(\frac{X_{ni}}{\delta}\right)\right|\right)\right]\right)dt\\&\le C\sum_{n=1}^{\infty}c_n\sum_{i=1}^{k_n}\widehat{\mathbb{E}}\left[f\left(|X_{ni}|\left(1-g\left(\frac{X_{ni}}{\mu\delta}\right)\right)\right)\right]\int_{f(\delta)}^{\infty}h^{-\eta}(t)s(t)\,dt<\infty.\end{aligned}$$

Finally, we will show that $I_{11}<\infty$. For $t\ge f(\delta)$, by the proof of (3.6), for n large enough, we have that

$$\sum_{i=1}^{k_n}\mathbb{V}(|X_{ni}|>\mu h(t)))<1.$$

Thus by the fact $\eta\ge1$ and conditions $(a^{'})$, we have

$$\begin{aligned}I_{11}&\le C\sum_{n=1}^{\infty}c_n\sum_{i=1}^{k_n}\int_{f(\delta)}^{\infty}\mathbb{V}(|X_{ni}|>\mu h(t)))\,dt\\&\le\sum_{n=1}^{\infty}c_n\sum_{i=1}^{k_n}C_{\mathbb{V}}\left[f\left(|X_{ni}|\left(1-g\left(\frac{X_{ni}}{\mu\delta}\right)\right)\right)\right]<\infty.\end{aligned}$$

□

Declarations

Conflict of interest The authors declare no conflict of interest.

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