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Multiple parametric Marcinkiewicz integrals with mixed homogeneity along surfaces

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Abstract. In this paper, the multiple parametric Marcinkiewicz integral operators with mixed homogeneity along surfaces are studied. The L^p-mapping properties for such operators are obtained under the rather weakened size conditions on the integral kernels both on the unit sphere and in the radial direction. The main results essentially improve and extend certain previous results.

§1 Introduction

Let $\mathbb{R}^d$ ($d = m$ or n), $d \geq 2$, be the d-dimensional Euclidean space and S^{d-1} be the unit sphere in $\mathbb{R}^d$ equipped with the induced Lebesgue measure $d\sigma_d$. Let $\alpha_{d,j} \geq 1 (j = 1, \cdots, d)$ be fixed real numbers. Define the function $F : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ by $F(x, \rho_d) = \sum_{j=1}^{d} x_j^2 \rho_d^{-2\alpha_{d,j}}$. It is clear that for each fixed $x \in \mathbb{R}^d$, the function $F(x, \rho_d)$ is a decreasing function in $\rho_d > 0$. Let $\rho_d(x)$ be the unique solution of the equation $F(x, \rho_d) = 1$. Fabes and Rivière [21] showed that $(\mathbb{R}^d, \rho_d)$ is a metric space which is often called the mixed homogeneity space related to $\{\alpha_{d,j}\}_{j=1}^{n}$. For $\lambda > 0$, let $A_{d,\lambda}$ be the diagonal $d \times d$ matrix, namely, $A_{d,\lambda} = \text{diag}\{\lambda^{\alpha_{d,1}}, \cdots, \lambda^{\alpha_{d,d}}\}$. For a function φ: $\mathbb{R}^+ \to (0, \infty)$, we shall let $A_d^{\varphi} : \mathbb{R}^d \to \mathbb{R}^d$ be the mapping

$$A_d^{\varphi}(y) = A_{d,\varphi(\rho(y))} y'$$

where $y' = A_{d,\rho(y)^{-1}} y \in \mathbb{S}^{d-1}$.

We would like to remark that if $\alpha_{d,1} = \alpha_{d,2} = \cdots \alpha_{d,d} = 1$, then $\rho_d(x) = |x|$. Indeed, the

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change of variables related to the space $(\mathbb{R}^d, \rho_d)$ is given by the transformation

$$\begin{aligned} x_1 &= \rho_d^{\alpha_{d,1}} \cos\theta_1 \cdots \cos\theta_{d-2}\cos\theta_{d-1},\\ x_2 &= \rho_d^{\alpha_{d,2}} \cos\theta_1 \cdots \cos\theta_{d-2}\sin\theta_{d-1},\\ &\cdots\cdots,\\ x_{d-1} &= \rho_d^{\alpha_{d,d-1}} \cos\theta_1 \sin\theta_2,\\ x_d &= \rho_d^{\alpha_{d,d}} \sin\theta_1, \end{aligned}$$

where $x \in \mathbb{R}^d$. Therefore, it is easy to check that

$$dx = \rho_d^{\alpha_d - 1} J_d(x') d\rho d\sigma_d(x'),$$

where $\rho_d^{\alpha_d-1} J_d(x')$ is the Jacobian of the above transform and $\alpha_d = \sum_{j=1}^d \alpha_{d,j}$. Furthermore, it was proved that in [21] that $J_d(x') \in C^\infty(\mathbb{S}^{d-1})$ and that there exists $M_d > 0$ such that

$$1 \le J_d(x') \le M_d, \quad x' \in S^{d-1}.$$

Let Ω be a real valued and measurable function on $\mathbb{R}^d$ with $\Omega \in L^1(S^{d-1})$ and satisfy

$$\Omega(A_{d,\lambda}x) = \Omega(x), \forall \lambda > 0, \text{ and } \int_{S^{d-1}} \Omega(y')J(y')d\sigma_d(y') = 0. \tag{1}$$

For a suitable function h defined on $(0,\infty)$, we define the parabolic Marcinkiewicz integral operator

$$\mu_{h,\Omega}(f)(x) = \Big(\int_0^\infty \Big|\frac{1}{t^\tau}\int_{\rho_d(y)\le t} \frac{\Omega(y)h(\rho_d(y))}{\rho_d(y)^{\alpha_d-\tau}} dy\Big|^2 \frac{dt}{t}\Big)^{\frac{1}{2}},) \tag{2}$$

where $\tau = a + ib$ ($a, b \in \mathbb{R}$ with $a > 0$) and $h \in \Delta_\gamma(\mathbb{R}^+)$. Here $\Delta_\gamma(\mathbb{R}^+)$ for $\gamma \ge 1$ denotes the set of all measurable functions satisfying the condition

$$\sup_{R>0} \big(R^{-1}\int_0^R |h(t)|^\gamma dt\big)^{1/\gamma} < \infty.$$

We would like to note that the class of the operators $\mu_{h,\Omega}$ is related to the class of the parabolic singular integral operators

$$T_{h,\Omega}(f)(x) = p.v. \int_{\mathbb{R}^d} \frac{\Omega(y)h(\rho(y))}{\rho(y)^\alpha} f(x-y)dy.$$

When $h \equiv 1$, we denote $\mu_{h,\Omega}$ by μ_Ω. Clearly, if $\alpha_1 = \alpha_2 = \cdots = \alpha_d = 1$ and $\tau = 1$, then the operator μ_Ω is a natural analogy of higher-dimensional Marcinkiewicz integral introduced by Stein [30], which has been investigated by many authors (see [8, 10, 18, 34]). When $\alpha_j \ge 1, j = 1, \cdots, d$, and $\tau = 1$, Xue ,Ding and Yabuta [37] first established that μ_Ω is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, provided that $\Omega \in L^q(\mathbb{R}^d)$ for $q > 1$. Subsequently, Chen and Ding [12, 13] extended the result of [37] to the case $\Omega \in L(\log^+ L)^{\frac{1}{2}}(S^{d-1})$ and $\Omega \in H^1(S^{d-1})$ respectively. Moreover, it follows from Wang, Chen and Yu's work [31] (also see [6]) that μ_Ω is bounded on $L^p(\mathbb{R}^d)$ for $\frac{2\beta}{2\beta-1} < p < 2\beta$ if $\Omega \in \mathcal{F}_\beta(S^{d-1})$ for some $\beta > 1$, where

$$\mathcal{F}_\beta(S^{d-1}) := \Big\{\Omega \in L^1(S^{d-1}) : \sup_{\xi\in S^{d-1}} \int_{S^{d-1}} |\Omega(y')|\big(\log\frac{1}{|\xi\cdot y'|}\big)^\beta d\sigma(y') < \infty\Big\}, \quad \forall \beta > 0.$$

For the general operator $\mu_{h,\Omega}$, the kernel of $\mu_{h,\Omega}$ has the additional roughness in the radial direction, which has received a large moment of interest of many authors in the Euclidean setting, for instance, see e.g. [15–17, 19]. In order to extend the results in [23] to the singular integral operator $T_{h,\Omega}$ in the Euclidean setting, Fan and Stao [22] introduced the function class $\mathcal{WF}_\beta(S^{d-1})$ in more general form, namely, the set of all functions $\Omega \in L^1(S^{d-1})$ satisfying

$$\sup_{\xi' \in S^{d-1}} \iint_{S^{d-1}\times S^{d-1}} |\Omega(\theta)\Omega(w)|\big(\log \frac{1}{|(\theta - w)\cdot \xi'|}\big)^\beta d\sigma(\theta)d\sigma(w) < \infty, \quad \beta > 0.$$

Furthermore, they showed that $\mathcal{F}_\beta(S^1) \subset \mathcal{WF}_\beta(S^1)$. However, for $d > 2$, the relation between $\mathcal{F}_\beta(S^{d-1})$ and $\mathcal{WF}_\beta(S^{d-1})$ remains to be open. Recently, Liu, Wu and Zhang [27] showed that $\mu_{h,\Omega}$ is bounded on $L^p(\mathbb{R}^d)$ for some $\beta > \max\{2,\gamma'\}/2$ with $|1/p - 1/2| < \min\{1/\gamma', 1/2\} - \min\{1/\gamma' + 1/2, 1\}/(\beta+1)$, provided that $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$ and $\Omega \in \mathcal{WF}_\beta(S^{d-1})$.

In this paper, we will focus our attention on the multiple Marcinkiewicz integrals with mixed homogeneity on product spaces. Suppose that $\Omega \in L^1(S^{m-1}\times S^{n-1})$ and satisfies the conditions

$$\Omega(A_{m,s}x, A_{n,t}) = \Omega(x,y), \quad \forall s,t > 0, \tag{3}$$

$$\int_{S^{m-1}} \Omega(u',\cdot)J_m(u')d\sigma_m(u') = \int_{S^{n-1}} \Omega(\cdot,v')J_n(v')d\sigma_n(v') = 0. \tag{4}$$

Let $\tau_i = a_i + ib_i$, $a_i, b_i \in \mathbb{R}$ with $a_i > 0, i = 1,2$. For $\gamma \geq 1$, let Δ_γ denote the set of measurable functions $h : \mathbb{R}^+ \times \mathbb{R}^+ \to C$ satisfying

$$\sup_{R_1>0,R_2>0} \big(\frac{1}{R_1R_2}\int_0^{R_1}\int_0^{R_2} |h(r,t)|^\gamma drdt\big)^{1/\gamma} < \infty.$$

Observe that $\Delta_{\gamma_1} \subsetneq \Delta_{\gamma_2}$ for $\gamma_1 > \gamma_2$ and $L^\infty = \Delta_\infty$. We consider the multiple parabolic Marcinkiewicz integral operators defined by

$$\mathcal{M}_{\Omega,h}(f)(x,y) = \left(\int_0^\infty\int_0^\infty |F_{s,t}(x,y)|^2 \frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{2}}, \tag{5}$$

where

$$F_{s,t}(x,y) = \frac{1}{s^{\tau_1}}\frac{1}{t^{\tau_2}}\int_{\rho_m(u)\leq s}\int_{\rho_n(v)\leq t} \frac{\Omega(u,v)h(\rho_m(u),\rho_n(v))}{\rho_m(u)^{\alpha_m-\tau_1}\rho_n(v)^{\alpha_n-\tau_2}} f(x-u, y-v)dudv.$$

When $h \equiv 1$, $\tau_1 = \tau_2 = 1$, and $\alpha_{m,i} = \alpha_{n,j} = 1, i = 1,\cdots,m, j = 1,\cdots,n$, the operator $\mathcal{M}_{\Omega,h}$ (denoted by $\mathcal{M}_\Omega$) is just the classical Marcinkiewicz integral on product domains, which studied extensively by many authors (see [2, 7, 9, 11, 14, 24, 25, 32, 33, 35, 36] among others). In particular, Al-Qassem et al. [2] proved that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ for $1 < p < \infty$ if $\Omega \in L\log^+ L(S^{m-1}\times S^{n-1})$. It should be pointed out that the condition $\Omega \in L\log^+ L(S^{m-1}\times S^{n-1})$ is optimal in the sense that the operator $\mathcal{M}_\Omega$ may fail L^2 boundedness if Ω is assumed to be in $L(\log^+ L)^{1-\epsilon}(S^{m-1}\times S^{n-1})$ for some $\epsilon > 0$. Afterward, Al-Salman extends the results in [2] to Marcinkiewicz integrals with mixed homogeneity. On the other hand, Hu, Lu and Yan [24] (also see Wu's work [33, 36]) obtained that $\mathcal{M}_\Omega$) is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ for $1 + \frac{1}{2\beta} < p < 2\beta$ and $\beta > \frac{1}{2}$, provided that Ω satisfies the following condition:

$$\sup_{(\xi',\eta')\in S^{m-1}\times S^{n-1}} \iint_{S^{m-1}\times S^{n-1}} |\Omega(u',v')|\{G(\xi',\eta')\}^\beta d\sigma_m(u')d\sigma_n(v') < \infty, \tag{6}$$

where

$$G(\xi',\eta') = \log\frac{1}{|\xi'\cdot u'|} + \log\frac{1}{|\eta'\cdot v'|} + \log\frac{1}{|\xi'\cdot u'|}\log\frac{1}{|\eta'\cdot v'|}.$$

For the sake of simplicity, we denote that for $\beta > 0$,

$$\mathcal{F}_\beta(S^{m-1}\times S^{n-1}) = \{\Omega \in L^1(S^{m-1}\times S^{n-1}) : \Omega \text{ satisfies } (6)\}.$$

In 2013, Liu and Wu [26] extends the result of [24] to the case: $h \equiv 1$, $\tau_1 = \tau_2 = 1$, $\alpha_{m,i} \geq 1$ and $\alpha_{n,j} \geq 1, i = 1, \cdots, m, j = 1, \cdots, n$. To study singular integral operator on product domains with rough kernels both along a radial direction and on the spherical surface, Ma, Fan and Wu [29] introduced the following size condition:

$$\sup_{(\xi',\eta')\in S^{m-1}\times S^{n-1}} \iint_{(S^{m-1}\times S^{n-1})^2} |\Omega(u',v')||\Omega(\theta,w)| \times \{G_{\xi',\eta'}(\theta,w)\}^{\beta} d\sigma_m(u')d\sigma_n(v')d\sigma_m(\theta)d\sigma_n(w) < \infty, \tag{7}$$

where

$$G_{\xi',\eta'}(\theta,w) = \log\frac{1}{|\langle u'-\theta,\xi'\rangle|}\log\frac{1}{|\langle v'-w,\eta'\rangle|} + \log\frac{1}{|\langle u'-\theta,\xi'\rangle|} + \log\frac{1}{|\langle v'-w,\eta'\rangle|}.$$

We set

$$\mathcal{WF}_\beta(S^{m-1}\times S^{n-1}) = \{\Omega \in L^1(S^{m-1}\times S^{n-1}) : \Omega \text{ satisfies (7)}\}.$$

We note that the condition (7) introduced by Ma et al. in a more general form in [29]. Employing the ideas in [22], one can check that $\mathcal{F}_\beta(S\times S) \subset \mathcal{WF}_\beta(S\times S)$ (see Proposition 2.1 in [29]). When $m > 2$ or $n > 2$, the relation between $\mathcal{F}_\beta(S^{m-1}\times S^{n-1})$ and $\mathcal{WF}_\beta(S^{m-1}\times S^{n-1})$ remains to be open.

A natural question, which arises from the above results, is the following:

Question: For the general case $\alpha_{m,i} \geq 1$ $(i = 1, \cdots, m)$ and $\alpha_{n,j} \geq 1$ $(j = 1, \cdots, n)$, determine whether the L^p boundedness of the operator $\mathcal{M}_{\Omega,h}$ holds under the condition in the form of $\Omega \in \mathcal{WF}_\beta(S^{m-1}\times S^{n-1})$ with $h \in \Delta_\gamma$ for $\gamma > 1$.

The main purpose of this paper is to settle this question. We will study a family of operators broader than $\mathcal{M}_{\Omega,h}$. More precisely, let P_{N_1} and P_{N_2} be two non-negative polynomials on $\mathbb{R}$ with $P_{N_i}(0)$=0 and $\deg(P_{N_i}) = N_i (i = 1, 2)$. For suitable functions $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$, we define the multiple singular integral operator $\mathcal{M}^{\varphi,\psi}_{\Omega,h}$ along surfaces $S(P_{N_1}(\varphi), P_{N_2}(\psi))$ by

$$\mathcal{M}^{\varphi,\psi}_{\Omega,h}(f)(x,y) = \left(\int_0^\infty\int_0^\infty |F^{P_{N_1}(\varphi),P_{N_2}(\psi)}_{s,t}(x,y)|^2\frac{ds}{s}\frac{dt}{t}\right)^{\frac12}, \tag{8}$$

where

$$F^{P_{N_1}(\varphi),P_{N_2}(\psi)}_{s,t}(x,y) := \frac{1}{s^{\tau_1}}\frac{1}{t^{\tau_2}}\int_{\rho_m(u)\leq s}\int_{\rho_n(v)\leq t}\frac{\Omega(u,v)h(\rho_m(u),\rho_n(v))}{\rho_m(u)^{\alpha_m-\tau_1}\rho_n(v)^{\alpha_n-\tau_2}}f(x - A_m^{P_{N_1}(\varphi)}(u), y - A_n^{P_{N_2}(\psi)}(v))dudv$$

and

$$S(P_{N_1}(\varphi), P_{N_2}(\psi)) := \{(A_m^{P_{N_1}(\varphi)}(u), A_n^{P_{N_2}(\psi)}(v)) : (u,v) \in \mathbb{R}^m\times\mathbb{R}^n\}.$$

Clearly, $\mathcal{M}_{\Omega,h}$ is the special case of $\mathcal{M}^{\varphi,\psi}_{\Omega,h}$ for $P_{N_i} = \varphi = \psi = 1, i = 1, 2$. It was verified in [1] that $\mathcal{M}_{\Omega,h}$ is bounded on L^p with $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$ if $\Omega \in L^q(S^{m-1}\times S^{n-1})$ and $h \in \Delta_\gamma$ for some $\gamma > 1$.

Our main results can be formulated as follows:

Theorem 1.1. *Let P_{N_1} and P_{N_2} be two real valued polynomials on $\mathbb{R}$ satisfying $P_{N_i}(0) = 0$ and $P_{N_i}(t) > 0$ for $t \neq 0$, where N_i is the degree of P_{N_i}, $i = 1, 2$. Let $\varphi, \psi \in \mathfrak{F}$, where $\mathfrak{F}$ is the set of functions ϕ satisfying the following properties:*

1. *$\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous strictly increasing and $\phi \in C^1(\mathbb{R}^+)$ satisfying that ϕ' is*

monotonous;

2. *there exist constants C_ϕ and c_ϕ such that $t\phi'(t) \geq C_\phi \phi(t)$ and $\phi(2t) \leq c_\phi \phi(t)$ for all $t > 0$.*

Suppose that $h \in \Delta_\gamma$ for some $\gamma > 1$ and $\Omega \in \mathcal{WF}_\beta(S^{m-1} \times S^{n-1})$ for some $\beta > \max\{2, \gamma'\}/2$ satisfying (3)-(4). Then $\mathcal{M}^{\varphi,\psi}_{\Omega,h}$ defined as in (8) is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $|1/p - 1/2| < min\{1/\gamma'\} - \min\{1/\gamma' + 1/2, 1\}/(\beta+1)$. Furthermore, the bound is independent of the coefficients of P_{N_1} and P_{N_2} , but depends on φ, ψ, N_1, N_2, m, n and β.

Remark 1.1. *It should be pointed out that the introduction of the class $\mathfrak{F}$ is greatly motivated by Al-Salman's works [5–7]. There are some model examples in the class $\mathfrak{F}$, such as $t^\alpha (\alpha > 0)$, $t^\alpha(\ln(1+t))^\beta(\alpha, \beta > 0), t\ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. In addition, for any $\phi \in \mathfrak{F}$, there exists a constant $B_\phi > 1$ such that $\phi(2r) > B_\phi \phi(r)$ for all $r > 0$. In [6], Al-Salman established the L^p-boundedness of the parabolic Marcinkiewicz integrals with $h(t) \equiv 1$ and $P_N(t) = t$ along surfaces defined by the functions ϕ in $\mathfrak{F}$, provided $\Omega \in \mathcal{F}_\beta(S^{n-1})$ for $\beta > 1$ and $\frac{2\beta}{2\beta-1} < p < 2\beta$ (see [7] for the multiple-parameter case). In the current paper, our theorems show that the L^p-boundedness of the operator $\mathcal{M}^{\varphi,\psi}_{\Omega,h}$, whose kernel has the additional roughness in the radial direction due to the presence of h, depends on the index γ, which characterize the roughness of h.*

Theorem 1.2. *Let P_{N_1}, P_{N_2}, φ and ψ be as in Theorem 1.1. Suppose that $h \in \Delta_\gamma$ for some $\gamma > 1$, Ω satisfies (3)-(4) with $\Omega \in \mathcal{F}_\beta(S \times S)$ for some $\beta > \max\{2, \gamma'\}/2$. Then $\mathcal{M}^{\varphi,\psi}_{\Omega,h}$ defined as in (8) is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $|1/p - 1/2| < min\{1/\gamma'\} - \min\{1/\gamma' + 1/2, 1\}/(\beta+1)$, and the bound is independent of the coefficients of P_{N_1} and P_{N_2} , but depend on φ, ψ, N_1, N_2, m, n and β.*

Obviously, Theorem 1.2 follows from Theorem 1.1 and the relation $\mathcal{F}_\beta(S \times S) \subset \mathcal{WF}_\beta(S \times S)$. Therefore, it suffices to prove Theorem 1.1.

We end this introduction with the following remarks. First of all, all of our results are new, even in the special case: $P_{N_1} = P_{N_2} = \varphi = \psi = 1$, moreover, even in the case where $\alpha_{m,i} = \alpha_{n,j} = 1, i = 1, \cdots, m, j = 1, \cdots, n$, namely, the Euclidean setting. Second, since $\bigcup_{q>1} L^q(S^{m-1} \times S^{n-1})$ is a proper subset of $\mathcal{WF}_\beta(S^{m-1} \times S^{n-1})$ for any $\beta > 0$, Theorem 1.1 gives an essential improvement of the result in [1]. However, we don't know whether the ranges of β and p in Theorems 1.1 and 1.2 are sharp, which is interesting. Third, we note that the main ingredient of Theorem 1.1 is based on the use of Fourier transform estimates and Littlewood-Paley theory which was originally introduced in [20]. Employing the ideas in [20], there are many works to investigate the L^p-boundedness of Marcinkiewicz integrals and related operators, for example, see [3, 4, 6, 7, 26, 36] and the references therein. Due to the presence of h, our methods and techniques are more delicate and complex than those of [3, 4, 6, 7, 26, 36].

This paper is organized as follows. In Section 2, we will introduce some notation and give some preliminary lemmas. Section 3 is devoted to proving Theorem 1.1.

Finally, we make some conventions. Throughout this paper, we let p' denote the conjugate index of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. The letter C, sometimes with additional parameters, stands for a positive constant which is independent of the essential variables, but whose value may vary from line to line.

§2 Some notations and preliminary lemmas

For given positive polynomials $P_{N_1}(t)=\sum_{i=1}^{N_1}\beta_i t^i$, $P_{N_2}(t)=\sum_{j=1}^{N_2}\gamma_j t^j$, and for $l\in\{1,2,\cdots,m\}$, $k\in\{1,2,\cdots,n\}$, we denote $(P_{N_1}(t))^{\alpha_{m,l}}=\sum_{i=1}^{N_1\alpha_{m,l}}a_{i,l}t^i$ and $(P_{N_2}(t))^{\alpha_{n,k}}=\sum_{j=1}^{N_2\alpha_{n,k}}b_{j,k}t^j$. Then for $x,\xi\in\mathbb{R}^m$, $y,\eta\in\mathbb{R}^n$ and $\varphi,\psi\in\mathfrak{F}$, we write

$$A_m^{P_{N_1}(\varphi)}(x)\cdot\xi=\sum_{l=1}^{m}P_{N_1}(\varphi(\rho_m(x)))^{\alpha_{m,l}}x_l'\cdot\xi_l=\sum_{l=1}^{m}\sum_{i=1}^{N_1\alpha_{m,l}}a_{i,l}\varphi(\rho_m(x))^i x_l'\cdot\xi_l,$$

$$A_n^{P_{N_2}(\psi)}(y)\cdot\eta=\sum_{k=1}^{n}P_{N_2}(\psi(\rho_n(y)))^{\alpha_{n,k}}y_k'\cdot\eta_k=\sum_{k=1}^{n}\sum_{j=1}^{N_2\alpha_{n,k}}b_{j,k}\psi(\rho_n(y))^j y_k'\cdot\eta_k.$$

Let $\mathcal{N}_1=\max\{N_1\alpha_{m,l}:1\le l\le m\}$, $\mathcal{N}_2=\max\{N_2\alpha_{n,k}:1\le k\le n\}$, let $a_{i,l}=0$ when $i>N_1\alpha_{m,l}$, $b_{j,k}=0$ when $j>N_2\alpha_{n,k}$. Thus

$$A_m^{P_{N_1}(\varphi)}(x)\cdot\xi=\sum_{l=1}^{m}\sum_{i=1}^{N_1\alpha_{m,l}}a_{i,l}\varphi(\rho_m(x))^i x_l'\cdot\xi_l=\sum_{i=1}^{\mathcal{N}_1}(L_i(\xi)\cdot y')\varphi((\rho_m(x)))^i,$$

where $L_i(\xi)=(a_{i,1}\xi_1,\cdots,a_{i,m}\xi_m)$. Similarly,

$$A_n^{P_{N_2}(\psi)}(y)\cdot\eta=\sum_{k=1}^{n}\sum_{j=1}^{N_2\alpha_{n,k}}b_{j,k}\psi(\rho_n(y))^j y_k'\cdot\eta_k=\sum_{j=1}^{\mathcal{N}_2}(I_j(\eta)\cdot y')\psi((\rho_n(y)))^j,$$

where $I_j(\eta)=(b_{j,1}\eta_1,\cdots,b_{j,j}\eta_n)$. For any $\mu\in\{0,1,\cdots,\mathcal{N}_1\}$ and $\nu\in\{0,1,\cdots,\mathcal{N}_2\}$, we set

$$Q_\mu(x)=\Big(\sum_{i=1}^{\mu}a_{i,1}x_1'\varphi(\rho_m(x))^i,\cdots,\sum_{i=1}^{\mu}a_{i,m}x_m'\varphi(\rho_m(x))^i\Big),$$

$$R_\nu(y)=\Big(\sum_{i=1}^{\nu}b_{j,1}y_1'\psi(\rho_n(y))^j,\cdots,\sum_{j=1}^{\nu}b_{j,n}y_n'\psi(\rho_n(y))^j\Big).$$

Then

$$Q_\mu(x)\cdot\xi=\sum_{i=1}^{\mu}(L_i(\xi)\cdot x')\varphi(\rho_m(x))^i,0\le\mu\le\mathcal{N}_1,$$

$$R_\nu(y)\cdot\eta=\sum_{j=1}^{\nu}(I_j(\eta)\cdot y')\psi(\rho_n(y))^j,0\le\mu\le\mathcal{N}_2.$$

For $i,j\in\mathbb{Z},s,t\in\mathbb{R}^+$ and $0\le\mu\le\mathcal{N}_1,0\le\nu\le\mathcal{N}_2$, we define the measures $\{\sigma_{i,j;s,t}^{\mu,\nu}\}$ and $\{|\sigma_{i,j;s,t}^{\mu,\nu}|\}$ by

$$\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)=\frac{1}{(2^is)^{\tau_1}(2^jt)^{\tau_2}}\iint_{\Delta_{i,j}^{s,t}}\frac{\Omega(x,y)h(\rho_m(x),\rho_n(y))}{\rho_m(x)^{\alpha_m-\tau_1}\rho_n(y)^{\alpha_n-\tau_2}}e^{-i(Q_\mu(x)\cdot\xi+R_\nu(y)\cdot\eta)}dxdy,$$

$$|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}|(\xi,\eta)=\frac{1}{(2^is)^{\tau_1}(2^jt)^{\tau_2}}\iint_{\Delta_{i,j}^{s,t}}\frac{|\Omega(x,y)||h(\rho_m(x),\rho_n(y))|}{\rho_m(x)^{\alpha_m-\tau_1}\rho_n(y)^{\alpha_n-\tau_2}}e^{-i(Q_\mu(x)\cdot\xi+R_\nu(y)\cdot\eta)}dxdy,$$

where $\Delta_{i,j}^{s,t}=\{2^{i-1}s<\rho_m(x)\le 2^is,2^{j-1}t<\rho_n(y)\le 2^jt\}$. One can easily check that for each $\mu\in\{0,1,\cdots,\mathcal{N}_1\}$ and $\nu\in\{0,1,\cdots,\mathcal{N}_2\}$,

$$\widehat{\sigma_{i,j;s,t}^{0,\nu}}(\xi,\eta)=\widehat{\sigma_{i,j;s,t}^{\mu,0}}(\xi,\eta)=0$$

and

$$F_{s,t}^{P_{N_1}(\varphi),P_{N_1}(\psi)}(x,y)=s^{\tau_1}t^{\tau_2}\sum_{i,j=-\infty}^{0}2^{i\tau_1}2^{j\tau_2}\sigma_{i,j;s,t}^{\mu,\nu}*f(x,y). \tag{9}$$

First we need the following estimate, which plays curial role in the proof of our main result.

Lemma 2.1. *([25], Lemma 2.5) Let $\varphi,\psi\in\mathcal{F}$. Then for $\mu\in\{1,\cdots,\mathcal{N}_1\}$, $\nu\in\{1,\cdots,\mathcal{N}_2\}$ and $r>0$,*

$$|\int_{r/2}^{r}e^{-i\sum_{i=1}^{\mu}L_i(\xi)\cdot x'\varphi(\rho_m)^i}\frac{d\rho_m}{\rho_m}|\le C|\varphi(r)^{\mu}L_{\mu}(\xi)\cdot x'|^{-1/\mu},$$
$$|\int_{r/2}^{r}e^{-i\sum_{j=1}^{\nu}L_j(\eta)\cdot y'\psi(\rho_n)^j}\frac{d\rho_n}{\rho_n}|\le C|\psi(r)^{\nu}L_{\nu}(\eta)\cdot y'|^{-1/\nu}.$$

Lemma 2.2. *Let $\varphi,\psi\in\mathcal{F}$, let $h\in\Delta_\gamma(\mathbb{R}^+)$ for some $1<\gamma\le\infty$ and $\tilde\gamma=\max\{2,\gamma'\}$. Suppose that $\Omega\in\mathcal{WF}_\beta(S^{m-1}\times S^{n-1})$ for some $\beta>0$ and satisfies (1). Then for $\mu\in\{1,\cdots,\mathcal{N}_1\}$, $\nu\in\{1,\cdots,\mathcal{N}_2\}$, there exists a constant $C>0$ such that*

(i)

$$|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{i,j;s,t}^{\mu-1,\nu}}(\xi,\eta)|\le C|\varphi(2^is)^{\mu}L_{\mu}(\xi)|\min\left\{1,(\ln|\psi(2^jt)I_{\nu}(\eta)|)^{-\frac{\beta}{\tilde\gamma}}\right\}; \tag{10}$$

(ii)

$$|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{i,j;s,t}^{\mu,\nu-1}}(\xi,\eta)|\le C|\psi(2^jt)^{\mu}I_{\nu}(\eta)|\min\left\{1,(\ln|\varphi(2^is)L_{\mu}(\xi)|)^{-\frac{\beta}{\tilde\gamma}}\right\}; \tag{11}$$

(iii)

$$\begin{aligned}|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)|\le C\min\Big\{&1,(\ln|\varphi(2^is)L_{\mu}(\xi)|)^{-\frac{\beta}{\tilde\gamma}},(\ln|\psi(2^jt)I_{\nu}(\eta)|)^{-\frac{\beta}{\tilde\gamma}},\\&(\ln|\varphi(2^is)L_{\mu}(\xi)|)^{-\frac{\beta}{\tilde\gamma}}\cdot(\ln|\psi(2^jt)I_{\nu}(\eta)|)^{-\frac{\beta}{\tilde\gamma}}\Big\};\end{aligned} \tag{12}$$

(iv)

$$\begin{aligned}&|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{i,j;s,t}^{\mu-1,\nu}}(\xi,\eta)-\widehat{\sigma_{i,j;s,t}^{\mu,\nu-1}}(\xi,\eta)+\widehat{\sigma_{i,j;s,t}^{\mu-1,\nu-1}}(\xi,\eta)|\\&\le C\min\left\{1,|\varphi(2^is)^{\mu}L_{\mu}(\xi)|,|\psi(2^jt)^{\mu}I_{\nu}(\eta)|,|\varphi(2^is)^{\mu}L_{\mu}(\xi)|\cdot|\psi(2^jt)^{\mu}I_{\nu}(\eta)|\right\}.\end{aligned} \tag{13}$$

Proof. By a change of variable, we have

$$\begin{aligned}&\left|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{i,j;s,t}^{\mu-1,\nu}}(\xi,\eta)\right|\\&\le\frac{1}{(2^is)^{\tau_1}(2^jt)^{\tau_2}}\iint_{\Delta_{i,j}^{s,t}}|e^{-iQ_{\mu}(x)\cdot\xi}-e^{-iQ_{\mu-1}(x)\cdot\xi}|\frac{|\Omega(x,y)h(\rho_m(x),\rho_n(y))|}{\rho_m(x)^{\alpha_m-\tau_1}\rho_n(y)^{\alpha_n-\tau_2}}dxdy\\&\le C|\varphi(2^is)^{\mu}L_{\mu}(\xi)|\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}h(r_1,r_2)\\&\quad\times\frac{dr_1dr_2}{r_1r_2}\iint_{S^{m-1}\times S^{n-1}}|\Omega(\theta_1,\theta_2)|J_m(\theta_1)J_n(\theta_2)d\sigma_m(\theta_1)d\sigma_n(\theta_2)\\&\le C|\varphi(2^is)^{\mu}L_{\mu}(\xi)|.\end{aligned}$$

On the other hand, by the Hölder's inequality, we have

$$\begin{aligned}
&\big|\widehat{\sigma^{\mu,\nu}_{i,j;s,t}}(\xi,\eta)-\widehat{\sigma^{\mu-1,\nu}_{i,j;s,t}}(\xi,\eta)\big|\\
&=\Big|\frac{1}{(2^is)^{\tau_1}(2^jt)^{\tau_2}}\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\iint_{S^{m-1}\times S^{n-1}}e^{-i\sum_{j=1}^{\nu}(I_j(\eta)\cdot\theta_2)\psi(r_2)^j}[e^{-i\sum_{i=1}^{\mu}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}\\
&\quad-e^{-i\sum_{i=1}^{\mu-1}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}]\Omega(\theta_1,\theta_2)J_m(\theta_1)J_n(\theta_2)d\sigma_m(\theta_1)d\sigma_n(\theta_2)h(r_1,r_2)\frac{dr_1dr_2}{r_1^{1-\tau_1}r_2^{1-\tau_2}}\Big|\\
&\le C\Big(\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\Big|\iint_{S^{m-1}\times S^{n-1}}e^{-i\sum_{j=1}^{\nu}(I_j(\eta)\cdot\theta_2)\psi(r_2)^j}[e^{-i\sum_{i=1}^{\mu}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}\\
&\quad-e^{-i\sum_{i=1}^{\mu-1}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}]\Omega(\theta_1,\theta_2)J_m(\theta_1)J_n(\theta_2)d\sigma_m(\theta_1)d\sigma_n(\theta_2)\Big|^{\gamma'}\frac{dr_1dr_2}{r_1r_2}\Big)^{\frac{1}{\gamma'}}\\
&\le C|\varphi(2^is)^\mu L_\mu(\xi)|^{\max\{1-\frac{2}{\gamma'},0\}}\Big(\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\Big|\iint_{S^{m-1}\times S^{n-1}}e^{-i\sum_{j=1}^{\nu}(I_j(\eta)\cdot\theta_2)\psi(r_2)^j}\\
&\quad\times[e^{-i\sum_{i=1}^{\mu}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}-e^{-i\sum_{i=1}^{\mu-1}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}]\\
&\quad\times\Omega(\theta_1,\theta_2)J_m(\theta_1)J_n(\theta_2)d\sigma_m(\theta_1)d\sigma_n(\theta_2)\Big|^2\frac{dr_1dr_2}{r_1r_2}\Big)^{\frac{1}{\gamma}}.
\end{aligned}\tag{14}$$

Let

$$\begin{aligned}
R^{\mu,\nu}_{i,j;s,t}(\xi,\eta):=&\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\Big|\iint_{S^{m-1}\times S^{n-1}}e^{-i\sum_{j=1}^{\nu}(I_j(\eta)\cdot\theta_2)\psi(r_2)^j}[e^{-i\sum_{i=1}^{\mu}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}\\
&-e^{-i\sum_{i=1}^{\mu-1}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}]\Omega(\theta_1,\theta_2)J_m(\theta_1)J_n(\theta_2)d\sigma_m(\theta_1)d\sigma_n(\theta_2)\Big|^2\frac{dr_1dr_2}{r_1r_2}
\end{aligned}$$

Then we can write

$$\begin{aligned}
|R^{\mu,\nu}_{i,j;s,t}(\xi,\eta)|=&\Big|\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\iint_{(S^{m-1}\times S^{n-1})^2}[e^{-iL_\mu(\xi)\cdot\theta_1\varphi(r_1)^\mu}-1][e^{iL_\mu(\xi)\cdot w_1\varphi(r_1)^\mu}-1]\\
&\times e^{-i\sum_{i=1}^{\mu-1}(L_i(\xi)\cdot(\theta_1-w_1))\varphi(r_1)^i}e^{-i\sum_{j=1}^{\nu}(I_j(\eta)\cdot(\theta_2-w_2)\psi(r_2)^j)}\Omega(\theta_1,\theta_2)\overline{\Omega(w_1,w_2)}\\
&\times J_m(\theta_1)J_n(\theta_2)\overline{J_m(w_1)J_n(w_2)}d\sigma_m(\theta_1)d\sigma_n(\theta_2)d\sigma_m(w_1)d\sigma_n(w_2)\frac{dr_1dr_2}{r_1r_2}\Big|.
\end{aligned}\tag{15}$$

Let

$$L_{j,\nu}(\theta_2,w_2,\eta)=\int_{2^{j-1}t}^{2^jt}e^{-i\sum_{j=1}^{\nu}I_j(\eta)\cdot(\theta_2-w_2)\psi(r_2)^j}\frac{dr_2}{r_2}.$$

Applying Lemma 2.1, we have

$$|L_{j,\nu}(\theta_2,w_2,\eta)|\le C\min\{1,|\psi(2^jt)^\nu L_\nu(\eta)\cdot(\theta_2-w_2)|^{-1/\nu}\}.$$

Since $\frac{t}{(\ln t)^\beta}$ is increasing in (e^β,∞), it must satisfy the estimate

$$|L_{j,\nu}(\theta_2,w_2,\eta)|\le C\frac{(\ln|(L_\nu(\eta))'\cdot(\theta_2-w_2)|^{-1}|)^\beta}{(\ln|\psi(2^jt)^\nu L_\nu(\eta)|)^\beta}.$$

Combining (14)-(15) with the fact that $\Omega\in\mathcal{WF}_\beta(S^{m-1}\times S^{n-1})$ yields (10). Similarly, (11) holds. Next we return to prove (iii). By a change of variable and Hölder's inequality, we have

$$
\begin{aligned}
|\widehat{\sigma^{\mu,\nu}_{i,j;s,t}}(\xi,\eta)| = &\Big|\frac{1}{(2^is)^{\tau_1}(2^jt)^{\tau_2}}\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\iint_{S^{m-1}\times S^{n-1}} e^{-i\sum_{i=1}^{\mu}(L_i(\xi)\cdot\theta)\varphi(r)^i}\\
&\times e^{-i\sum_{j=1}^{\nu}(I_j(\eta)\cdot\theta_2)\psi(r_2)^j}\times\Omega(\theta_1,\theta_2)J(\theta_1,\theta_2)d\sigma_1(\theta_1)d\sigma_2(\theta_2)h(r_1,r_2)\frac{dr_1dr_2}{r_1^{1-\tau_1}r_2^{1-\tau_2}}\Big|\\
\le &C\|h\|_{\Delta_\gamma}\Big(\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\Big|\iint_{S^{m-1}\times S^{n-1}} e^{-i\sum_{i=1}^{\mu}(L_i(\xi)\cdot\theta)\varphi(r)^i}e^{-i\sum_{j=1}^{\nu}(I_j(\eta)\cdot\theta_2)\psi(r_2)^j}\\
&\times\Omega(\theta_1,\theta_2)J(\theta_1,\theta_2)d\sigma_1(\theta_1)d\sigma_2(\theta_2)\Big|^{\gamma'}\frac{dr_1dr_2}{r_1r_2}\Big)^{\frac{1}{\gamma'}}\\
\le &C\Big(\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\Big|\iint_{S^{m-1}\times S^{n-1}} e^{-i\sum_{i=1}^{\mu}(L_i(\xi)\cdot\theta)\varphi(r)^i}e^{-i\sum_{j=1}^{\nu}(I_j(\eta)\cdot\theta_2)\psi(r_2)^j}\\
&\times\Omega(\theta_1,\theta_2)J(\theta_1,\theta_2)d\sigma_1(\theta_1)d\sigma_2(\theta_2)\Big|^{2}\frac{dr_1dr_2}{r_1r_2}\Big)^{\frac{1}{\tilde\gamma}}
\end{aligned}
$$

Repeating the same argument as in (i), we get (12). (13) follows from the inequality

$$
\begin{aligned}
&|\widehat{\sigma^{\mu,\nu}_{i,j;s,t}}(\xi,\eta)-\widehat{\sigma^{\mu-1,\nu}_{i,j;s,t}}(\xi,\eta)-\widehat{\sigma^{\mu,\nu-1}_{i,j;s,t}}(\xi,\eta)+\widehat{\sigma^{\mu-1,\nu-1}_{i,j;s,t}}(\xi,\eta)|\\
\le&\frac{1}{(2^is)^{\tau_1}(2^jt)^{\tau_2}}\iint_{S^{m-1}\times S^{n-1}}\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\big|e^{-i\sum_{i=1}^{\mu}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}-e^{-i\sum_{i=1}^{\mu-1}(L_i(\xi)\cdot\theta_1)\varphi(r_1)^i}\big|\\
&\times\big|e^{-i\sum_{j=1}^{\nu}(I_j(\eta)\cdot\theta_2)\psi(r_2)^j}-e^{-i\sum_{j=1}^{\nu-1}(I_j(\eta)\cdot\theta_2)\psi(r_2)^j}\big|\big|\Omega(\theta_1,\theta_2)J_m(\theta_1)J_n(\theta_2)\big|\\
&\times|h(r_1,r_2)|\frac{dr_1dr_2}{r_1^{1-\tau_1}r_2^{1-\tau_2}}d\sigma_m(\theta_1)d\sigma_n(\theta_2).
\end{aligned}
$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $\Omega\in L^1(S^{m-1}\times S^{n-1})$ satisfy (3) and (4). Suppose that $h\in\Delta_\gamma$ for some $\gamma>1$ and $\varphi,\psi\in\mathcal{F}$, then for any $\mu\in\{1,\cdots,\mathcal{N}_1\}$ and any $\nu\in\{1,\cdots,\mathcal{N}_2\}$, the maximal operator defined by*

$$\sigma^*_{\mu,\nu}(f)(x,y)=\sup_{i,j\in\mathbb{Z}}\sup_{s,t>0}\big||\sigma^{\mu,\nu}_{i,j;s,t}|*f(x,y)\big|$$

is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ for $\gamma'<p\le\infty$.

Proof. We first define the measures $\{\Lambda^{\mu,\nu}_{i,j;s,t}\}$ and maximal operator $\{\Lambda^{\mu,\nu}\}$ as

$$|\widehat{\Lambda^{\mu,\nu}_{i,j;s,t}}|(\xi,\eta)=\frac{1}{(2^is)^{\tau_1}(2^jt)^{\tau_2}}\iint_{\Delta^{s,t}_{i,j}}\frac{|\Omega(x,y)|}{\rho_m(x)^{\alpha_m-\tau_1}\rho_n(y)^{\alpha_n-\tau_2}}e^{-i(Q_\mu(x)\cdot\xi+R_\nu(y)\cdot\eta)}dxdy$$

and

$$\Lambda^*_{\mu,\nu}(f)(x,y)=\sup_{i,j\in\mathbb{Z}}\sup_{s,t>0}\big||\Lambda^{\mu,\nu}_{i,j;s,t}|*f(x,y)\big|$$

.

By a change of variable, we have

$$\Lambda^*_{\mu,\nu}(f)(x,y)\le C\sup_{i,j\in\mathbb{Z}}\int_{2^{i-1}s}^{2^is}\int_{2^{j-1}t}^{2^jt}\iint_{S^{m-1}\times S^{n-1}}|f(x-Q_\mu(A_{m,r_1}u'),y-R_\nu(A_{n,r_2}v'))|$$

$$\times |\Omega(u',\theta_2)| J_m(u') J_n(v') d\sigma_m(\theta_1) d\sigma_n(\theta_2) \frac{dr_1 dr_2}{r_1 r_2}$$

$$\leq C \iint_{S^{m-1}\times S^{n-1}} \sup_{i,j\in\mathbb{Z}} \int_{2^{i-1}s}^{2^i s} \int_{2^{j-1}t}^{2^j t} |f(x - Q_\mu(A_{m,r_1}u'), y - R_\nu(A_{n,r_2}v'))| \frac{dr_1 dr_2}{r_1 r_2}$$

$$\times |\Omega(u',\theta_2)| J_m(u') J_n(v') d\sigma_m(\theta_1) d\sigma_n(\theta_2).$$

Following the same arguments as those in the proof of Lemma 5 in [28], we obtain that

$$\|\Lambda^*_{\mu,\nu}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)}, 1 < p < \infty. \tag{16}$$

By Hölder's inequality,

$$|\sigma^*_{\mu,\nu}(f)(x,y)| \leq C\big(\Lambda^*_{\mu,\nu}(|f|^{\gamma'})(x,y)\big)^{\frac{1}{\gamma'}}.$$

Plugging this estimate into (16) yields Lemma 2.3. □

Now we take two radial functions $\phi_1 \in C_0^\infty(\mathbb{R}^m)$ and $\phi_2 \in C_0^\infty(\mathbb{R}^n)$ such that $\phi_i(t) = 1$ for $|t| \leq 1$ and $\phi_i(t) = 0$ for $|t| > \min\{B_\varphi, B_\psi\}, i = 1, 2$, where B_φ, B_ψ are as in Remark 1.1. We define the measures $\{w^{\mu,\nu}_{i,j;s,t}\}$ as

$$\widehat{w^{\mu,\nu}_{i,j;s,t}}(\xi,\eta) = \widehat{\sigma^{\mu,\nu}_{i,j;s,t}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu) - \widehat{\sigma^{\mu-1,\nu}_{i,j;s,t}}(\xi,\eta)\Pi_1(\mu-1)\Pi_2(\nu)$$
$$- \widehat{\sigma^{\mu,\nu-1}_{i,j;s,t}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu-1) + \widehat{\sigma^{\mu-1,\nu-1}_{i,j;s,t}}(\xi,\eta)\Pi_1(\mu-1)\Pi_2(\nu-1),$$

where $\Pi_1(\mu) = \Pi_{\kappa=\mu+1}^{\mathcal{N}_1}\phi_1(\varphi(2^i s)^\kappa L_\kappa(\xi))$, $\Pi_2(\nu) = \Pi_{\ell=\nu+1}^{\mathcal{N}_2}\phi_2(\psi(2^j t)^\ell I_\ell(\eta))$. Here we use the convention $\Pi_{j\in\varnothing} a_j = 1$. Observe that

$$\sigma^{\mathcal{N}_1,\mathcal{N}_2}_{i,j;s,t} = \sum_{\mu=1}^{\mathcal{N}_1}\sum_{\nu=1}^{\mathcal{N}_2} w^{\mu,\nu}_{i,j;s,t}. \tag{17}$$

Applying Lemma 2.2 and the same arguments as in the proof of Lemma 2.7 in [25], we get

Lemma 2.4. *For $\mu = 1, 2, \cdots, \mathcal{N}_1$ and $\nu = 1, 2, \cdots, \mathcal{N}_2$,*

(i) $|\widehat{w^{\mu,\nu}_{i,j;s,t}}(\xi,\eta)| \leq C|\varphi(2^i s)^\mu L_\mu(\xi)||\psi(2^j t)^\mu I_\nu(\eta)|$;

(ii) if $|\varphi(2^i s)^\mu L_\mu(\xi)| > B_\varphi$, *then*

$$|\widehat{w^{\mu,\nu}_{i,j;s,t}}(\xi,\eta)| \leq C(\ln|\varphi(2^i s)L_\mu(\xi)|)^{-\frac{\beta}{\gamma}}|\psi(2^j t)^\mu I_\nu(\eta)|;$$

(iii) if $|\psi(2^j t)^\mu I\nu(\xi)| > B_\psi$, *then*

$$|\widehat{w^{\mu,\nu}_{i,j;s,t}}(\xi,\eta)| \leq C|\varphi(2^i s)^\mu L_\mu(\xi)|(\ln|\psi(2^j t)I_\nu(\eta)|)^{-\frac{\beta}{\gamma}};$$

(iv) if $|\varphi(2^i s)^\mu L_\mu(\xi)| > B_\varphi$ *and* $|\psi(2^j t)^\mu I\nu(\xi)| > B_\psi$, *then*

$$|\widehat{w^{\mu,\nu}_{i,j;s,t}}(\xi,\eta)| \leq C(\ln|\varphi(2^i s)L_\mu(\xi)|)^{-\frac{\beta}{\gamma}}(\ln|\psi(2^j t)I_\nu(\eta)|)^{-\frac{\beta}{\gamma}}.$$

From the definition of $w^{\mu,\nu}_{i,j;s,t}$ and Lemma 2.3, we get

$$\big\|\sup_{i,j\in\mathbb{Z}}\sup_{s,t>0}|w^{\mu,\nu}_{i,j;s,t} * f|\big\|_p \leq C\|f\|_p \tag{18}$$

for $p \in (\gamma', \infty)$.

Lemma 2.5. *For arbitrary functions $\{g_{j,k}\}$,*

$$\Big\|\big(\sum_{i,j\in\mathbb{Z}}\int_1^2\int_1^2 |w^{\mu,\nu}_{i,j;s,t} * g_{i,j}|^2 ds dt\big)^{\frac{1}{2}}\Big\|_p \leq C\Big\|\big(\sum_{i,j\in\mathbb{Z}}|g_{i,j}|^2\big)^{\frac{1}{2}}\Big\|_p$$

and

$$\Big\|\Big(\sum_{i,j\in\mathbb{Z}}|w^{\mu,\nu}_{i,j;s,t}*g_{i,j}|^2dsdt\Big)^{\frac{1}{2}}\Big\|_p\le C\Big\|\Big(\sum_{i,j\in\mathbb{Z}}|g_{i,j}|^2\Big)^{\frac{1}{2}}\Big\|_p$$

for $|\frac{1}{p}-\frac{1}{2}|<\frac{1}{\gamma'}$, *where the constant is independent of the coefficients of* P_{N_1} *and* P_{N_2}.

Applying (18), the proof of Lemma 2.5 follows from the arguments similar to the proof of Lemma 4 in [27]. Here, we omit it.

§3 Proof of Theorem 1.1

Take two collections of C^∞ functions $\{\Phi_k\}_{k\in\mathbb{Z}}$ and $\{\Psi_l\}_{\ell\in\mathbb{Z}}$ such that

(i)supp $\Phi_k\subset[\varphi(2^{k+1})^{-\mu},\varphi(2^{k-1})^{-\mu}]$, supp $\Psi_l\subset[\psi(2^{l+1})^{-\nu},\psi(2^{l-1})^{-\nu}]$;

(ii) $0\le\Phi_k,\Psi_l\le1$, $\sum_{k\in\mathbb{Z}}\Phi_k^2(t)=\sum_{l\in\mathbb{Z}}\Psi_l^2(t)=1$;

(iii) $|d\Phi_k(t)/dt|\le C/t, |d\Psi_l(t)/dt|\le C/t$.

For $k,l\in\mathbb{Z}$, we define the multiplier operator $S_{k,l}$ on $\mathbb{R}^m\times\mathbb{R}^n$ by

$$\widehat{S_{k,l}f}(\xi,\eta)=\Phi_k(|L_\mu(\xi)|)\Psi_l(|I_\nu(\eta)|)\hat{f}(\xi,\eta).$$

By taking Fourier transform, it is easy to see that for any test function f,

$$f(x,y)=\sum_{k,l}S^2_{k,l}(f)(x,y). \tag{19}$$

To show Theorem 1.1, we first consider the mapping $\mathcal{G}$ defined by

$$\mathcal{G}:\{g^{s,t}_{i,j;k,l}(x,y)\}_{i,j\in\mathbb{Z},k,l\in\mathbb{Z}}\to\Big\{\sum_{k,l\in\mathbb{Z}}S_{i+k,j+l}(g^{s,t}_{i,j;k,l})(x,y)\Big\}_{i,j\in\mathbb{Z}}.$$

Then we have the following result.

Lemma 3.1. *(i) For* $1<p<2$ *and* $1<q<p$,

$$\Big\|\Big(\sum_{i,j\in\mathbb{Z}}\int_1^2\int_1^2|\sum_{k,l\in\mathbb{Z}}S_{i+k,j+l}(g^{s,t}_{i,j;k,l})|^2dsdt\Big)^{\frac{1}{2}}\Big\|_p^q\le C\sum_{k,l\in\mathbb{Z}}\Big\|\Big(\sum_{i,j\in\mathbb{Z}}\int_1^2\int_1^2|g^{s,t}_{i,j;k,l}|^2dsdt\Big)^{\frac{1}{2}}\Big\|_p^q. \tag{20}$$

(i) For $2<p<\infty$ *and* $1<q<p'$,

$$\Big\|\Big(\sum_{i,j\in\mathbb{Z}}\int_1^2\int_1^2|\sum_{k,l\in\mathbb{Z}}S_{i+k,j+l}(g^{s,t}_{i,j;k,l})|^2dsdt\Big)^{\frac{1}{2}}\Big\|_p^q$$
$$\le C\sum_{k,l\in\mathbb{Z}}\left(\int_1^2\int_1^2\|\Big(\sum_{i,j\in\mathbb{Z}}|g^{s,t}_{i,j;k,l}|^2\Big)^{\frac{1}{2}}\|_p^2dsdt\right)^{\frac{q}{2}}. \tag{21}$$

By the arguments similar to those used in [24], one can easily establish the above lemma. The details are omitted.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By (9), (17) and Minkowski's inequality, it follows that

$$\begin{aligned}\mathcal{M}^{\varphi,\psi}_{\Omega,h}(f)(x,y) &= \Big(\int_0^\infty\int_0^\infty\Big|\sum_{i,j=-\infty}^{0}2^{i\tau_1}2^{j\tau_2}\sigma^{\mu,\nu}_{i,j;s,t}*f(x,y)\Big|^2\frac{dsdt}{st}\Big)^{\frac12}\\ &\le \sum_{i,j=-\infty}^{0}2^{i\tau_1}2^{j\tau_2}\Big(\int_0^\infty\int_0^\infty|\sigma^{\mu,\nu}_{i,j;s,t}*f(x,y)|^2\frac{dsdt}{st}\Big)^{\frac12}\\ &= \sum_{i,j=-\infty}^{0}2^{i\tau_1}2^{j\tau_2}\Big(\int_0^\infty\int_0^\infty|\sigma^{\mu,\nu}_{0,0;s,t}*f(x,y)|^2\frac{dsdt}{st}\Big)^{\frac12}\\ &\le C\Big(\int_1^2\int_1^2\sum_{k,l\in\mathbb{Z}}|\sigma^{\mu,\nu}_{k,l;s,t}*f(x,y)|^2\frac{dsdt}{st}\Big)^{\frac12}\\ &\le C\sum_{\mu}^{\mathcal{N}_1}\sum_{\nu}^{\mathcal{N}_2}\Big(\int_1^2\int_1^2\sum_{k,l\in\mathbb{Z}}|w^{\mu,\nu}_{k,l;s,t}*f(x,y)|^2dsdt\Big)^{\frac12}.\end{aligned}$$

Thus, it suffices to consider the $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ boundedness for the operator

$$\tilde{\mathcal{M}}^{\varphi,\psi}_{\Omega,h}(f)(x,y) = \Big(\int_1^2\int_1^2\sum_{k,l\in\mathbb{Z}}|w^{\mu,\nu}_{k,l;s,t}*f(x,y)|^2dsdt\Big)^{\frac12}.$$

By (19), one can write

$$\tilde{\mathcal{M}}^{\varphi,\psi}_{\Omega,h}(f)(x,y) = \Big(\sum_{k,l\in\mathbb{Z}}\int_1^2\int_1^2\Big|\sum_{i,j\in\mathbb{Z}}S_{i+k,j+l}(w^{\mu,\nu}_{k,l;s,t}*S_{i+k,j+l}f(x,y))\Big|^2dsdt\Big)^{\frac12}.$$

We now establish the $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ boundedness for $\tilde{\mathcal{M}}^{\varphi,\psi}_{\Omega,h}$. We consider two cases.

Case 1: $\frac{2\tilde\gamma(\beta+1)}{(\tilde\gamma+2)\beta}<p<2$.

By (20), we have that, for any $1<q<p$,

$$\|\tilde{\mathcal{M}}^{\varphi,\psi}_{\Omega,h}(f)\|_p^q \le C\sum_{i,j\in\mathbb{Z}}\Big\|\Big(\sum_{k,l\in\mathbb{Z}}\int_1^2\int_1^2|w^{\mu,\nu}_{k,l;s,t}*S_{i+k,j+l}f|^2dsdt\Big)^{\frac12}\Big\|_p^q. \tag{22}$$

For every fixed $i,j\in\mathbb{Z}$, define the operator

$$U_{i,j}f(x,y) = \Big(\sum_{k,l\in\mathbb{Z}}\int_1^2\int_1^2|w^{\mu,\nu}_{k,l;s,t}*S_{i+k,j+l}f|^2dsdt\Big)^{\frac12}.$$

By Plancherel's theorem, we have

$$\begin{aligned}\|U_{i,j}f\|_2^2 &= \int_1^2\int_1^2\sum_{k,l\in\mathbb{Z}}\iint_{\mathbb{R}^m\times\mathbb{R}^n}|\widehat{w^{\mu,\nu}_{k,l;s,t}}(\xi,\eta)|^2|\Phi_{i+k}(|L_\mu(\xi)|)|^2|\Psi_{j+l}(|I_\nu(\eta)|)|^2\\ &\quad\times|\hat f(\xi,\eta)|^2d\xi d\eta dsdt\\ &\le C\sum_{k,l\in\mathbb{Z}}\iint_{E_{i+k,j+l}}|\hat f(\xi,\eta)|^2\int_1^2\int_1^2|\widehat{w^{\mu,\nu}_{k,l;s,t}}(\xi,\eta)|^2dsdtd\xi d\eta,\end{aligned}$$

where $E_{i+k,j+l}=\{(\xi,\eta)\in\mathbb{R}^m\times\mathbb{R}^n:\varphi(2^{i+k+1})^{-\mu}\le L_\mu(\xi)\le\varphi(2^{i+k-1})^{-\mu},\psi(2^{i+l+1})^{-\nu}\le L_\nu(\eta)\le\varphi(2^{j+l-1})^{-\nu}\}$. Then by Lemma 2.4 and Remark 1.1, we have that

$$\|U_{i,j}f\|_2\le CB_{ij}\|f\|_{L^2}, \tag{23}$$

where

$$B_{ij}=\begin{cases} B_{\varphi}^{-i\mu}B_{\psi}^{-j\nu}, i,j>-2;\\ |i|^{-\beta/\tilde{\gamma}}B_{\psi}^{-j\nu}, i\le -2, j>-2;\\ B_{\varphi}^{-i\mu}|j|^{-\beta/\tilde{\gamma}}, i>-2, j\le -2;\\ |i|^{-\beta/\tilde{\gamma}}|j|^{-\beta/\tilde{\gamma}}, i,j\le -2.\end{cases}$$

On the other hand, applying Lemma 2.5 and the Littlewood-Paley theory, we get

$$\|U_{i,j}f\|_p\le C\|(\sum_{k,l\in\mathbb{Z}}|S_{i+k,j+l}f|^2)^{\frac12}\|_p\le C\|f\|_p,\quad |\frac1p-\frac12|<\frac{1}{\gamma'}. \tag{24}$$

By interpolating between (23) and (24), there exists a $\theta_p\in(\frac{\tilde{\gamma}+2}{2(\beta+1)},1)$ such that

$$\|U_{i,j}f\|_p\le CB_{i,j}^{\theta_p}\|f\|_p,\ \text{for}\ \frac{2\tilde{\gamma}(\beta+1)}{(\tilde{\gamma}+2)\beta}<p<2.$$

Then for fixed $\frac{2\tilde{\gamma}(\beta+1)}{(\tilde{\gamma}+2)\beta}<p<2$, we can choose $1<q<p$ such that $\frac{q\theta_p\beta}{\tilde{\gamma}}>1$. Therefore,

$$\begin{aligned}\sum_{i,j\in\mathbb{Z}}\|U_{i,j}f\|_p^q\le C\Big(&\sum_{i,j>-2}B_{\varphi}^{-i\mu\theta_pq}B_{\psi}^{-j\nu\theta_pq}+\sum_{i\le-2,j>-2}|i|^{-q\theta_p\beta/\tilde{\gamma}}B_{\psi}^{-j\nu\theta_pq}\\ &+\sum_{i>-2,j\le-2}B_{\varphi}^{-i\mu\theta_pq}|j|^{-q\theta_p\beta/\tilde{\gamma}}+\sum_{i,j\le-2}|i|^{-q\theta_p\beta/\tilde{\gamma}}|j|^{-q\theta_p\beta/\tilde{\gamma}}\Big)\|f\|_p^q\\ \le{}& C\|f\|_p^q,\end{aligned}$$

which yields

$$\|\tilde{\mathcal{M}}_{\Omega,h}^{\varphi,\psi}(f)\|_p\le C\|f\|_p,\ \frac{2\tilde{\gamma}(\beta+1)}{(\tilde{\gamma}+2)\beta}<p<2. \tag{25}$$

Case 2: $2<p<\frac{2\tilde{\gamma}(\beta+1)}{(\tilde{\gamma}-2)\beta+2\tilde{\gamma}}$.

By (21), we have that, for $2<p<\infty$ and $1<q<p'$,

$$\|\tilde{\mathcal{M}}_{\Omega,h}^{\varphi,\psi}(f)\|_p^q\le C\sum_{i,j\in\mathbb{Z}}\left(\int_1^2\int_1^2\big\|(\sum_{k,l\in\mathbb{Z}}|w_{k,l;s,t}^{\mu,\nu}*S_{i+k,j+l}f|^2)^{\frac12}\big\|_p^2dsdt\right)^{\frac q2}.$$

For each fixed $i,j\in\mathbb{Z}$, let

$$V_{i,j}^{s,t}f(x,y)=(\sum_{k,l\in\mathbb{Z}}|w_{k,l;s,t}^{\mu,\nu}*S_{i+k,j+l}(x,y)|^2)^{\frac12}.$$

Applying the Lemma 2.5 and the Littlewood-Paley theory, we get

$$\|V_{i,j}^{s,t}f\|_p\le C\big\|\big(\sum_{k,l\in\mathbb{Z}}|S_{i+k,j+l}f|^2\big)^{\frac12}\big\|_p\le C\|f\|_p,\ |\frac1p-\frac12|<\frac{1}{\tilde{\gamma}}. \tag{26}$$

Also, by Plancherel's theorem as in Case 1, we have

$$\|V_{i,j}^{s,t}f\|_2\le CB_{i,j}\|f\|_2, \tag{27}$$

where $B_{i,j}$ is the same as before. By interpolating between (26) and (27), for $2<p<\frac{2\tilde{\gamma}(\beta+1)}{(\tilde{\gamma}-2)\beta+2\tilde{\gamma}}$, we can choose $q\in(1,p')$ and $\vartheta_p\in(\frac{\tilde{\gamma}+2}{2(\beta+1)},1)$ such that $\frac{q\vartheta_p\beta}{\tilde{\gamma}}>1$ and

$$\|V_{i,j}^{s,t}f\|_p\le CB_{i,j}^{\theta_p}\|f\|_p$$

. This shows that

$$\begin{aligned}\|\tilde{\mathcal{M}}^{\varphi,\psi}_{\Omega,h}(f)\|_p^q \le C\Bigg(&\sum_{i,j>-2} B_\varphi^{-i\mu\theta_p q}B_\psi^{-j\nu\theta_p q} + \sum_{i\le-2,j>-2}|i|^{-q\theta_p\beta/\tilde{\gamma}}B_\psi^{-j\nu\theta_p q}\\ &+\sum_{i>-2,j\le-2}B_\varphi^{-i\mu\theta_p q}|j|^{-q\theta_p\beta/\tilde{\gamma}}+\sum_{i,j\le-2}|i|^{-q\theta_p\beta/\tilde{\gamma}}|j|^{-q\theta_p\beta/\tilde{\gamma}}\Bigg)\|f\|_p^q\\ \le{}& C\|f\|_p^q\end{aligned}$$

for $2<p<\frac{2\tilde{\gamma}(\beta+1)}{(\tilde{\gamma}-2)\beta+2\tilde{\gamma}}$. This together with (25) finishes the proof of Theorem 1.1.

Declarations

Conflict of interest The authors declare no conflict of interest.

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