Appl. Math. J. Chinese Univ.
2023, 38(3): 403-412

A fixed point theorem for Proinov mappings with a contractive iterate

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Abstract. In this paper, we consider the fixed point theorem for Proinov mappings with a contractive iterate at a point. In other words, we combine and unify the basic approaches of Proinov and Sehgal in the framework of the complete metric spaces. We consider examples to illustrate the validity of the obtained result.

§1 Introduction and Preliminaries

Fixed point theory took its place in the literature with the well-known results of Banach [1] in 1922: Every self-mapping over a complete metric space possesses a unique fixed point if it is Lipschitz with the Lipschitz constant strictly less than one. Such mappings are called contractions, and by definition, a contraction mapping is continuous. Indeed, it is an abstraction of successive approximation method that was introduced by Picard [2] to solve some certain differential equations. For this reason, Banach Fixed point theorem is also called the Picard-Banach theorem. On the other hand, we emphasize the strong relation between applied mathematics and fixed point theory which is the common research field of functional analysis and topology.

There are two main reasons why fixed point theory has caught the attention of researchers: First, it is a relatively new research area, and the other is that it has wide application potential in different disciplines, including theoretical computer science, engineering, and economics. Under these motivations, a huge number of research papers on the extension and generalization of Banach's fixed point theorem has rebeen leased. Among all, we emphasize the outstanding results of Sehgal [3] who successfully removed the requirement of the continuity from Banach's fixed point theorem. In what follows, we mention another selected result in the fixed point theory was given by Proinov [4], very recently. In his distinguished paper, Proinov [4] proved that most of the well-known results are equivalent to each other. Further, some recent fixed point results belong to Wardowski [5] and Jleli and Samet [6] are not only equivalent to each other, but also they are a special case of the magnificent fixed point theorem of Skof [7].

Received: 2020-09-06.   Revised: 2020-12-14.
MR Subject Classification: 47H09, 47H10, 54H25.
Keywords: contractive iterate at a point, Proinov mappings, fixed point theorems.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-023-4258-y.

In this paper, we consider the fixed point theorem for Proinov mappings with a contractive iterate at a point. In other words, we combine and unify the basic approaches of Proinov [4] and Sehgal [3] in the framework of the complete metric spaces.

For the sake of the completeness of the paper, we recollect the famous result of Sehgal [3]:

Theorem 1. *[3] On a complete metric space $(\mathcal{X}, d)$, a continuous mapping $\mathscr{g} : \mathcal{X} \to \mathcal{X}$ possesses a unique fixed point provided that there exits $\mathscr{k} \in [0, 1)$ such that for each $z \in \mathcal{X}$ there exists a positive integer $p(z)$ such that for each $v \in \mathcal{X}$*

$$d(\mathscr{g}^{p(z)}z, \mathscr{g}^{p(z)}v) \leq \mathscr{k}d(z, v).$$

Before state Proinov's theorem, we state the following significant lemma.

Lemma 1. *[4] For any function $\psi : (0, \infty) \to \mathbb{R}$ the following conditions are equivalent:*

(1) $\inf_{u>\mathscr{e}} \psi(u) > -\infty$ *for every* $\mathscr{e} > 0$;

(2) $\liminf_{u\to \mathscr{e}+} \psi(u) > -\infty$ *for every* $\mathscr{e} > 0$;

(3) $\lim_{n\to\infty} \psi(u_n) = -\infty$ *implies* $\lim_{n\to\infty} u_n = 0$.

Next, we recall the basic and crucial result of Proinov [4].

Theorem 2. *[4] Let $(\mathcal{X}, d)$ be a metric space and $\mathscr{g} : \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$\psi(d(\mathscr{g}z, \mathscr{g}v)) \leq \phi(d(z, v)), \tag{1.1}$$

for all $z, v \in \mathcal{X}$ with $d(\mathscr{g}z, \mathscr{g}v) > 0$ where the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ are such that the following conditions are satisfied:

(1) $\phi(u) < \psi(u)$ *for any* $u > 0$;

(2) ψ *is nondecreasing;*

(3) $\limsup_{u\to u_0+} \phi(u) < \psi(u_0+)$ *for any* $u_0 > 0$.

Then $\mathscr{g}$ admits a unique fixed point.

A self-mapping $\mathscr{g} : \mathcal{X} \to \mathcal{X}$ is called Proinov mapping if it satisfies the condition (1.1).

In this paper, we consider the fixed point theorem for Proinov mappings with a contractive iterate at a point. In other words, we combine and unify the basic approaches of Proinov [4] and Sehgal [3] in the framework of the complete metric spaces.

§2 Main results

The main observation of this paper is the following result:

Theorem 3. *Let $(\mathcal{X}, d)$ be a complete metric space and $g : \mathcal{X} \to \mathcal{X}$ be a mapping such that for all $z \in \mathcal{X}$ there exists $p(z) \in \mathbb{N}$ such that $d(g^{p(z)}z, g^{p(z)}v) > 0$ and*

$$\psi(d(g^{p(z)}z, g^{p(z)}v)) \leq \phi(d(z,v)), \tag{2.1}$$

for every $v \in \mathcal{X}$, where the functions $\psi, \phi : (0,\infty) \to \mathbb{R}$ are such that the following conditions are satisfied:

(c_0) *$\phi(u) < \psi(u)$ for any $u > 0$ and ψ is nondecreasing;*

(c_1) *$\inf_{u>e} \psi(u) > -\infty$ for any $e > 0$;*

(c_2) *if the sequences $\{\psi(u_n)\}$ and $\{\phi(u_n)\}$ are both convergent with the same limit and $\{\psi(u_n)\}$ is strictly decreasing, then $\lim_{n\to\infty} u_n = 0$;*

(c_3) *$\limsup_{u\to 0+} \phi(u) < \liminf_{u\to e+} \psi(u)$ for any $e > 0$;*

Then g admits a unique fixed point.

Proof. Let $z_0 \in \mathcal{X}$, be an arbitrary point and starting from this point, we construct the sequence $\{z_n\}$ by:

$$z_n = g^{p(z_{n-1})}z_{n-1} \text{ for all } n \in \mathbb{N}. \tag{2.2}$$

Denoting by $p_n = p(z_n)$, we have

$$z_1 = g^{p_0}z_0, z_2 = g^{p_1}z_1 = g^{p_1+p_0}z_0, \tag{2.3}$$

and then $z_n = g^{p_{n-1}+\ldots+p_1+p_0}z_0$. Moreover, for any $n, k \in \mathcal{X}$ such that $k \geq 1$ and $n \geq n_0$ we have

$$z_{n+k} = g^m z_n$$

where $m = p_{n+k-1} + p_{n+k-2} + \ldots + p_n$.

If there exists $l \in \mathbb{N} \cup \{0\}$ such that $z_{l+1} = z_l$, then we have $g^l z_l = z_{l+1} = z_l$. Thus, z_l is a fixed point of g^l. Moreover, if for some n there exists $k \in \mathbb{N} \cup \{0\}$ such that $d(g^{p_n}z_n, g^{p_n}(g^k z_n)) = 0$ then

$$z_{n+1} = g^{p_n}z_n = g^{p_n}(g^k z_n) = g^k(g^{p_n}z_n) = g^k z_{n+1},$$

which means that z_{n+1} is a fixed point of g^k. Therefore, we will suppose that $d(z_n, z_\kappa) > 0$ for every $n, \kappa \in \mathbb{N} \cup \{0\}$, $q \geq n$. Now, taking $z = z_n$ and $v = z_\kappa = g^q z_n$ with $q \geq m$ in (2.1) we have

$$\begin{aligned} \psi(d(z_n, g^q z_n)) &= \psi(d(g^{p_{n-1}}z_{n-1}, g^{p_{n-1}}(g^q z_{n-1})) \\ &\leq \phi(d(z_{n-1}, g^q z_{n-1})) < \psi(d(z_{n-1}, g^q z_{n-1})). \end{aligned} \tag{2.4}$$

Given be $s \in \mathbb{N}$, we set

$$a_n = \lim_{s\to\infty} \sup d(z_n, g^s z_0)$$

for any $n \in \mathbb{N}$. Since $r(z_0) < \infty$, the sequence $\{a_n\}$ is bounded, so we can find a sequence $\{s(i)\}$ of positive integer numbers such that the subsequence $\left\{d(z_n, g^{s(i)}z_0)\right\}$ is convergent. Hence, there exists A_n such that

$$\lim_{i\to\infty} d(z_{n-1}, g^{s(i)}z_0) = A_{n-1} < a_{n-1}.$$

Thus, the inequality (2.4) becomes

$$\begin{aligned}\psi(d(z_n, g^{p_{n-1}+s(i)}z_0)) &= \psi(d(g^{p_{n-1}}z_{n-1}, g^{p_{n-1}}(g^{s(i)}z_0)) \\ &\le \phi(d(z_{n-1}, g^{s(i)}z_0)) < \psi(d(z_{n-1}, g^{s(i)}z_0)).\end{aligned} \tag{2.5}$$

Letting $i \to \infty$, from (c_1) it follows

$$\psi(a_n) < \psi(A_{n-1}) < \psi(a_{n-1}). \tag{2.6}$$

This means the sequence $\{\psi(a_n)\}$ is strictly decreasing and positive, and we have to consider two cases.

(a) If the sequence $\{\psi(a_n)\}$ is not bounded, by taking into account (c_1) and Lemma 1 we have that $\lim_{n\to\infty} a_n = 0$.

(b) If, on the contrary, the sequence $\{\psi(a_n)\}$ is suppose to be bounded below then it is convergent, so, there exists $\alpha \ge 0$ such that $\lim_{n\to\infty} \psi(a_n) = \alpha$. Thus, in view of (2.6) the sequence $\{\phi(a_n)\}$ is also convergent and $\lim_{n\to\infty} \phi(a_n) = \alpha$. Taking into account (c_2) it follows that $\lim_{n\to\infty} a_n = 0$. Thus,

$$\lim_{n\to\infty} d(z_n, g^q z_n) = \lim_{n\to\infty} d(z_n, z_\kappa) = 0 \tag{2.7}$$

that is, $\{z_n\}$ is Cauchy sequence on a complete metric space. Therefore, there exists $\mathbf{z} \in \mathcal{X}$ such that

$$\lim_{n\to\infty} d(z_n, \mathbf{z}) = 0. \tag{2.8}$$

We claim that $\mathbf{z}$ is a fixed point for g and will prove this in few steps. First of all we show that

$$\lim_{n\infty} d(g^{p(\mathbf{z})}z_n, z_n) = 0. \tag{2.9}$$

Indeed, from (2.1) we have Indeed, from (2.1) we have

$$\begin{aligned}\psi(d(g^{p(\mathbf{z})}z_n, z_n) &= \psi(d(g^{p_{n-1}}(g^{p(\mathbf{z})}z_{n-1}), g^{p_{n-1}}z_{n-1}) \le \phi(d(g^{p(\mathbf{z})}z_{n-1}, z_{n-1})) \\ &< \psi(d(g^{p(\mathbf{z})}z_{n-1}, z_{n-1}))\end{aligned}$$

which shows that the sequence $\left\{\psi(d(g^{p(\mathbf{z})}z_n, z_n))\right\}$ is decreasing and using similar arguments to the above we get (2.9).

As a second step, we prove that

$$\lim_{n\to\infty} d(g^{p(\mathbf{z})}z_n, \mathbf{z}) = 0, \tag{2.10}$$

using the method of Reductio ad absurdum. So, we suppose that there exists $\varepsilon > 0$ and $k_0 \ge 1$ such that $d(g^{p(\mathbf{z})}z_n, \mathbf{z}) > \varepsilon$, for any $n > k_0$. On the other hand, from (2.8) and (2.9) ....

$$d(z_n, \mathbf{z}) < \varepsilon/2 \text{ and } d(g^{p(\mathbf{z})}z_n, z_n) < \varepsilon/2$$

and from the triangle inequality we have

$$\varepsilon < d(g^{p(\mathbf{z})}z_n, \mathbf{z}) \le d(g^{p(\mathbf{z})}z_n, z_n) + d(z_n, \mathbf{z}) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which is a contradiction. Consequently, $\lim_{n\to\infty} d(g^{p(\mathbf{z})}z_n, \mathbf{z}) = 0$. Now, taking in (2.1) $z = g^{p(\mathbf{z})}$ $\mathbf{z}$ and $v = g^{p(\mathbf{z})}z_n$ we obtain

$$\psi(d(g^{p(\mathbf{z})}\mathbf{z}, g^{p(\mathbf{z})}z_n)) \le \phi(d(\mathbf{z}, z_n)) < \psi(d(\mathbf{z}, z_n)).$$

Denoting $a_n = d(g^{p(\mathbf{z})}\mathbf{z}, g^{p(\mathbf{z})}z_n)$, $a = d(g^{p(\mathbf{z})}\mathbf{z}, \mathbf{z})$ and $b_n = d(\mathbf{z}, z_n)$ and presuming that $a > 0$, since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = 0$, from (2.1) it follows

$$\liminf_{s\to a} \psi(s) \leq \liminf_{n\to\infty} \psi(a_n) \leq \limsup_{n\to\infty} \phi(b_n) < \limsup_{s\to 0} \psi(s).$$

This is a contradiction with the assumption (c_3). Therefore,

$$d(g^{p(\mathtt{z})}\mathtt{z}, \mathtt{z}) = 0,$$

which means that $\mathtt{z}$ is a fixed point of the mapping $g^{p(\mathtt{z})}$. Let's suppose now that there is another point $\mathtt{v} \neq \mathtt{z}$ such that $g^{p(\mathtt{z})}\mathtt{v} = \mathtt{v}$. Thus, by (2.1) and (c_0) we have

$$0 < \psi(d(\mathtt{v}, \mathtt{z})) = \psi(d(g^{p(\mathtt{z})}\mathtt{v}, g^{p(\mathtt{z})}\mathtt{z})) \leq \phi(d(\mathtt{v}, \mathtt{z})) < \psi(d(\mathtt{v}, \mathtt{z})),$$

which is a contradiction. Therefore $g^{p(\mathtt{z})}$ admits a unique fixed point. Moreover,

$$g(\mathtt{z}) = g(g^{p(\mathtt{z})}\mathtt{z})) = g^{p(\mathtt{z})}(g\mathtt{z}),$$

which due to the uniqueness of the fixed point leads us to $g\mathtt{z} = \mathtt{z}$. □

Example 1. *Let $\mathcal{X} = [0,1] \cup \{e^2\}$ and $d : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ where $d(z,v) = |z - v|$. Let the mapping $g : \mathcal{X} \to \mathcal{X}$,*

$$gz = \begin{cases} e^z, & \text{for } z \in [0, \frac{1}{2}] \\ 1, & \text{for } z \in (\frac{1}{2}, 1) \\ \frac{-\ln z + 2}{2}, & \text{for } z \in [1, e] \\ \frac{1}{2}, & \text{for } z = e^2 \end{cases}$$

and the functions $\psi, \phi : (0,\infty) \to \mathbb{R}$ defined by

$$\psi(u) = e^u, \ \phi(u) = 1 + \ln(1 + u).$$

Taking into account this choice of the functions ψ and ϕ, it remains to verify that (2.1) holds. Since $g^3 z = \begin{cases} 1, & \text{for } z \in [0,e] \\ \frac{3}{4}, & \text{for } z = e^2, \end{cases}$, we get $d(g^3 z, g^3 e^2) = d(1, \frac{3}{4}) = \frac{1}{4} > 0$ and

$$\psi(d(g^3 z, g^3 e^2)) = \psi(\frac{1}{4}) = \sqrt[4]{e} < 2 < 1 + \ln(1 + e^2 - z) = \phi(d(z, e^2)),$$

for any $z \in [0,e]$. Thus, according to the Theorem 3, the mapping g has a unique fixed point.

Let now the triplet $(\mathcal{X}, d, g)$ be represents the metric space $(\mathcal{X}, d)$ with a self-mapping g on it. For an arbitrary point z_0 in $\mathcal{X}$, the set $O(z_0) = \{g^m z_0 : m = 1,2,3,...\}$ is called an orbit of z_0 and we denote by $\rho(z_0) = \sup\{d(z,v) : z, v \in O(z_0)\}$, the diameter of $O(z_0)$. Moreover, we indicate by $(\mathcal{X}_o, d, g)$ the corresponding orbitally complete space, which means that any Cauchy sequence from $O(z_0)$ converges in $\mathcal{X}$.

Theorem 4. *On $(\mathcal{X}_o, d, g)$ let $g : \mathcal{X} \to \mathcal{X}$ be a mapping such that for all z there exists $p(z) \in \mathbb{N}$ such that for every $v \in \mathcal{X}$ such that $d(g^{p(z)}z, g^{p(z)}v) > 0$ and*

$$\psi(d(g^{p(z)}z, g^{p(z)}v)) \leq \phi(d(z,v)), \tag{2.11}$$

for all $z, v \in \mathcal{X}$ with $d(g^{p(z)}z, g^{p(z)}v) > 0$, where the functions $\psi, \phi : (0,\infty) \to \mathbb{R}$ are such that the following conditions are satisfied:

(C_0) *$\phi(u) < \psi(u)$ for any $u > 0$, ψ is nondecreasing and $\lim_{u\to e+} \phi(u) < \psi(e+)$ for any $e > 0$;*

(C_1) *the orbit $O(z_0)$ is bounded.*

Then g admits a unique fixed point.

Proof. Let $z_0 \in \mathcal{X}$, $p_0 = p(z_0) \in \mathbb{N}$ and the sequence z_n defined by (2.3). Let $q = p_0 + p_1 + ... + p_{n-1}$ and $m = p_n + p_{n+1} + ... + p_{n+\kappa-1}$, where $p_l = p(z_l)$. First of all, we show that

$$d(z_n, z_{n+\kappa}) < \rho(z_0), \tag{2.12}$$

where $\kappa \geq 0$. Since $z_n = g^{p_0+p_1+...+p_{n-1}} z_0 = g^q z_0$ and $z_{n+\kappa} = g^q(g^m z_0)$. From the similar considerations as in Theorem 3 we have $d(z_n, z_{n+\kappa}) = d(g^q z_0, g^q(g^m z_0)) > 0$. Thus, from (3.6) we have

$$\begin{aligned}\psi(d(z_n, z_{n+\kappa})) &= \psi(d(g^q z_0, g^q(g^m z_0))) \leq \phi(d(z_0, g^m z_0)) \\ &< \psi(d(z_0, g^m z_0))\end{aligned}$$

and taking into account the monotony of the function ψ, we get

$$d(z_n, z_{n+\kappa}) < d(z_0, g^m z_0) \leq \rho(z_0). \tag{2.13}$$

Let now the sequence $\{s(i)\}$ of natural numbers, defined by

$$s(0) = 0, \ s(i+1) = s(i) + p(z_{s(i)}) = s(i) + p_{s(i)}, \ \text{for } i = 0, 1, 2, ...$$

and the sequence $\{\delta_i\}$ defined by

$$\delta_i = d(z_{s(i)}, z_{s(i)+\eta}),$$

where $\eta \geq 0$. We show that the sequence $\{\delta_i\}$ is decreasing. Indeed, for $i = 0$,

$$\delta_1 = d(z_{s(1)}, z_{s(1)+\eta}) = d(z_{p_0}, z_{p_0+\eta}) < d(z_0, g^\eta z_0) \leq \rho(z_0),$$

which is identical with (2.13). Let now i be arbitrary and

$$\begin{aligned}\psi(\delta_{i+1}) &= \psi(d(z_{s(i+1)}, z_{s(i+1)+\eta})) = \psi(g^{p(z_{s(i)})} z_{s(i)}, g^{p(z_{s(i)})}(z_{s(i)+\eta})) \\ &\leq \phi(d(z_{s(i)}, z_{s(i)+\eta})) = \phi(\delta_i).\end{aligned}$$

Thus, $\delta_{i+1} < \delta_i$ since the function ψ is decreasing. Therefore, being strictly decreasing and positive, the sequence $\{\delta_i\}$ is convergent and there exists $\delta \geq 0$ such that $\delta_i \searrow \delta$. If we assume that $\delta > 0$, taking the limit (the superior limit) in the above inequality and keeping in mind C_0, we get

$$\psi(\delta+) = \lim_{i\infty} \psi(\delta_{i+1}) \leq \limsup_{i\to\infty} \phi(\delta_i) \leq \limsup_{s\to\delta+} < \psi(\delta+),$$

which is a contradiction. Consequently, $d(z_{s(i)}, z_{s(i)+\eta}) = d(g^{p_{s(i)}} z_0, g^{p_{s(i+\eta)}} z_0) \to 0$ so that $\{z_n\} \subset O(z_0)$ is a Cauchy sequence in an orbitally complete metric space. Thus, there is $\mathsf{z} \in \mathcal{X}$ such that $\lim_{n\to\infty} z_n = \mathsf{z}$.

We will use indirect proof, so, we suppose that $d(g^{p(\mathsf{z})}, \mathsf{z}) = e > 0$ and

1 .We claim: $d(g^{p(\mathsf{z})} z_n, z_n) \to 0$ as $n \to \infty$. If there exists $k_0 \geq 1$ such that $z_{k_0} = g^{p(\mathsf{z})} z_{k_0}$ we have z_{k_0} is a fixed point of $g^{p(\mathsf{z})}$.

Indeed, if we denote $c_n = d(g^{p(\mathsf{z})} z_n, z_n)$, it is easy to see that $\{c_n\}$ is a subsequence of the sequence $\{\delta_n\}$, so that $c_n \to 0$, that means

$$\lim_{n\to\infty} d(g^{p(\mathsf{z})} z_n, z_n) = 0. \tag{2.14}$$

2. We claim: $d(g^{p(\mathsf{z})} z_n, g^{p(\mathsf{z})} \mathsf{z}) \to 0$ We can suppose that $0 < d(g^{p(\mathsf{z})} z_n, g^{p(\mathsf{z})} \mathsf{z})$ and by (3.6) we have

$$\psi(d(g^{p(\mathsf{z})} z_n, g^{p(\mathsf{z})} \mathsf{z})) \leq \phi(d(z_n, \mathsf{z})) < \psi(d(z_n, \mathsf{z})).$$

This implies

$$d(g^{p(\mathtt{z})}z_n, g^{p(\mathtt{z})}\mathtt{z}) < d(z_n, \mathtt{z})$$

and taking the limit as $n \to \infty$ we have

$$\lim_{n\to\infty} d(g^{p(\mathtt{z})}z_n, g^{p(\mathtt{z})}\mathtt{z}) = 0. \tag{2.15}$$

Thus, from (2.14) and (2.15), for j sufficiently large, we have

$$d(z_i, \mathtt{z}) < e/3,\ d(g^{p(\mathtt{z})}z_j, z_j) < e/3,\ d(g^{p(\mathtt{z})}z_j, g^{p(\mathtt{z})}\mathtt{z}) < e/3. \tag{2.16}$$

Finally, by the triangle inequality,

$$\begin{aligned} e = d(g^{p(\mathtt{z})}\mathtt{z}, \mathtt{z}) &\le d(g^{p(\mathtt{z})}\mathtt{z}, g^{p(\mathtt{z})}z_j) + d(g^{p(\mathtt{z})}z_j, z_j) + d(z_j, \mathtt{z}) \\ &< e/3 + e/3 + e/3 = e \end{aligned}$$

which is a contradiction. Thus, $g^{p(\mathtt{z})}\mathtt{z} = \mathtt{z}$,and consequently, $\mathtt{z}$ is a fixed point of $g^{p(\mathtt{z})}\mathtt{z}$. Again, if we can find another point $\mathtt{v} \in \mathcal{X}$ such that $g^{p(\mathtt{z})}\mathtt{v} = \mathtt{v} \neq \mathtt{z}$, from (3.6) we have

$$\psi(d(\mathtt{z}, \mathtt{v})) = \psi(d(g^{p(\mathtt{z})}\mathtt{z}, g^{p(\mathtt{z})}\mathtt{v})) \le \phi(d(\mathtt{z}, \mathtt{v}))$$

and from the assumption $(\mathcal{C}_0)$ it follows the uniqueness of the fixed point of $g^{p(\mathtt{z})}$. Moreover, since $g\mathtt{z} = g^{p(\mathtt{z})}(g\mathtt{z})$, the above lines yield that $\mathtt{z} = g\mathtt{z}$. □

§3 Consequences

In this section, we will justify the importance of the main result, showing that famous results can be obtained as particular (special) cases.

(a) Setting $\psi(u) = u$ and $\phi(u) = \kappa u$, with $\kappa \in [0, 1]$ then both Theorems 3 and 4 reduce to the famous theorem of V. M. Sehgal 1, in fact an improved version, because the condition of continuity of the mapping g is omitted.

(b) Let $\mathcal{I}$ be an open interval and the sets:

$$\begin{aligned} \mathcal{H} &= \{H : \mathcal{I} \to \mathbb{R} \,|\, H \text{ is upper semicontinuous and } H(u) < u, \text{ for all } u \in \mathcal{I}\} \\ \Psi_{\mathcal{I}} &= \{\psi : (0, \infty) \to \mathcal{I} \,|\, \psi \text{ is nondecreasing }\}. \end{aligned}$$

Setting $\phi(u) = H(\psi(u))$ in Theorem 4, we get the following result:

Corollary 1. *Let the space $(\mathcal{X}_o, d, g)$ and two functions $H \in \mathcal{H}$, $\psi \in \Psi_{\mathcal{I}}$. Let $g : \mathcal{X} \to \mathcal{X}$ be a mapping such that for all z there exists $p(z) \in \mathbb{N}$ such that $d(g^{p(z)}z, g^{p(z)}v) > 0$ and*

$$\psi(d(g^{p(z)}z, g^{p(z)}v)) \le H(\psi(d(z, v))), \tag{3.1}$$

for every $v \in \mathcal{X}$. Then, g admits a unique fixed point provided that for some z_0, the orbit $O(z_0)$ is bounded.

(c) Taking $\phi(u) = \psi(u) - \tau$ in Theorem 4 we obtain a Wardowski-type fixed point result.

Corollary 2. *On $(\mathcal{X}_o, d, g)$ let $g : \mathcal{X} \to \mathcal{X}$ be a mapping such that for all $z \in \mathcal{X}$ there exists $p(z) \in \mathbb{N}$ such that for every $v \in \mathcal{X}$*

$$d(g^{p(z)}z, g^{p(z)}v) > 0 \ \Rightarrow\ \tau + \psi(d(g^{p(z)}z, g^{p(z)}v)) \le \psi(d(z, v)), \tag{3.2}$$

where $\tau > 0$ and $\psi : (0, \infty) \to \mathbb{R}$ is a nondecreasing function. Then, g has a unique fixed point whenever for some z_0, the orbit $O(z_0)$ is bounded.

(d) Let the sets

$$\mathcal{B} = \left\{\beta : (0,\infty) \to (0,1) \mid \limsup_{u \to e+} \beta(u) < 1 \text{ for any } e > 0\right\}$$
$$\Psi = \{\psi : (0,\infty) \to (0,\infty) \mid \psi \text{ is nondecreasing } \}.$$

Setting $\phi(u) = \beta(u)\psi(u)$ in Theorem 4, we obtain the next result:

Corollary 3. *On $(\mathcal{X}_o, d, g)$, a mapping $g : \mathcal{X} \to \mathcal{X}$ such that for all $z \in \mathcal{X}$ there exists $p(z) \in \mathbb{N}$ such that*

$$\psi(d(g^{p(z)}z, g^{p(z)}v)) \le \beta(d(z,v))\psi(d(z,v)), \tag{3.3}$$

for every $v \in \mathcal{X}$ with $d(g^{p(z)}z, g^{p(z)}v) > 0$, where $\psi \in \Psi$ and $\beta \in \mathcal{B}$ has a unique fixed point whenever for some z_0, the orbit $O(z_0)$ is bounded.

Moreover, if we take $\psi(u) = u$, Corollary 3 turns into Kincses-Totik result [**?**].

Corollary 4. *Let g be a self-mapping $(\mathcal{X}_o, d, g)$. If there exists $\beta \in \mathcal{B}$ such that for all $z \in \mathcal{X}$ there exists $p(z) \in \mathbb{N}$ such that*

$$d(g^{p(z)}z, g^{p(z)}v) \le \beta(d(z,v))d(z,v), \tag{3.4}$$

for every $v \in \mathcal{X}$ with $d(g^{p(z)}z, g^{p(z)}v) > 0$ then, g has a unique fixed point whenever for some z_0, the orbit $O(z_0)$ is bounded.

(e) Setting $\phi(u) = k\psi(u)$, $k \in (0,1)$ in Theorem 3 we obtain the follow result:

Corollary 5. *Let $(\mathcal{X}, d, g)$ be a complete metric space, $k \in (0,1)$ and a nondecreasing function $\psi : (0,\infty) \to (0,\infty)$. Let $g : \mathcal{X} \to \mathcal{X}$ be a mapping such that for all $z \in \mathcal{X}$ there exists $p(z) \in \mathbb{N}$ such that*

$$\psi(d(g^{p(z)}z, g^{p(z)}v)) \le k\psi(d(z,v)), \tag{3.5}$$

for every $v \in \mathcal{X}$ with $d(g^{p(z)}z, g^{p(z)}v) > 0$. Then g has a unique fixed point.

(f) Letting $\phi = \psi - \alpha$, where $\psi : (0,\infty) \to \mathbb{R}$ and $\alpha : (0,\infty) \to (0,\infty)$ in Theorem 4, we have:

Corollary 6. *On $(\mathcal{X}_o, d, g)$ let $g : \mathcal{X} \to \mathcal{X}$ be a mapping such that for all z there exists $p(z) \in \mathbb{N}$ such that for every $v \in \mathcal{X}$ such that $d(g^{p(z)}z, g^{p(z)}v) > 0$ and*

$$\psi(d(g^{p(z)}z, g^{p(z)}v)) \le \psi(d(z,v)) - \alpha(d(z,v)), \tag{3.6}$$

where the functions $\psi : (0,\infty) \to \mathbb{R}$ and $\alpha : (0,\infty) \to (0,\infty)$ are such that the following conditions are satisfied:

(a_0) *ψ is nondecreasing and $\liminf_{u \to e+} \phi(u) > 0$ for any $e > 0$;*

(a_1) *the orbit $O(z_0)$ is bounded.*

Then g admits a unique fixed point.

If we consider that the function ψ we get:

Corollary 7. *On $(\mathcal{X}_o, d, g)$ let $g : \mathcal{X} \to \mathcal{X}$ be a mapping such that for all z there exists $p(z) \in \mathbb{N}$ such that for every $v \in \mathcal{X}$*

$$\psi(d(g^{p(z)}z, g^{p(z)}v)) \leq \psi(d(z,v)) - \alpha(d(z,v)), \tag{3.7}$$

where the functions $\psi, \alpha : [0,\infty) \to [0,\infty)$ are such that the following conditions are satisfied:

(b_0) $\psi(u) = \alpha(u) = 0 \Leftrightarrow u = 0$;

(b_1) *ψ is continuous and $\lim_{n\to\infty} \alpha(u_n) = 0$ implies that $\lim_{n\to\infty} u_n = 0$;*

(b_2) *the orbit $O(z_0)$ is bounded.*

Then g admits a unique fixed point.

Example 2. *Consider the linear system $Mz = b$, where $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -39 & 1 & 0 \\ 8 & -4 & -2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -4 \\ 8 \\ 7 \\ -21 \end{bmatrix}$ and $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$, which can be rewrite as*

$$z = Kz + b, \tag{3.8}$$

where $K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 2 & 39 & 0 & 0 \\ -8 & 4 & 2 & 0 \end{bmatrix}$.

Let the mapping $g : \mathbb{R}^4 \to \mathbb{R}^4$ defined as

$$gz = Kz + b,$$

the metric $d(z,v) = \|z - v\|$ in $\mathbb{R}^4$ and the matrix norm $\|K\| = \max_{1\leq j\leq 4} \sum_{i=1}^{4} |k_{ij}|$. Thus, for any $z, v \in \mathbb{R}^4$ we have

$$\left\|gz - gv\right\| = \|Kz - Kv\| \leq \|K\| \, \|z - v\| = 43 \, \|z - v\|$$

so the Banach principle can not be applied. But, since $K^4 = 0_4$ (null matrix), we have $g^4 z = K^3 b + K^2 b + Kb + b$ and then $d(g^4 z, g^4 v) = 0$ for every $z, v \in \mathbb{R}^4$. Consequently, choosing for example $\psi(u) = ln(1+u)$ and $\alpha(u) = u$, the assumptions of the Corollary 7 hold. Thus, the mapping g has a unique fixed point, that means that the system (3.8) has a unique solution, that is $z = \begin{bmatrix} -4 \\ 16 \\ 623 \\ 1321 \end{bmatrix}$.

Declarations

Conflict of interest The authors declare no conflict of interest.

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