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Solutions for Schrödinger-Poisson system involving nonlocal term and critical exponent

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Abstract. In this paper, we consider the following Kirchhoff-Schrödinger-Poisson system:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\triangle u+u+\phi u=\mu Q(x)|u|^{q-2}u+K(x)|u|^4u, & \text{in } \mathbb{R}^3,\\ -\triangle\phi=u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

the nonlinear growth of $|u|^4u$ reaches the Sobolev critical exponent. By combining the variational method with the concentration-compactness principle of Lions, we establish the existence of a positive solution and a positive radial solution to this problem under some suitable conditions. The nonlinear term includes the nonlinearity $f(u)\sim|u|^{q-2}u$ for the well-studied case $q\in[4,6)$, and the less-studied case $q\in(2,3)$, we adopt two different strategies to handle these cases. Our result improves and extends some related works in the literature.

§1 Introduction

In this paper, we study the following Kirchhoff-Schrödinger-Poisson system

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\triangle u+u+\phi u=\mu Q(x)|u|^{q-2}u+K(x)|u|^4u, & \text{in } \mathbb{R}^3,\\ -\triangle\phi=u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $q\in(2,3)$ or $q\in[4,6)$, a,b are positive constants, $\mu>0$ is a parameter. In recent years, the following elliptic problem

$$-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2\Big)\triangle u+V(x)u=f(x,u),\qquad u\in H^1(\mathbb{R}^N) \tag{1.2}$$

has been extensively studied by many researchers, where $V:\mathbb{R}^N\to\mathbb{R}$, $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$, $N=1,2,3$ and $a,b>0$ are constants. Such problems are viewed as being nonlocal because of the term $\int_{\mathbb{R}^N}|\nabla u|^2$, which implies that (1.2) is not longer a pointwise identity and is very different from classical elliptic equations. Similar nonlocal problems model several biological systems where u describes a process that depends on the average of itself, for example, that of

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the population density. Problem (1.2) also arises in an interesting physical context, we refer to [1,8,9,17] and the references therein. There have been many works about the existence of nontrivial solutions to (1.2) by using variational methods only after Lions [13,14] introduced an abstract framework to this problem. Ma and Rivera [16] proved the existence and nonexistence of positive solutions of a class of Kirchhoff type system via variational methods. In [17], Mao and Zhang proved the existence of at least a positive solution, a negative solution and a sign-changing solution to (1.2) by using the method of invariant sets of descent flow. For the case of an unbounded domain, by using the (symmetric) mountain pass theorem, Nie and Wu in [18] studied the existence of infinitely many high-energy solutions for the following Kirchhoff type problem on $\mathbb{R}^N$:

$$\begin{cases} -(\varepsilon^2 a + b\varepsilon \int_{\mathbb{R}^N} |\nabla u|^2)\triangle u + V(x)u = f(x,u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), & \text{in } \mathbb{R}^N. \end{cases}$$

If $a = 1$ and $b = 0$, then (1.1) reduces to the following Schrödinger-Poisson systems

$$\begin{cases} -\triangle u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

many authors studied (1.3) under various assumptions on V and f. If $V = 1$ and $f = |u|^{p-2}u$, equation (1.3) has been studied sufficiently as p varies. In [3], Azzollini and Pomponio obtained the existence of ground state solutions for the subcritical case $3 < p < 6$ and the critical case $f = |u|^{p-2}u + u^5$ with $4 < p < 6$. We refer the readers to [2,11,23,26] and the references therein for further related results.

Recently, a great deal of attention has been focused on nonlocal problems involving critical exponent. Since Brezis and Nirenberg in [6] first studied a critical growth problem in a bounded domain, many researchers considered kinds of critical problems by either pulling the energy level down below some critical energy to recover certain compactness or using a combination of the idea above with the concentration compactness principle of Lions [15]. In [22], Wang et al. proved the existence of positive ground state solution for the following Kirchhoff problem with critical growth

$$\begin{cases} -(\varepsilon^2 a + b\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2)\triangle u + V(x)u = \lambda f(u) + u^5, & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3, \end{cases}$$

by using the Nehari manifold and pulling the energy level down below the critical level

$$c_1 \triangleq \frac{1}{3}(aS)^{\frac{3}{2}} + \frac{1}{12}b^3 S^6. \tag{1.4}$$

For more related results, see [10,12,22,25].

Motivated by the above works, this paper aims to study Kirchhoff-Schrödinger-Poisson system (1.1) which is more general than the above-mentioned problems. To our best knowledge, there are few results on the problem (1.1). In this paper, by combining the variational method with the concentration-compactness principle of Lions, we focus on the critical case and existence of positive solutions, what's more, the nonlinear term includes the nonlinearity $f(u) \sim |u|^{q-2}u$ for the well-studied case $q \in [4, 6)$, and the less-studied case $q \in (2, 3)$, we will adopt two different strategies to handle these cases.

Before state our main results, we make some assumptions on K and Q.

$(H_1)(i)$ $K \in C(\mathbb{R}^3, \mathbb{R}), \lim_{|x|\to\infty} K(x) = K_\infty \in (0, \infty)$ and $K(x) \geq K_\infty$ for $x \in \mathbb{R}^3$,

(ii) $Q \in C(\mathbb{R}^3, \mathbb{R}), \lim_{|x|\to\infty} Q(x) = Q_\infty \in (0, \infty)$ and $Q(x) \geq Q_\infty$ for $x \in \mathbb{R}^3$.

(H_2) There exist $x_0 \in \mathbb{R}^3$, $\delta > 0$ and $\rho > 0$ such that $K(x_0) = \max_{x\in\mathbb{R}^3} K(x)$ and $|K(x) - K(x_0)| \leq \delta|x - x_0|^\alpha$ for $|x - x_0| < \rho$ with $1 \leq \alpha < 3$.

Our main results read as follows.

Theorem 1.1. *Assume (H_1) (H_2) hold. Then, for $4 < q < 6$, problem* (1.1) *has at least a positive solution $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ for each $\mu > 0$; for $q = 4$, problem* (1.1) *still possesses a positive solution provided that μ is sufficiently large.*

Theorem 1.2. *Assume $(H_1)(H_2)$ hold, $2 < q < 3$ and assume furthermore that K and Q are radial functions, then problem* (1.1) *has at least a positive radial solution $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ for $\mu > 0$ sufficiently large.*

Remark 1.3. As mentioned above, our main results extend and improve the related works in [4,7,18,22]. The main difficulty is the lack of compactness since the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)(2 \leq s < 6)$ is not compact. When $4 \leq q < 6$, to overcome this difficulty, we make use of the concentration-compactness principle of Lions [5] and the methods of Brezis and Nirenberg [6], which allow us to determine the energy level of the functional for which the (PS) condition holds. When $2 < q < 3$, we restrict ourselves to $H_r^1(\mathbb{R}^3)$ since K and Q are radial functions. In this case, the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)(2 < s < 6)$ is compact. Moreover, a function in $H_r^1(\mathbb{R}^3)$ has a decay like $|x|^{-1}$ for $x \in \mathbb{R}^3$ far away from the origin, this fact plays an important role in for boundedness of the (PS) sequence, similar argument once appeared in [20].

Remark 1.4. It is not easy to see that I' is weakly continuous by direct calculations since equation (1.1) is no longer a pointwise identity. Indeed, in general, we do not know $\int_{\mathbb{R}^3} |\nabla u_n|^2 \to \int_{\mathbb{R}^3} |\nabla u|^2$ from $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. We succeed in doing so by using the method used in [9], which strongly relies on $q \in [4, 6)$. Considering the nonlocal effect, the critical level c^* is given as follows

$$c^* = \frac{ab}{4\|K\|_\infty} S^3 + \frac{[b^2S^4 + 4\|K\|_\infty aS]^{\frac{3}{2}}}{24\|K\|_\infty^2} + \frac{b^3S^6}{24\|K\|_\infty^2},$$

$\|K\|_\infty$ is the usual L^∞ norm. When $\|K\|_\infty^2 < \frac{1}{2}$, $c^* > c_1$, which means that we can verify the (PS) condition in a wider scope.

The rest of this paper is organized as follows. The variational framework of our problem and some preliminaries are given in Section 2. Section 3 is devoted to the proof of Theorem 1.1. Finally, the proof of Theorem 1.2 is presented in Section 4.

§2 Preliminaries and functional setting

Throughout this paper, we make use of the following notations

$$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}, \quad \|u\|_{H^1}^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2),$$

$$D^{1,2}(\mathbb{R}^3) := \{u \in L^{2^*}(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}, \quad \|u\|_{D^{1,2}}^2 := \int_{\mathbb{R}^3} |\nabla u|^2.$$

For fixed $a > 0$, we also use the notation $\|u\|_a = \left(\int_{\mathbb{R}^3}(a|\nabla u|^2 + u^2)\right)^{1/2}$, which is a norm equivalent to $\|u\|_{H^1}$. $L^p \triangleq L^p(\mathbb{R}^3)(1 \le p < +\infty)$ is the usual Lebesgue space with the standard norm $|u|_p$. $B_r(x) \triangleq \{y \in \mathbb{R}^3 | |x-y| < r\}$. C will denote a positive constant unless specified. The best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$ is denoted by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{|u|_6^2}.$$

For $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi_u = u^2$. Moreover, ϕ_u can be expressed as

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|},$$

then

$$-(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\triangle u + u + \phi_u u = K(x)|u|^4 u + \mu Q(x)|u|^{q-2}u. \tag{2.1}$$

Thus we can define a smooth functional $F : H^1(\mathbb{R}^3) \to \mathbb{R}$ by setting

$$F(u) = \int_{\mathbb{R}^3} \phi_u(x)u^2(x).$$

It turns out that the functional

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\Big)^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3}(\frac{1}{6}K(x)|u|^6 + \frac{\mu}{q}Q(x)|u|^q)$$

is of class C^1 and its critical points are classical solutions of (2.1); see for instance [3,13]. Define the associated functional $I_\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$I_\infty(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\Big)^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3}(\frac{1}{6}K_\infty|u|^6 + \frac{\mu}{q}Q_\infty|u|^q).$$

Set

$$N := \{u \in H^1(\mathbb{R}^3)\setminus\{0\} : \langle I'(u), u\rangle = 0\}, \quad m := \inf_{u\in N} I(u),$$
$$N_\infty := \{u \in H^1(\mathbb{R}^3)\setminus\{0\} : \langle I'_\infty(u), u\rangle = 0\}, \quad m_\infty := \inf_{u\in N_\infty} I_\infty(u).$$

The rest of this section is to estimate level sets of I for which the (PS) condition holds.

Lemma 2.1.([25]) *(i) $F : H^1(\mathbb{R}^3) \to \mathbb{R}$ is weakly continuous in $H^1(\mathbb{R}^3)$.*

(ii) $F(u(\cdot + y)) = F(u)$, for $y \in \mathbb{R}^3$, $u \in H^1(\mathbb{R}^3)$.

(iii) Let $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then

$$F(u_n - u) = F(u_n) - F(u) + o(1) \qquad as \quad n \to \infty.$$

Lemma 2.2.([24]) *Let $r > 0$ and $2 \le q < 2^*$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^q \to 0, \qquad n \to +\infty,$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$.

Lemma 2.3. *For $t, s > 0$ and λ is a positive constant, the following system*

$$\begin{cases} f(t,s) \triangleq t - aS\lambda^{-\frac{1}{3}}(t+s)^{\frac{1}{3}} = 0, \\ g(t,s) \triangleq s - bS^2\lambda^{-\frac{2}{3}}(t+s)^{\frac{2}{3}} = 0, \end{cases}$$

has a unique solution (t_0, s_0). *Moreover,*

$$\begin{cases} f(t,s) \geq 0, \\ g(t,s) \geq 0, \end{cases} \Longrightarrow \begin{cases} t \geq t_0, \\ s \geq s_0. \end{cases}$$

Proof. If $f(t_0, s_0) = g(t_0, s_0) = 0$, then $t_0 + s_0 = \frac{\lambda t_0^3}{a^3 S^3}$. It is enough to solve the following

$$\Big(\frac{\lambda t_0^3 - a^3 S^3 t_0}{a^3 S^3}\Big)^3 = b^3 \lambda^{-2} S^6 \Big(\frac{\lambda t_0^3}{a^3 S^3}\Big)^2,$$

then

$$t_0 = \frac{abS^3 + a\sqrt{b^2 S^6 + 4a\lambda S^3}}{2\lambda}$$

and

$$s_0 = \frac{b^3 S^6 + 2\lambda abS^3 + b^2 S^3 \sqrt{b^2 S^6 + 4aS^3}}{2\lambda^2}.$$

If $f(t,s) \geq 0$ and $g(t,s) \geq 0$, then

$$t + s \geq aS(t+s)^{\frac{1}{3}} + bS^2 (t+s)^{\frac{2}{3}},$$

hence,

$$t + s \geq t_0 + s_0,$$

where we have used the fact that the function $h(l) \triangleq l - aS\lambda^{-\frac{1}{3}} l^{\frac{1}{3}} - bS^2 \lambda^{-\frac{2}{3}} l^{\frac{2}{3}}$, $l > 0$ has a unique zero point $l_0 > 0$ and $h(l) \geq 0$, then $l \geq l_0$. Suppose that $t < t_0$, then

$$f(t,s) = t - aS\lambda^{-\frac{1}{3}}(t+s)^{\frac{1}{3}} < t_0 - aS\lambda^{-\frac{1}{3}}(t_0+s_0)^{\frac{1}{3}} = 0,$$

which is impossible, so $t \geq t_0$. Similarly, $s \geq s_0$. The proof is completed.

Lemma 2.4. *Assume* (H_1) *holds and* $4 \leq q < 6$, *then the following statements hold.*

(i) *For every* $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, *there exists a unique* $t_u = t(u) > 0$ *such that* $t_u u \in N$ *and*

$$I(t_u u) = \max_{t \geq 0} I(tu).$$

(ii) Let $\{u_n\} \subset H^1(\mathbb{R}^3)$ *be a sequence such that* $\langle I'(u_n), u_n\rangle \to 0$ *and* $\int_{\mathbb{R}^3}(K(x)|u_n|^6 + Q(x)|u_n|^q)$ $\to a > 0$ *as* $n \to \infty$. *Then up to a subsequence there exists* $t_n > 0$ *such that* $\langle I'(t_n u_n), t_n u_n\rangle = 0$, *and* $t_n \to 1$ *as* $n \to \infty$.

Proof.(i) For every $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, define $g(t) := I(tu)$ and

$$f(t) := \|u\|^2 + bt^2 \Big(\int_{\mathbb{R}^3} |\nabla u|^2\Big)^2 + t^2 F(u) - t^{q-2}\mu \int_{\mathbb{R}^3} Q(x)|u|^q - t^4 \int_{\mathbb{R}^3} K(x)|u|^6.$$

Then we have $g'(t) = tf(t)$, and for $t > 0$,

$$g'(t) = \langle I'(tu), u\rangle = 0 \Leftrightarrow tu \in N. \tag{2.2}$$

Since $\sup_{t>0} g(t)$ is achieved at some $t_u = t(u) > 0$, one gets $g'(t_u) = 0$, by (2.2), $t_u u \in N$. It's easy to check that t_u is unique and

$$I(t_u u) = \max_{t \geq 0} I(tu).$$

(ii) The proof is standard, see e.g. [244]. We omit it here.

The following concentration-compactness principle is due to P.Lions.

Lemma 2.5.([15]) *Let* $\{\rho_n\}$ *be a sequence of nonnegative* L^1 *functions on* $\mathbb{R}^N$ *satisfying* $\int_{\mathbb{R}^N} \rho_n = l$, *where* $l > 0$ *is fixed. There exists a subsequence, still denoted by* $\{\rho_n\}$, *satisfying one of the following three possibilities:*

(i) (Vanishing) for all $R > 0$, it holds

$$\limsup_{n\to+\infty y\in\mathbb{R}^N}\int_{B_R(y)}\rho_n = 0;$$

(ii) (Dichotomy) there exist $\alpha \in (0, l)$ and $\{y_n\} \subset \mathbb{R}^N$ such that for every $\varepsilon > 0$, $\exists R > 0$, for all $r \geq R$ and $r' \geq r$, it holds

$$\limsup_{n\to+\infty}\left|\alpha - \int_{B_r(y_n)}\rho_n\right| + \left|(l-\alpha) - \int_{\mathbb{R}^N\setminus B_{r'}(y_n)}\rho_n\right| < \varepsilon;$$

(iii) (Compactness) there exists $\{y_n\} \subset \mathbb{R}^N$ such that, for all $\varepsilon > 0$, there exists $R > 0$ satisfying

$$\liminf_{n\to+\infty}\int_{B_R(y_n)}\rho_n \geq l - \varepsilon.$$

Lemma 2.6. *Assume $(H_1) - (H_2)$ hold and $q \in [4, 6)$. If $c < min\{m_\infty, c^*\}$, then I satisfies the $(PS)_c$ condition, where $c^* := \frac{ab}{4\|K\|_\infty}S^3 + \frac{[b^2S^4+4\|K\|_\infty aS]^{\frac{3}{2}}}{24\|K\|_\infty^2} + \frac{b^3S^6}{24\|K\|_\infty^2}$.*

Proof. Let $\{u_n\} \subset H^1$ be a $(PS)_c$ sequence of I at the level c, i.e.

$$I(u_n) \to c, \quad I'(u_n) \to 0 \quad in \quad H^{-1}. \tag{2.3}$$

Then, for n large enough,

$$c + 1 + \|u_n\| \geq I(u_n) - \frac{1}{4}I'(u_n)u_n \geq \frac{1}{4}\|u_n\|^2,$$

which implies that $\{u_n\}$ is bounded in H^1.

Set

$$\rho_n(x) := \frac{a}{4}|\nabla u_n|^2 + \frac{1}{4}u_n^2 + \frac{1}{12}K(x)|u_n|^6 + (\frac{\mu}{4} - \frac{\mu}{q})Q(x)|u_n|^q,$$

Then $\{\rho_n\}$ is bounded in $L^1(\mathbb{R}^3)$ and, we may assume

$$\Phi(u_n) := \|\rho_n\|_1 \to c, \quad as \quad n \to \infty.$$

Now, we will apply Lemma 2.5 to $\{\rho_n\}$. To get the compactness of $\{\rho_n\}$, it is sufficient to show that neither vanishing nor dichotomy occurs. If $\{\rho_n\}$ vanishes, then $\{u_n^2\}$ also vanishes, i.e. there exists $R > 0$ such that

$$\limsup_{n\to\infty y\in\mathbb{R}^3}\int_{B_R(y)}|u_n|^2 = 0.$$

In view of Lemma 2.2, one has $u_n \to 0$ in $L^s(\mathbb{R}^3)$, $2 < s < 6$. Thus,

$$\int_{\mathbb{R}^3}Q(x)|u_n|^q \leq \|Q\|_\infty\int_{\mathbb{R}^3}|u_n|^q \to 0,$$

$$F(u_n) \leq C\|u_n\|^4_{\frac{12}{5}} \to 0,$$

as $n \to \infty$. Furthermore,

$$I(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2 - \frac{1}{6}\int_{\mathbb{R}^3}K(x)|u_n|^6 + o(1), \tag{2.4}$$

$$I'(u_n)u_n = \|u_n\|^2 + b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2 - \int_{\mathbb{R}^3}K(x)|u_n|^6 + o(1). \tag{2.5}$$

We may therefore assume that there exist $l_i \geq 0$ $(i = 1, 2, 3)$ such that

$$\|u_n\|^2 \to l_1, \quad b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2 \to l_2, \quad \int_{\mathbb{R}^3}K(x)|u_n|^6 \to l_3, \quad as \quad n \to \infty,$$

then by $l_1 + l_2 = l_3$, it is easy to see that $l_1 > 0$ and hence that $l_2, l_3 > 0$. By the Sobolev inequality, we have

$$a^3\int_{\mathbb{R}^3}K(x)|u_n|^6\le a^3\|K\|_\infty\int_{\mathbb{R}^3}|u_n|^6\le a^3\|K\|_\infty\Big(S^{-1}\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^3\le\|K\|_\infty S^{-3}\|u_n\|^6$$

and

$$b\Big(\int_{\mathbb{R}^3}K(x)|u_n|^6\Big)^{\frac{2}{3}}\le b\|k\|_\infty^{\frac{2}{3}}\Big(S^{-1}\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2=b\|k\|_\infty^{\frac{2}{3}}S^{-2}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2,$$

and so

$$l_1\ge aS\|K\|_\infty^{\frac{-1}{3}}(l_1+l_2)^{\frac{1}{3}}\quad and\quad l_2\ge bS^2\|K\|_\infty^{\frac{-2}{3}}(l_1+l_2)^{\frac{2}{3}}.$$

By Lemma 2.3, we have

$$\begin{aligned}\frac{1}{3}l_1+\frac{1}{12}l_2\ge&\frac{1}{3}\frac{abS^3+a\sqrt{b^2S^6+4a\|K\|_\infty S^3}}{2\|K\|_\infty}\\&+\frac{1}{12}\Big(\frac{b^2S^3\sqrt{b^2S^6+4a\|K\|_\infty S^3}+b^3S^6+2\|K\|_\infty abS^3}{2\|K\|_\infty^2}\Big)\\=&\frac{ab}{4\|K\|_\infty}S^3+\frac{[b^2S^4+4\|K\|_\infty aS]^{\frac{3}{2}}}{24\|K\|_\infty^2}+\frac{b^3S^6}{24\|K\|_\infty^2}=c^*.\end{aligned}\tag{2.6}$$

Hence, it follows from (2.3) – (2.5) that

$$c=I(u_n)-\frac{1}{6}I'(u_n)u_n+o(1)=\frac{1}{3}\|u_n\|^2+\frac{b}{12}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2+o(1)=\frac{1}{3}l_1+\frac{1}{12}l_2\ge c^*,$$

which contradicts with $c<c^*$. Hence, vanishing does not occur.

Now, we show that dichotomy does not occur. Suppose by contradiction that there exist $\alpha\in(0,c)$ and $\{y_n\}\subset\mathbb{R}^3$ such that for $\varepsilon_n\to 0$, we can choose $\{R_n\}\subset\mathbb{R}^+$ with $R_n\to+\infty$ satisfying

$$\limsup_{n\to+\infty}\Big|\alpha-\int_{B_{R_n}(y_n)}\rho_n\Big|+\Big|(c-\alpha)-\int_{\mathbb{R}^3\setminus B_{2R_n}(y_n)}\rho_n\Big|<\varepsilon_n.\tag{2.7}$$

Let $\xi:\mathbb{R}^+\to[0,1]$ be a cut-off function satisfying $\xi(s)\equiv 1$ for $s\le 1$, $\xi(s)\equiv 0$ for $s\ge 2$ and $|\xi'(s)|\le 2$. Set

$$v_n(x):=\xi\Big(\frac{|x-y_n|}{R_n}\Big)u_n(x),\qquad w_n(x):=\Big(1-\xi\Big(\frac{|x-y_n|}{R_n}\Big)\Big)u_n(x),$$

then by (2.7), we see that

$$\liminf_{n\to+\infty}\Phi(v_n)\ge\alpha\quad and\quad\liminf_{n\to+\infty}\Phi(w_n)\ge c-\alpha.\tag{2.8}$$

In fact,

$$\begin{aligned}\Phi(v_n)=&\int_{\mathbb{R}^3}\Big(\frac{a}{4}|\nabla v_n|^2+\frac{1}{4}v_n^2+\frac{1}{12}K(x)|v_n|^6+(\frac{\mu}{4}-\frac{\mu}{q})Q(x)|v_n|^q\Big)\\&=\int_{B_{R_n}(y_n)}\rho_n+C\int_{\mathbb{R}^3\setminus B_{R_n}(y_n)}\rho_n,\end{aligned}$$

then by (2.7),

$$\liminf_{n\to\infty}\Phi(v_n)\ge\liminf_{n\to\infty}\int_{B_{R_n}(y_n)}\rho_n+\liminf_{n\to\infty}C\int_{\mathbb{R}^3\setminus B_{R_n}(y_n)}\rho_n\ge\alpha.$$

Similarly, one has $\liminf_{n\to+\infty}\Phi(w_n)\ge c-\alpha$.

Denote $\Omega_n:=B_{2R_n}(y_n)\setminus B_{R_n}(y_n)$, then $\int_{\Omega_n}\rho_n\to 0$ as $n\to+\infty$. Therefore, by direct computations, we have

$$\int_{\Omega_n}(a|\nabla v_n|^2+v_n^2)\to 0\quad and\quad\int_{\Omega_n}(a|\nabla w_n|^2+w_n^2)\to 0,$$

as $n \to +\infty$. Hence, we conclude that

$$\begin{aligned} a\int_{\mathbb{R}^3}|\nabla u_n|^2 &= a\int_{\mathbb{R}^3}|\nabla v_n|^2 + a\int_{\mathbb{R}^3}|\nabla w_n|^2 + o(1),\\ \int_{\mathbb{R}^3}u_n^2 &= \int_{\mathbb{R}^3}v_n^2 + \int_{\mathbb{R}^3}w_n^2 + o(1),\\ \int_{\mathbb{R}^3}Q(x)u_n^q &= \int_{\mathbb{R}^3}Q(x)v_n^q + \int_{\mathbb{R}^3}Q(x)w_n^q + o(1),\\ \int_{\mathbb{R}^3}K(x)u_n^6 &= \int_{\mathbb{R}^3}K(x)v_n^6 + \int_{\mathbb{R}^3}K(x)w_n^6 + o(1). \end{aligned} \tag{2.9}$$

Moreover,

$$\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2 \geq \Big(\int_{\mathbb{R}^3}|\nabla v_n|^2\Big)^2 + \Big(\int_{\mathbb{R}^3}|\nabla w_n|^2\Big)^2 + o(1) \tag{2.10}$$

and

$$F(u_n) \geq F(v_n) + F(w_n) + o(1). \tag{2.11}$$

Hence, from (2.9), we deduce that

$$\Phi(u_n) = \Phi(v_n) + \Phi(w_n) + o(1).$$

It follows from (2.8) that

$$c = \lim_{n\to+\infty}\Phi(u_n) \geq \liminf_{n\to+\infty}\Phi(v_n) + \liminf_{n\to+\infty}\Phi(w_n) \geq \alpha + c - \alpha = c.$$

Furthermore,

$$\lim_{n\to+\infty}\Phi(v_n) = \alpha, \qquad \lim_{n\to+\infty}\Phi(w_n) = c-\alpha. \tag{2.12}$$

By (2.3) and (2.9) – (2.11),

$$0 = I'(u_n)u_n + o(1) \geq I'(v_n)v_n + I'(w_n)w_n + o(1). \tag{2.13}$$

Now, we distinguish two cases.

Case 1. Up to a subsequence, we may assume that either $I'(v_n)v_n \leq 0$ or $I'(w_n)w_n \leq 0$. Without loss of generality, suppose that $I'(v_n)v_n \leq 0$, then

$$a\int_{\mathbb{R}^3}|\nabla v_n|^2 + \int_{\mathbb{R}^3}v_n^2 + F(v_n) + b\Big(\int_{\mathbb{R}^3}|\nabla v_n|^2\Big)^2 - \mu\int_{\mathbb{R}^3}Q(x)v_n^q - \int_{\mathbb{R}^3}K(x)v_n^6 \leq 0. \tag{2.14}$$

By Lemma 2.4, for each $n \in \mathbb{N}$, there exists $t_n > 0$ such that $t_nv_n \in N$ and $I'(t_nv_n)t_nv_n = 0$, i.e.,

$$at_n^2\int_{\mathbb{R}^3}|\nabla v_n|^2 + t_n^2\int_{\mathbb{R}^3}v_n^2 + t_n^4F(v_n) + bt_n^4\Big(\int_{\mathbb{R}^3}|\nabla v_n|^2\Big)^2 - t_n^q\mu\int_{\mathbb{R}^3}Q(x)v_n^q - t_n^6\int_{\mathbb{R}^3}K(x)v_n^6 = 0. \tag{2.15}$$

Combining (2.14) and (2.15), we have

$$(t_n^2 - t_n^q)\|v_n\|^2 + (t_n^4 - t_n^q)F(v_n) + b(t_n^4 - t_n^q)\Big(\int_{\mathbb{R}^3}|\nabla v_n|^2\Big)^2 + (t_n^q - t_n^6)\int_{\mathbb{R}^3}K(x)v_n^6 \geq 0,$$

which implies that $t_n \leq 1$. Then, by (2.12)

$$\begin{aligned} c \leq I(t_nv_n) &= I(t_nv_n) - \frac{1}{4}I'(t_nv_n)t_nv_n\\ &= \frac{1}{4}t_n^2\|v_n\|^2 + \Big(\frac{1}{4} - \frac{1}{q}\Big)t_n^q\int_{\mathbb{R}^3}Q(x)v_n^q + \frac{t_n^6}{12}\int_{\mathbb{R}^3}K(x)v_n^6\\ &\leq \Phi(v_n) \to \alpha < c \end{aligned}$$

which is a contradiction.

Case 2. Up to a subsequence, we may assume that $I'(v_n)v_n > 0$ and $I'(w_n)w_n > 0$. By (2.13), we see that $I'(v_n)v_n \to 0$ and $I'(w_n)w_n \to 0$ as $n \to +\infty$. In view of (2.9) – (2.11), we have

$$I(u_n) \geq I(v_n) + I(w_n) + o(1). \tag{2.16}$$

If $\{y_n\} \subset \mathbb{R}^3$ is bounded. we deduce a contradiction by comparing $I(w_n)$ and m_∞. In this case,

$$\int_{\mathbb{R}^3} (Q(x) - Q_\infty)|w_n|^q \leq \sup_{x \in \mathbb{R}^3} |Q(x) - Q_\infty||w_n|_q^q = \sup_{|x-y_n| \geq R_n} |Q(x) - Q_\infty||w_n|_q^q \to 0 \quad as \quad n \to \infty,$$

and

$$\int_{\mathbb{R}^3} (K(x) - K_\infty)|w_n|^6 \to 0, \quad as \quad n \to \infty.$$

It follows that

$$I(w_n) = I_\infty(w_n) + o(1) \quad and \quad o(1) = I'(w_n)w_n = I'_\infty(w_n)w_n + o(1).$$

Then by Lemma 2.4, there exist two sequence positive constants $t_n \to 1$ and $s_n \to 1$, such that $t_n w_n \in N_\infty$, $s_n v_n \in N$. Thus,

$$I(w_n) = I_\infty(w_n) + o(1) = I_\infty(t_n w_n) + o(1) \geq m_\infty + o(1) \tag{2.17}$$

and

$$I(v_n) = I(s_n v_n) + o(1) \geq m + o(1). \tag{2.18}$$

Therefore, from (2.16) – (2.18), we have $c \geq m_\infty + m \geq m_\infty$, a contradiction.

If $\{y_n\} \subset \mathbb{R}^3$ is unbounded, in a similar way, we are led to a contradiction by comparing $I(v_n)$ and m_∞. Thus, dichotomy does not happen.

So far, by Lemma 2.5, we know that the sequence $\{\rho_n\}$ is compact, i.e., there exist $\{y_n\} \subset \mathbb{R}^3$ such that for every $\varepsilon > 0$, there exists $R > 0$, we have $\int_{B_R^c(y_n)} \rho_n(x) < \varepsilon$, which implies that

$$\int_{B_R^c(y_n)} (K(x)|u_n|^6 + Q(x)|u_n|^q) < \varepsilon.$$

That is to say, the sequence $\{K(x)|u_n|^6 + Q(x)|u_n|^q\}$ is also compact. Then $\{y_n\}$ must be bounded. Otherwise,

$$\int_{\mathbb{R}^3} (K(x) - K_\infty)|u_n|^6 = \int_{\mathbb{R}^3} (Q(x) - Q_\infty)|u_n|^q = o(1)$$

and thus $I(u_n) = I_\infty(u_n) + o(1)$ and $\langle I'_\infty(u_n), u_n \rangle = o(1)$. By Lemma 2.4, there exist $t_n \to 1$ such that $t_n u_n \in N_\infty$. Hence,

$$c = I(u_n) + o(1) = I_\infty(u_n) + o(1) = I_\infty(t_n u_n) + o(1) \geq m_\infty + o(1),$$

a contradiction. Let $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. Since $\{y_n\}$ is bounded, it is easy to see that $u_n \to u$ in $L^q(\mathbb{R}^3)$.

Set $h_n = u_n - u$, then

$$F(h_n) \leq C\|h_n\|_{\frac{12}{5}}^4 \to 0.$$

By the Brezis-Lieb Lemma [28] and Lemma 2.1, we have

$$\|u_n\| = \|u\| + \|h_n\| + o(1),$$

$$\int_{\mathbb{R}^3} K(x)u_n^6 = \int_{\mathbb{R}^3} K(x)u^6 + \int_{\mathbb{R}^3} K(x)h_n^6 + o(1),$$

$$F(h_n) = F(u_n) - F(u) + o(1).$$

Therefore, we have

$$\begin{aligned}I(u_n)-I(u)=&\frac{1}{2}\|h_n\|^2+\frac{b}{4}\Big(\Big(\int_{\mathbb{R}^3}|\nabla h_n|^2\Big)^2+2\int_{\mathbb{R}^3}|\nabla h_n|^2\int_{\mathbb{R}^3}|\nabla u|^2\Big)\\&-\frac{1}{6}\int_{\mathbb{R}^3}K(x)h_n^6+o(1)\end{aligned}\tag{2.19}$$

and

$$\begin{aligned}o(1)&=I'(u_n)u_n-I'(u)u\\&=\|h_n\|^2+b\Big(\int_{\mathbb{R}^3}|\nabla h_n|^2\Big)^2+2b\int_{\mathbb{R}^3}|\nabla h_n|^2\int_{\mathbb{R}^3}|\nabla u|^2-\int_{\mathbb{R}^3}K(x)h_n^6+o(1).\end{aligned}\tag{2.20}$$

Let

$$\|h_n\|^2\to\widetilde{l}_1,\quad b\Big(\int_{\mathbb{R}^3}|\nabla h_n|^2\Big)^2+2b\int_{\mathbb{R}^3}|\nabla h_n|^2\int_{\mathbb{R}^3}|\nabla u|^2\to\widetilde{l}_2,\quad \int_{\mathbb{R}^3}K(x)h_n^6\to\widetilde{l}_3.$$

then $\widetilde{l}_1+\widetilde{l}_2=\widetilde{l}_3$. If $\widetilde{l}_1>0$, then $\widetilde{l}_2,\widetilde{l}_3>0$. It follows from (2.19), (2.20) and $I(u)\geq 0$ that

$$\begin{aligned}c&\geq\frac{1}{3}\|h_n\|^2+\frac{b}{12}\Big(\Big(\int_{\mathbb{R}^3}|\nabla h_n|^2\Big)^2+2\int_{\mathbb{R}^3}|\nabla h_n|^2\int_{\mathbb{R}^3}|\nabla u|^2\Big)+o(1)\\&=\frac{1}{3}\widetilde{l}_1+\frac{1}{12}\widetilde{l}_2+o(1).\end{aligned}$$

then like in the proof of (2.6), we have $c\geq c*$, a contradiction. Thus $\widetilde{l}_1=0$, that is $u_n\to u$ in $H^1(\mathbb{R}^3)$.

§3 Proof of Theorem 1.1

First of all, it is easy to verify that the functional I possesses the mountain pass geometry, that is, $I(0)=0$ and

(i) there exists $\rho'>0$ such that $\inf_{\|u\|=\rho'}I(u)>0$,

(ii) there exists $e\in H^1(\mathbb{R}^3)$ such that $\|e\|>\rho'$ and $I(e)<0$.

Now we define

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t)),$$

where $\Gamma=\{\gamma\in C([0,1],H^1(\mathbb{R}^3))|\gamma(0)=0,\gamma(1)=e\}$.

We will estimate the mountain pass value c. For this purpose, we introduce the function $u_{\varepsilon,x_0}\in D^{1,2}(\mathbb{R}^3)$ defined by

$$u_{\varepsilon,x_0}:=C\frac{\varepsilon^{1/4}}{(\varepsilon+|x-x_0/\varepsilon|^2)^{1/2}},$$

where C is normalizing constant and x_0 is given in (H_2). Set $v_\varepsilon:=\varphi(x-x_0/\varepsilon)u_{\varepsilon,x_0}$, where $\varphi\in C_0^\infty(\mathbb{R}^3)$ satisfies $0\leq\varphi(x)\leq 1$ and $\varphi(x)\equiv 1$ on $B_r(0)$, $supp\varphi\subset B_{2r}(0)$, where r is a positive constant. It is well known that S is attained by the functions $\frac{\varepsilon^{1/4}}{(\varepsilon+|x-x_0/\varepsilon|^2)^{1/2}}$. Direct calculation shows that

$$\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2=K_1+O(\varepsilon^{\frac{1}{2}}),\qquad \int_{\mathbb{R}^3}|v_\varepsilon|^6=K_2+O(\varepsilon^{\frac{3}{2}}).\tag{3.1}$$

$$\int_{\mathbb{R}^3}|v_\varepsilon|^t=\begin{cases}O(\varepsilon^{\frac{t}{4}}), & t\in[2,3),\\ O(\varepsilon^{\frac{t}{4}}|\ln\varepsilon|), & t=3,\\ O(\varepsilon^{\frac{6-t}{4}}), & t\in(3,6),\end{cases}\tag{3.2}$$

where K_1, K_2 are positive constants. Moreover, $S = \frac{K_1}{K_2^{\frac{1}{3}}}$. Using (3.1) and (3.2), we have

$$\frac{\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2}{\Big(\int_{\mathbb{R}^3} |v_\varepsilon|^6\Big)^{\frac{1}{3}}} = S + O(\varepsilon^{\frac{1}{2}}).$$

Lemma 3.1. *Let* $4 < q < 6$*, then for any* $\mu > 0$*,*

$$c < c^* := \frac{ab}{4\|K\|_\infty} S^3 + \frac{[b^2S^4 + 4\|K\|_\infty aS]^{\frac{3}{2}}}{24\|K\|_\infty^2} + \frac{b^3S^6}{24\|K\|_\infty^2}.$$

Moreover, if $2 < q \leq 4$*, the above inequality still holds provided* μ *is sufficient large.*

Proof. It is easy to know that $c = c_1 := \inf\limits_{u\in H^1\setminus\{0\}} \max\limits_{t\geq 0} I(tu) > 0$, see for instance [10,24], we infer that $c \leq \max\limits_{t\geq 0} I(tv_\varepsilon)$. Define function

$$g(t) := \frac{t^2}{2}\|v_\varepsilon\|^2 + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2\Big)^2 - \frac{t^6}{6}\int_{\mathbb{R}^3} K(x_0)|v_\varepsilon|^6.$$

It is clear that $g(t)$ attains its maximum at

$$t_0 = \Big(\frac{b(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2)^2 + \sqrt{b^2(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2)^4 + 4\|K\|_\infty \|v_\varepsilon\|^2 \int_{\mathbb{R}^3} |v_\varepsilon|^6}}{2\|K\|_\infty \int_{\mathbb{R}^3} |v_\varepsilon|^6}\Big)^{\frac{1}{2}}$$

and

$$g(t_0) = \frac{ab}{4\|K\|_\infty} S^3 + \frac{[b^2S^4 + 4\|K\|_\infty aS]^{\frac{3}{2}}}{24\|K\|_\infty^2} + \frac{b^3S^6}{24\|K\|_\infty^2} + O(\varepsilon^{\frac{1}{2}}).$$

By (H_2),

$$|K(x) - K(x_0)| \leq \delta|x - x_0|^\alpha, \quad for \quad |x - x_0| < \rho,$$

thus

$$\begin{aligned}
\int_{\mathbb{R}^3} |K(x) - K(x_0)||v_\varepsilon|^6 &\leq C\delta \int_{B_\rho(x_0)} \frac{|x - x_0|^\alpha \varepsilon^{\frac{3}{2}}}{(\varepsilon + |x - x_0|^2)^3} + C\int_{B^c_\rho(x_0)} \frac{\varepsilon^{\frac{3}{2}}}{(\varepsilon + |x - x_0|^2)^3} \\
&\leq C\delta\varepsilon^{\frac{3}{2}} \int_0^\rho \frac{r^{2+\alpha}}{(\varepsilon + r^2)^3} dr + C\varepsilon^{\frac{3}{2}} \int_\rho^\infty r^{-4} dr \\
&= C\delta\varepsilon^{\frac{\alpha}{2}} \int_0^{\rho\varepsilon^{-\frac{1}{2}}} \frac{\varrho^{2+\alpha}}{(1 + \varrho^2)^3} d\varrho + C\rho^{-3}\varepsilon^{\frac{3}{2}} \\
&\leq C\delta\varepsilon^{\frac{\alpha}{2}} + C\varepsilon^{\frac{3}{2}} \leq O(\varepsilon^{\frac{1}{2}}).
\end{aligned} \tag{3.3}$$

where we use the fact that $1 \leq \alpha < 3$. Thus, for v_ε, there exists $t_\varepsilon > 0$, such that $t_\varepsilon v_\varepsilon \in N$ and

$$\begin{aligned}
I(t_\varepsilon v_\varepsilon) &\leq \sup_{t>0} g(t) + \frac{t_\varepsilon^4}{4} F(v_\varepsilon) - \frac{t_\varepsilon^q \mu}{q} \int_{\mathbb{R}^3} Q(x)|v_\varepsilon|^q + \frac{t_\varepsilon^6}{6} \int_{\mathbb{R}^3} (K(x_0) - K(x))|v_\varepsilon|^6 \\
&\leq \frac{ab}{4\|K\|_\infty} S^3 + \frac{[b^2S^4 + 4\|K\|_\infty aS]^{\frac{3}{2}}}{24\|K\|_\infty^2} + \frac{b^3S^6}{24\|K\|_\infty^2} + O(\varepsilon^{\frac{1}{2}}) + C_3\|v_\varepsilon\|^4_{\frac{12}{5}} - \mu C_4 \int_{\mathbb{R}^3} |v_\varepsilon|^q,
\end{aligned}$$

where $C_i (i = 3, 4)$ are positive constants, independent from ε. Thus to complete the proof, it suffice to show that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\frac{1}{2}}}\Big(\int_{\mathbb{R}^3} -\mu v_\varepsilon^q + \|v_\varepsilon\|^4_{\frac{12}{5}}\Big) = -\infty. \tag{3.4}$$

In fact, the following estimate holds as $\varepsilon \to 0$:

$$\int_{\mathbb{R}^3} -\mu v_\varepsilon^q + \Big(\int_{\mathbb{R}^3} |v_\varepsilon|^{\frac{12}{5}}\Big)^{\frac{5}{3}} \leq \begin{cases} -C_1\mu\varepsilon^{\frac{6-q}{4}} + C_2\varepsilon, & 3 < q < 6, \\ -C_1\mu\varepsilon^{\frac{q}{4}}|\ln\varepsilon| + C_2\varepsilon, & q = 3, \\ -C_1\mu\varepsilon^{\frac{q}{4}} + C_2\varepsilon, & 2 < q < 3, \end{cases} \tag{3.5}$$

where $C_i (i = 1, 2)$ are positive constants independent from ε. If $4 < q < 6$, (3.4) follows from (3.5) for any $\mu > 0$. If $2 < q \leq 4$, in the above inequality one can stress the parameter choosing $\mu = \varepsilon^{-\sigma}$, $\sigma > 0$, to obtain (3.4).

Lemma 3.2. *If $4 < q < 6$, then $c < m_\infty$ for any $\mu > 0$. If $q = 4$, then $c < m_\infty$ provided μ is sufficiently large.*

Proof. It is easy to check that I_∞ has the mountain pass geometry. Let

$$\Gamma = \Big\{\gamma \in C([0, 1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, I_\infty(\gamma(1)) < 0\Big\}$$

and

$$c_\infty = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\infty(\gamma(t)).$$

Since $4 \leq q < 6$, it is easy to verify that

$$m_\infty = \inf_{u\in H^1(\mathbb{R}^3)} \max_{t\geq 0} I_\infty(tu) > 0.$$

and then using an argument like that in [18], we have

$$c_\infty = m_\infty. \tag{3.6}$$

As a consequence of mountain pass principle [23], there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$I_\infty(u_n) \to c_\infty \qquad I'_\infty(u_n) \to 0.$$

It is easy to see that $\{u_n\}$ is bounded since $4 \leq q < 6$.

Define $w_n(x) := u_n(x + y_n)$, for $y_n \in \mathbb{R}^3$. We claim that there exists $y_n \in \mathbb{R}^3$ such that $w_n \rightharpoonup w \neq 0$ in $H^1(\mathbb{R}^3)$. Suppose, on the contrary, that for any $y_n \in \mathbb{R}^3$, $w_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. We will get a contradiction by proving $u_n \to 0$ in $L^s(\mathbb{R}^3)$, $2 < s < 6$. In this case, we claim for all $s \in [2, 6)$,

$$\limsup_{n\to\infty, y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^s = 0. \tag{3.7}$$

If this is not true, then there exists $s \in [2, 6)$, $\delta > 0$, such that

$$\limsup_{n\to\infty, y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^s > \delta > 0.$$

Thus there exists $y_n \in \mathbb{R}^3$ such that $\lim_{n\to\infty} \int_{B_1(y_n)} |u_n|^s > \frac{\delta}{2} > 0$. Therefore we have $\lim_{n\to\infty} \int_{B_1(0)} |w_n|^s > \frac{\delta}{2} > 0$, that is $w_n \rightharpoonup w \neq 0$, a contradiction. Thus (3.7) holds. By Lemma 2.2, we get $u_n \to 0$ in $L^s(\mathbb{R}^3)$, $2 < s < 6$. In particular, we have $u_n \to 0$ in $L^q(\mathbb{R}^3)$ and in $L^{\frac{12}{5}}(\mathbb{R}^3)$. By the same argument as in the proof of Lemma 2.6, we have $c \geq \frac{ab}{4\|K\|_\infty} S^3 + \frac{[b^2S^4 + 4\|K\|_\infty aS]^{\frac{3}{2}}}{24\|K\|_\infty^2} + \frac{b^3S^6}{24\|K\|_\infty^2}$, which contradicts Lemma 3.1, thus the claim holds. Since $I_\infty(w_n) = I_\infty(u_n) \to c_\infty$, $I'_\infty(w_n) =$

$I'_\infty(u_n)+o(1)\to 0$, it is standard to show that $I'_\infty(w)=0$, which implies

$$\begin{aligned} c_\infty &= \liminf_{n\to\infty}\Big(I_\infty(w_n)-\frac{1}{4}\langle I'_\infty(w_n),w_n\rangle\Big)\\ &= \liminf_{n\to\infty}\Big[\frac{1}{4}\|w_n\|^2+\frac{1}{12}\int_{\mathbb{R}^3}K_\infty|w_n|^6+\Big(\frac{\mu}{4}-\frac{\mu}{q}\Big)\int_{\mathbb{R}^3}Q_\infty|w_n|^q\Big]\\ &\geq \frac{1}{4}\|w\|^2+\frac{1}{12}\int_{\mathbb{R}^3}K_\infty|w|^6+\Big(\frac{\mu}{4}-\frac{\mu}{q}\Big)\int_{\mathbb{R}^3}Q_\infty|w|^q\\ &= I_\infty(w). \end{aligned}$$

Since $w\neq 0$, $I_\infty(w)\geq m_\infty$. From (3.6), $I_\infty(w)=m_\infty$. Let u_∞ be the minimizer of m_∞, then $I_\infty(u_\infty)=\max\limits_{t\geq 0}I_\infty(tu_\infty)$. Thus there exists $t^*>0$ such that

$$c\leq \sup_{t\geq 0}I(tu_\infty)=I(t^*u_\infty)<I_\infty(t^*u_\infty)\leq I_\infty(u_\infty)=m_\infty.$$

The proof is finished.

Now we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. It follows from Lemma 2.6, 3.1 and 3.2 that I has a nontrivial critical point $u\in H^1(\mathbb{R}^3)$. If we replace I by the following functional

$$I_+(u)=\frac{1}{2}\|u\|^2+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2\Big)^2+\frac{1}{4}F(u)-\int_{\mathbb{R}^3}(\frac{1}{6}K|u^+|^6+\frac{\mu}{q}Q|u^+|^q),$$

where $u^\pm=\max\{\pm u,0\}$, then we see that all the above calculations can be repeated word by word, so I^+ has a nontrivial critical point $u\in H^1(\mathbb{R}^3)$. Hence,

$$0=\langle I'_+(u),u^-\rangle=\|u^-\|^2+b\int_{\mathbb{R}^3}|\nabla u|^2\int_{\mathbb{R}^3}|\nabla u^-|^2+\int_{\mathbb{R}^3}\phi_u|u^-|^2\geq\|u^-\|^2,$$

which implies $u\geq 0$, and u is not identically zero, the maximum principle yield $u>0$. Thus (u,ϕ_u) is a positive solution of (1.1).

§4 Proof of Theorem 1.2

When $2<q<3$, the structure of the Nehari manifold N_∞ is delicate, so it seems to be impossible to compare c_∞ and m_∞, which is important to overcome the lack of compactness. Thus we restrict ourselves to radial functions space $H^1_r(\mathbb{R}^3)$. As shown in [3], I is invariant under the group action of $O(3)$, the orthogonal transform in $\mathbb{R}^3$, since we assume K and Q are radial symmetric. By the symmetry critical principle in [24], a critical point $u\in H^1_r(\mathbb{R}^3)$ for $I|_{H^1_r(\mathbb{R}^3)}$ is also a critical point for I.

Lemma 4.1. *Assume (H_1), (H_2) hold and K, Q are radial functions. If I has a bounded $(PS)_c$ sequence with $c\in(0,c^*)$, then I has a nontrivial critical point $u\in H^1_r(\mathbb{R}^3)$, for $2<q<3$, and $\mu>0$ large.*

Proof. Let $\{u_n\}\subset H^1_r(\mathbb{R}^3)$ be a bounded $(PS)_c$ sequence of I. Since $H^1_r(\mathbb{R}^3)$ is compactly embedded in $L^s(\mathbb{R}^3)(2<s<6)$, we may therefore assume that there exists $u\in H^1_r(\mathbb{R}^3)$ such that

$$u_n\rightharpoonup u \quad in \quad H^1_r(\mathbb{R}^3), \qquad u_n\to u \quad in \quad L^s(\mathbb{R}^3).$$

Plainly, $I'(u)=0$. In this case, we only show that $u\neq 0$. Suppose, on the contrary, that $u=0$. Then $u_n\to 0$ in $L^q(\mathbb{R}^3)$, and $F(u_n)\to 0$. Thus

$$I(u_n)=\frac{1}{2}\|u_n\|+\frac{b}{4}(\int_{\mathbb{R}^3}|\nabla u_n|^2)^2-\frac{1}{6}\int_{\mathbb{R}^3}K(x)|u_n^+|^6+o(1),$$

$$\langle I'(u_n),u_n\rangle=\|u_n\|^2+b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2-\int_{\mathbb{R}^3}K(x)|u_n^+|^6+o(1),$$

we also assume that as $n\to\infty$, there exist $l_i\geq 0 (i=1,2,3)$ such that

$$\|u_n\|^2\to l_1,\qquad b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2\to l_2,\qquad \int_{\mathbb{R}^3}K(x)|u_n^+|^6\to l_3.$$

then $l_1+l_2=l_3$. It is clear that $l_1>0$ and hence that $l_2,l_3>0$. then

$$\begin{aligned}c&=I(u_n)-\frac{1}{6}I'(u_n)u_n+o(1)\\&=\frac{1}{3}\|u_n\|^2+\frac{b}{12}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2+o(1)\\&=\frac{1}{3}l_1+\frac{1}{12}l_2+o(1).\end{aligned}$$

Similarly to the proof of (2.6), $c\geq c^*$, a contradiction. Thus $u\neq 0$.

Proof of Theorem 1.2. By Lemma 4.1, it is sufficient to construct a bounded $(PS)_c$ sequence $\{u_n\}$ with $c\neq 0$ for I. It is easy to see that I has the mountain pass geometry, then there exists a $(PS)_c$ sequence $\{u_n\}$ with $c>0$ for I, that is

$$I(u_n)\to c>0,\quad I'(u_n)\to 0,\quad in\ \ H^{-1}(\mathbb{R}^3).$$

Multiplying $-\triangle\phi_u=u^2$ by $|u_n|$ and integrating by parts, we have

$$\int_{\mathbb{R}^3}|u_n|^3=\int_{\mathbb{R}^3}-\triangle\phi_{u_n}|u_n|\leq\int_{\mathbb{R}^3}|\nabla u_n|^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2.$$

Thus we deduce that

$$\begin{aligned}c+1+\|u_n\|&\geq I(u_n)-\frac{1}{6}\langle I'(u_n),u_n\rangle\\&\geq\frac{1}{6}\|u_n\|^2+\frac{1}{24}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+g(u_n)\\&=\frac{1}{24}\|u_n\|^2+\frac{1}{8}\|u_n\|^2+\frac{1}{24}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+g(u_n).\end{aligned}$$

for $g(u):=\frac{1}{8}\int_{\mathbb{R}^3}|u|^3+\frac{1}{6}\int_{\mathbb{R}^3}u^2-(\frac{\mu}{q}-\frac{\mu}{6})|Q|_\infty\int_{\mathbb{R}^3}|u^+|^q$.

We claim that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. If not, the following inequality must hold for n large enough:

$$\frac{1}{8}\|u_n\|^2+\frac{1}{24}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+g(u_n)\leq 0.$$

Similar to the idea of [19]. Define $m=\min g$, clearly, $m<0$. Then, the set $\{u>0:g(u)<0\}$ is of the form (α,β), with $\alpha>0$. Note that α,β,m are constants depending only on q. For each function u_n, define $A_n=\{x\in\mathbb{R}^3:u_n(x)\in(\alpha,\beta)\}$. Note that A_n is spherically symmetric, and define $\rho_n=\sup\{|x|:x\in A_n\}$. Since

$$\begin{aligned}0&\geq\frac{1}{8}\|u_n\|^2+\frac{1}{24}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+g(u_n)\\&\geq\frac{1}{8}\|u_n\|^2+\frac{1}{24}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+\int_{u_n\in(\alpha,\beta)}g(u_n)\\&\geq\frac{1}{8}\|u_n\|^2+\frac{1}{24}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+m|A_n|,\end{aligned}\tag{4.1}$$

one has

$$|m||A_n|>\frac{1}{8}\|u_n\|^2,\tag{4.2}$$

which, in particular, implies that $|A_n|\to+\infty$.

We now recall the following general result due to Strauss [21]

$$|u(x)| \leq c_0|x|^{-1}\|u\| \qquad \forall u \in H_r^1. \tag{4.3}$$

for some $c_0 > 0$. Take $x \in \mathbb{R}^3$, $|x| = \rho_n$. Clearly, $u_n(x) = \alpha > 0$. We use inequalities (4.2), (4.3) to obtain

$$0 < \alpha = u_n(x) \leq c_0\rho_n^{-1}\|u_n\| \leq \sqrt{8}c_0\rho_n^{-1}(|m||A_n|)^{1/2} \quad \Rightarrow \quad c_1\rho_n \leq |A_n|^{1/2} \tag{4.4}$$

for some $c_1 > 0$. On the other hand, by (4.1), we have that $|m||A_n| \geq \frac{1}{24}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2$, then

$$\begin{aligned} 24|m||A_n| &\geq \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = \frac{1}{4\pi}\int_{\mathbb{R}^{\nvDash}}\int_{\mathbb{R}^{\nvDash}} \frac{u_n^2(x)u_n^2(y)}{|x-y|} \\ &\geq \frac{1}{4\pi}\int_{A_n}\int_{A_n} \frac{u_n^2(x)u_n^2(y)}{|x-y|} \geq \frac{1}{4\pi}\alpha^4\frac{A_n^2}{2\rho_n} \\ &\Rightarrow c_2\rho_n \geq |A_n| \end{aligned}$$

for some $c_2 > 0$, which is a contradiction with (4.4). Thus $\{u_n\}$ is a bounded $(PS)_c$ sequence with $c > 0$ for I, by Lemma 3.1, 4.1 and the symmetry critical principle [23], I has a nontrivial critical point u, and as in the proof of Theorem 1.1, (u, ϕ_u) is a positive solution to problem (1.1). The proof is completed.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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