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Reduced differential transform and Sumudu transform methods for solving fractional financial models of awareness

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Abstract. In that paper, our new study has been carried out on previous studies of one of the most important mathematical models that describe the global economic movement, and that is described as a non-linear fractional financial model of awareness, where the studies are represented at the steps following: **One**: The schematic of the model is suggested. **Two**: The disease-free equilibrium point (**DFE**) and the stability of the equilibrium point are discussed. **Three**: The stability of the model is fulfilled by drawing the Lyapunov exponents and Poincare map. **Fourth**: The existence of uniformly stable solutions have discussed. **Five**: The Caputo is described as the fractional derivative. **Six**: Fractional optimal control for **NFFMA** is discussed by clarifying the fractional optimal control through drawing before and after control. **Seven**: Reduced differential transform method **(RDTM)** and Sumudu Decomposition Method (**SDM**) are used to take the resolution of an **NFFMA**. Finally, we display that **SDM** and **RDTM** are highly identical.

§1 Foreword

It is recognized that the declaration's goal is to convince buyers to buy products, relying on the general necessity of these products to show that they differ as a distinctive brand from other products to support buyers to purchase them [3]. There are many ways to turn the customer's thinking about the products and services offered to them. One of them is advertising through messages. These messages are through physical media, such as newspapers, magazines televisions, and radios. This mission can be through simple media, such as websites and drawing out messages [3]. It is very important to study advertising strategies to increase sales to achieve the highest profit for the company [32]. Thus, it is much more useful to study and create an

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Table 1. The parameter values and their definition.

Parameter	Definition
$N(t)$	Population total with time
$x_1(t)$	Number of a set of persons who do not know the entity of the produce
$x_2(t)$	Number of a set of persons who know about the product but have not yet buy
$x_3(t)$	Number of the group of people who have bought the product
u	Knowledge, this changs $x_1(t)$ into the scope one $x_2(t)$
v	Try advertisement, this carries $x_2(t)$ into the purchased one $x_3(t)$
a	Trial rate
k	Connect ratio
δ	Transform ratio
μ_b^α	Birth ratio
μ_d^α	Death rate
$ux_1(t)$	Total number of persons carry to the aware group $x_2(t)$ via declaration
$(N(t)-x_1(t))$	Connect and report a total of $k(N(t)-x_1(t))$, out of which only $x_1(t)\ /N(t)$.

appropriate dynamic and to represent time-based selling and public opinion [5]. There are also many approach models to appear in the relationship between advertising that identifies snags from the marketing and economic management point of view. Advertising policies are analyzed over time by dynamic models described as differential equations where sell a lot, deals, and all severe conditions variables are constantly changing. Respect for time. The purpose of advertising is always different. For example, some advertisements are to compare two, three, or more brands, and for another purpose, such as introducing a novel product to the mart based on these goals, advertising types are created. Commonly, the action of advertising is forever late in time, and it is necessary to integrate the memory of different models of a declaration, so models that rely on the previous cases in the current cases have not only their initial previous cases appropriate to describe strategies for the declaration. Latterly, (**FC**) has acquired great circulation and significance due to its catchy implementation as a new model work in an assortment of engineering and scientific domains ([1]-[58]), such as viscoelasticity [1] and thermoelasticity ([2], [3], [59]). The best method fractional models are led as FDEs.

The prime object of this manuscript is to propose a prorated discuss among STM and RDTM for solving NFFMA [3]:

$$\begin{bmatrix} D^\alpha x_1 \\ D^\alpha x_2 \\ D^\alpha x_3 \end{bmatrix} = \begin{bmatrix} -u^\alpha-\mu_d^\alpha & 0 & 0 & -k^\alpha & \mu_b^\alpha \\ u^\alpha & -a^\alpha-v^\alpha-\mu_d^\alpha & \delta^\alpha & k^\alpha & 0 \\ 0 & a^\alpha+v^\alpha & -\delta^\alpha-\mu_d^\alpha & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \frac{x_1(N-x_1)}{N} \\ N \end{bmatrix}, \tag{1}$$

with given initial condition:

$$x_1(t)=x_{10},\ x_2(t)=x_{20},\ x_3(t)=x_{30}. \tag{2}$$

Definition 1 The D^α is Caputo fractional derivative is known ([4]-[6], [37]):

$$D^\alpha z\left(r\right)=\begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{z^{(n)}(\eta)}{(r-\eta)^{\alpha-n+1}}d\eta, & 0\le n-1<\alpha<n, \\ z^{(n)}\left(r\right), & \alpha=n\in N. \end{cases} \tag{3}$$

For further particular about the basal simplification and advantages of fractional derivatives see ([4]-[6], [37]).

The paper is structured into five sections. In section 2, we discuss the equilibrium points, stability, existence of uniformly stable solution **NFFMA**, clarify the dynamics of the model between Lyapunov exponents, and Poincare maps. Optimal control for **NFFMA** is discussed in Section 3. In section 4, we show an example to display the activity of using (**STM**) and (**RDTM**) to solve **NFFMA**. Finally, pertinent conclusions are drawn in section 5.

§2 Equilibrium and Stability of fractional financial models of awareness

In this section, we discuss the equilibrium point and the stability of **NFFMA** (1).

2.1 Equilibrium points

We discuss the equilibrium points of the **NFFMA**. The model has one equilibrium point, more details about the stability and equilibrium point of models fractional in ([27]-[31]).

Hence, we resolve the next equations to find the equilibrium point:

$$\begin{aligned} D^{\alpha}x_1 &= -u^{\alpha}x_1 - \frac{k^{\alpha}}{N}x_1(N-x_1) + \mu_b^{\alpha}N - \mu_d^{\alpha}x_1 = 0, \\ D^{\alpha}x_2 &= u^{\alpha}x_1 + \frac{k^{\alpha}}{N}x_1(N-x_1) - (a^{\alpha}+v^{\alpha})x_2 + \delta^{\alpha}x_3 - \mu_d^{\alpha}x_2 = 0, \\ D^{\alpha}x_3 &= (a^{\alpha}+v^{\alpha})x_2 - \delta^{\alpha}x_3 - \mu_d^{\alpha}x_3 = 0. \end{aligned} \tag{4}$$

Equation (4) leads to getting the one equilibrium point

$$E_0\left(QN, \frac{-\delta^{\alpha}Q + \mu_d^{(-\alpha)}\delta^{\alpha}\mu_b^{\alpha} - \mu_d^{\alpha}Q + \mu_b^{\alpha}}{T}, \frac{T_1a^{\alpha}\mu_b^{\alpha} - Nv^{\alpha}Q + N\mu_d^{(-\alpha)}v^{\alpha}\mu_b^{\alpha}}{T}\right),$$

where Q=$\sqrt{\mu_b^{\alpha} - u^{\alpha} - \mu_d^{\alpha}}$ and T=$(\delta^{\alpha} + a^{\alpha} + v^{\alpha} + \mu_d^{\alpha})\,, T_1 = -Na^{\alpha}Q + N\mu_d^{(-\alpha)}$.

2.2 Studying the stability

We calculate the Jacobian matrix J for the model (1) as next:

$$J = \begin{bmatrix} -u^{\alpha} - \mu_d^{\alpha} - k^{\alpha} + \frac{2k^{\alpha}}{N}x_1 & 0 & 0 \\ u^{\alpha} + k^{\alpha} - \frac{2k^{\alpha}}{N}x_1 & -a^{\alpha} - v^{\alpha} - \mu_d^{\alpha} & \delta^{\alpha} \\ 0 & a^{\alpha} + v^{\alpha} & -\delta^{\alpha} - \mu_d^{\alpha} \end{bmatrix},$$

at the equilibrium point E_0 the Jacobian matrix J(E_0) model (1) is given via

$$J(E_0) = \begin{bmatrix} -u^{\alpha} - \mu_d^{\alpha} - k^{\alpha} + 2k^{\alpha}Q & 0 & 0 \\ u^{\alpha} + k^{\alpha} - 2k^{\alpha}Q & -a^{\alpha} - v^{\alpha} - \mu_d^{\alpha} & \delta^{\alpha} \\ 0 & a^{\alpha} + v^{\alpha} & -\delta^{\alpha} - \mu_d^{\alpha} \end{bmatrix}.$$

Consequently, we have

$$|J - \lambda I| = \begin{vmatrix} -u^{\alpha} - \mu_d^{\alpha} - k^{\alpha} + 2k^{\alpha}Q - \lambda & 0 & 0 \\ u^{\alpha} + k^{\alpha} - 2k^{\alpha}Q & -a^{\alpha} - v^{\alpha} - \mu_d^{\alpha} - \lambda & \delta^{\alpha} \\ 0 & a^{\alpha} + v^{\alpha} & -\delta^{\alpha} - \mu_d^{\alpha} - \lambda \end{vmatrix} = 0.$$

Then, the eigenvalues given by

$$\lambda = -\mu_d^\alpha \quad , \quad \lambda_2 = -a^\alpha - \delta^\alpha - \mu_d^\alpha - \upsilon^\alpha \quad , \quad \lambda_3 = -(u^\alpha + \mu_d^\alpha + k^\alpha - 2k^\alpha Q),$$

the solution is stable.

2.3 Clarify Lyapunov exponents and Poincare map

Figures 2-4 clarify Lyapunov exponents in different time periods. Figures 5-10 clarify the system by Poincare map for 3 several values of a, k and delta. All of which included model stability. At the fixed point, eigenvalues have negative, which means stability, and we guarantee its security by drawing its Lyapunov. The calculation utilized for deciding Lyapunov examples has been suggested in [38], see Figures 2-4. In Figure 2-4, we see that all Lyapunov examples have negative after little transient time that infers the framework is steady and approaches its fixed point. Additionally, we guarantee its soundness by drawing the model's Poincare guide as shown in Figures 5-10. In Figures 5-10, plainly, all conditions of the framework go to its fixed point.

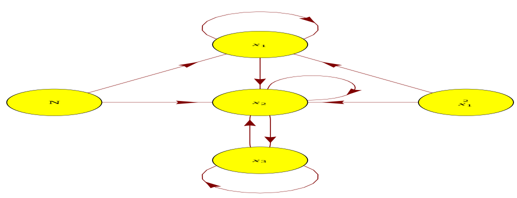

Figure 1. The suggested schematic of the model.

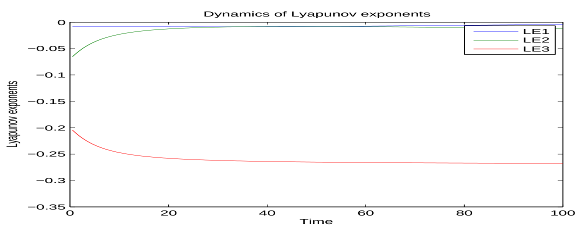

Figure 2. Represent the Lyapunov exponents' dynamics for the model.

2.4 Existence of Uniformly stable solution

Let

$$f_1(x_1, x_2, x_3) = -u^\alpha x_1 - \frac{k^\alpha}{N} x_1(N - x_1) + \mu_b^\alpha N - \mu_d^\alpha x_1,$$

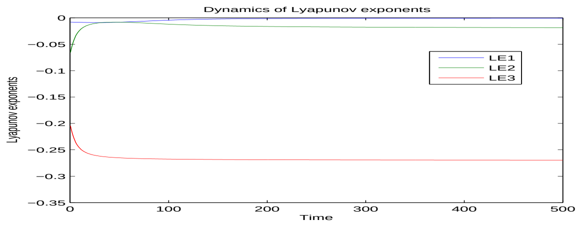

Figure 3. Represent the Lyapunov exponents' dynamics for the model.

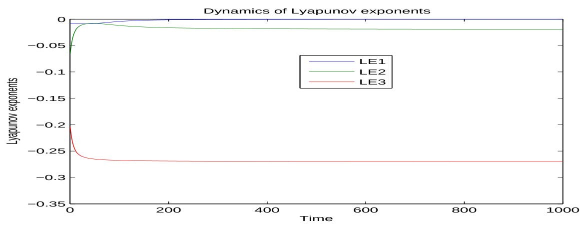

Figure 4. Represent the Lyapunov exponents' dynamics for the model.

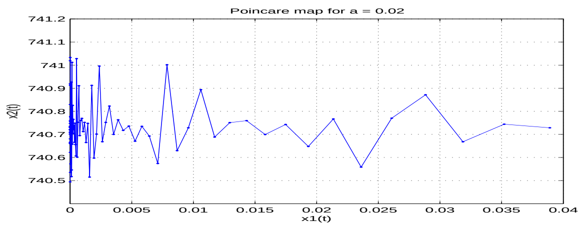

Figure 5. Clarify the Poincare map of the model between x_1, x_2 and a.

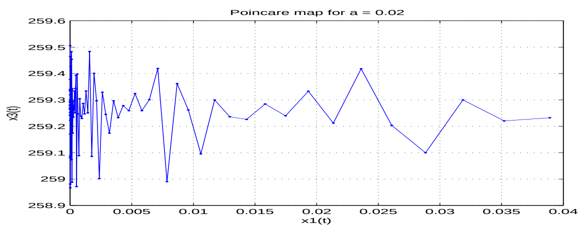

Figure 6. Clarify the Poincare map of the model between x_1,x_3 and a.

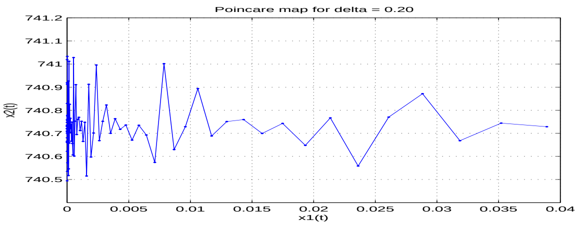

Figure 7. Clarify the Poincare map of the model between x_1, x_2 and δ.

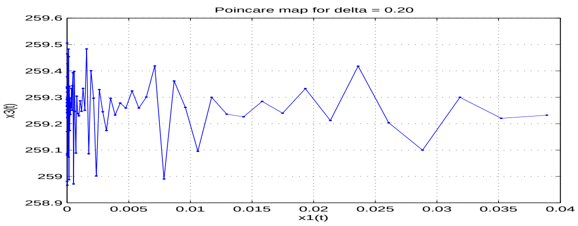

Figure 8. Clarify the Poincare map of the model between x_1, x_2 and δ.

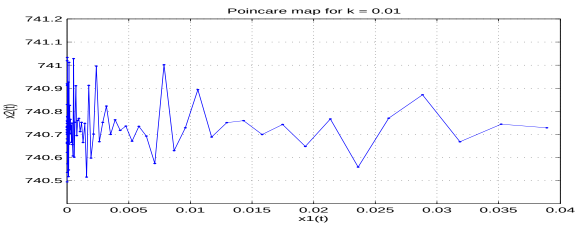

Figure 9. Clarify the Poincare map of the model between x_1, x_2 and k.

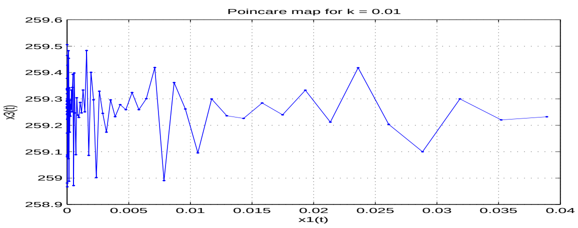

Figure 10. Clarify the Poincare map of the model between x_1, x_3 and k.

$$f_2(x_1,x_2,x_3) = u^\alpha x_1 + \frac{k^\alpha}{N} x_1(N-x_1) - (a^\alpha + v^\alpha)x_2 + \delta^\alpha x_3 - \mu_d^\alpha x_2,$$

$$f_3(x_1,x_2,x_3) = (a^\alpha + v^\alpha)x_2 - \delta^\alpha x_3 - \mu_d^\alpha x_3.$$

Let

$$D = \{x_1, x_2, x_3 \in \Re : |x_1, x_2, x_3| \leq a, \quad t \in [0,T]\}.$$

We have at D :

$$\frac{\partial f_1}{\partial x_1} = -u^\alpha - \mu_d^\alpha - k^\alpha + 2k^\alpha Q, \quad \frac{\partial f_1}{\partial x_2} = 0, \frac{\partial f_1}{\partial x_3} = 0,$$

$$\frac{\partial f_2}{\partial x_1} = u^\alpha + k^\alpha - 2k^\alpha Q, \quad \frac{\partial f_2}{\partial x_2} = -a^\alpha - v^\alpha - \mu_d^\alpha, \frac{\partial f_2}{\partial x_3} = \delta^\alpha,$$

$$\frac{\partial f_3}{\partial x_1} = 0, \quad \frac{\partial f_3}{\partial x_2} = a^\alpha + v^\alpha, \frac{\partial f_3}{\partial x_3} = -\delta^\alpha - \mu_d^\alpha,$$

$$\left|\frac{\partial f_1}{\partial x_1}\right| \leq k_1, \left|\frac{\partial f_2}{\partial x_2}\right| \leq k_2, \left|\frac{\partial f_3}{\partial x_3}\right| \leq k_3,$$

where $k_1 k_2$ and k_3 are positive constants. This means that each of the three functions f_1, f_2 and f_3 fulfill the condition Lipschitz with the three cases, and consequently any of the three functions are continuous absolutely with the three cases.

§3 Optimal control

Let us see the case model given in Eqs. (1), in $\Re^3$, with the set of accepted control functions for more details in ([33]-[36], [39]):

$$\Omega = \left\{u(.)v(.) \in (L^\infty(0,T_f)^2) \mid 0 \leq u(.)v(.) \leq 1, \forall t \in [0,T_f]\right\},$$

where T_f is the final time, $u(.)$ and $v(.)$ are controls functions.

The objective function is known as the next.

$$J(u(.),v(.)) = \int_0^{T_f} \left[Ax_1(t) + Bu^2(t) + Cv^2(t)\right] dt, \tag{5}$$

where A, B, and C illustrate the rule constants.

The premier point in FOCPs is to get the optimal controls $u(.)$ and $v(.)$, which minimizes the dependent objective function:

$$J(u,v) = \int_0^{T_f} \eta\left[x,y,z,u,v,t\right] dt, \tag{6}$$

subjected to the constraint

$$D^\alpha x_1 = \xi_1, D^\alpha x_2 = \xi_2, D^\alpha x_3 = \xi_3, D^\alpha x_1 = \xi_i = \xi(x_1,x_2,x_3,u,v,t), \quad i = 1,2,3. \tag{7}$$

The dependent initial cases are contented:

$$x_1(0) = x_{10}, x_2(0) = x_{20}, x_3(0) = x_{30}. \tag{8}$$

To realize the FOCP, let us think of a revised objective (cost) function as directed:

$$J = \int_0^{T_f} \left[H(x_1, x_2, x_3, u, v, t) - \sum_{i=1}^{3} \lambda_i \xi_i(x_1, x_2, x_3, u, v, t) \right] dt, \tag{9}$$

where the Hamiltonian for the goal functional (9) and the control financial models of awareness (1) are given as follows:

$$H(x_1, x_2, x_3, u, v, t) = \eta(x_1, x_2, x_3, u, v, t) + \sum_{i=1}^{3} \lambda_i \xi_i(x_1, x_2, x_3, u, v, t), \tag{10}$$

$$\begin{aligned} H = Ax_1 + Bu^2 + Cv^2 + \lambda_1 \left[-u^\alpha x_1 - \frac{k^\alpha}{N} x_1 (N - x_1) + \mu_b^\alpha N - \mu_d^\alpha x_1 \right] \\ + \lambda_2 \left[u^\alpha x_1 + \frac{k^\alpha}{N} x_1 (N - x_1) - (a^\alpha + v^\alpha) x_2 + \delta^\alpha x_3 - \mu_d^\alpha x_2 \right] \\ + \lambda_3 \left[(a^\alpha + v^\alpha) x_2 - \delta^\alpha x_3 - \mu_d^\alpha x_3 \right]. \end{aligned} \tag{11}$$

From (9) and (11), we can deduce the sufficient and necessary conditions for FOPC as

$$D^\alpha \lambda_1 = \frac{\partial H}{\partial x_1}, D^\alpha \lambda_2 = \frac{\partial H}{\partial x_2}, D^\alpha \lambda_3 = \frac{\partial H}{\partial x_3}, \tag{12}$$

$$\frac{\partial H}{\partial u} = 0, \frac{\partial H}{\partial v} = 0, \tag{13}$$

$$D^\alpha x_1 = \frac{\partial H}{\partial \lambda_1}, D^\alpha x_2 = \frac{\partial H}{\partial \lambda_2}, D^\alpha x_3 = \frac{\partial H}{\partial \lambda_3}, \tag{14}$$

$$\lambda_j(T_f) = 0, \tag{15}$$

where $\lambda_j, j = 1, 2, 3$ are the multipliers Lagrange. Eqs.(13) and (14) appear the conditions necessary in terms of a Hamiltonian for the FOPC.

We arrive at the following theorem:

Theorem 1.

If u and v are optimal controls with the uniform state $x_1^*, x_2^*, and\ x_3^*$ then the next there exist adjoint variables $\lambda_i^*\ i = 1, 2, 3$ satisfies:

(i) Adjoint (co-state) equations

Laying the conditions in the content hypothesis and putting conditions (12) see, ([33]- [36]), we obtain the accompanying three conditions, which can be composed as follows:-

$$D^\alpha \lambda_1^* = A + \lambda_1^* (-u^\alpha - k^\alpha + \frac{2k^\alpha}{N} x_1 - \mu_d^\alpha) + \lambda_2^* (u^\alpha + k^\alpha - \frac{2k^\alpha}{N} x_1), \tag{16}$$

$$D^\alpha \lambda_2^* = \lambda_2^* (-a^\alpha - v^\alpha - \mu_d^\alpha) + \lambda_3^* (a^\alpha + v^\alpha), \tag{17}$$

$$D^\alpha \lambda_3^* = \lambda_2^* (\delta^\alpha) + \lambda_3^* (-\delta^\alpha - \mu_d^\alpha), \tag{18}$$

(ii) With transversality conditions:

$$\lambda_i^*(T_f) = 0, i = 1, 2, 3. \tag{19}$$

(iii) Optimality conditions

$$H(x_1^*, x_2^*, x_3^*, u^*, v^*, \lambda^*) = \min_{0 \le u^*, v^* \le 1} H(x_1^*, x_2^*, x_3^*, u^*, v^*, \lambda^*). \tag{20}$$

As well, the control functions u^*, v^* are offered by

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u^{\alpha^{-2}} = \frac{2B}{\alpha x_1^*(\lambda_1^* - \lambda_2^*)} \Rightarrow u^\alpha = \frac{2Bu^2}{\alpha x_1^*(\lambda_1^* - \lambda_2^*)}, \tag{21}$$

$$\frac{\partial H}{\partial v} = 0 \Rightarrow v^{\alpha^{-2}} = \frac{2C}{\alpha x_2^*(\lambda_2^* - \lambda_3^*)} \Rightarrow v^\alpha = \frac{2Cv^2}{\alpha x_2^*(\lambda_2^* - \lambda_3^*)}, \tag{22}$$

$$u^* = \min\left\{1, \max\left\{0, \frac{\alpha x_1^*(\lambda_1^* - \lambda_2^*)}{2B}\right\}\right\}, \tag{23}$$

$$v^* = \min\left\{1, \max\left\{0, \frac{\alpha x_2^*(\lambda_2^* - \lambda_3^*)}{2C}\right\}\right\}, \tag{24}$$

Argument. The system co-state Eqs. (16)-(18) have set from Eq.(14) where the Hamiltonian H^* is presented

$$H^* = A_1 x_1^* + Bu^{*2} + Cv^{*2} + \lambda_1^* D^\alpha x_1^* + \lambda_2^* D^\alpha x_2^* + \lambda_3^* D^\alpha x_3^*. \tag{25}$$

Further, the condition in Eq.(15) also satisfied, and the optimal control written in Eqs.(23)–(24) can be derived from Eq.(13).

Putting u^*, v^* in (1), the following state system can be found as:

$$D^\alpha x_1^* = -u^\alpha x_1^* - \frac{k^\alpha}{N} x_1^*(N - x_1^*) + \mu_b^\alpha N - \mu_d^\alpha x_1^*,$$

$$D^\alpha x_2^* = u^\alpha x_1^* + \frac{k^\alpha}{N} x_1^*(N - x_1^*) - (a^\alpha + v^\alpha) x_2^* + \delta^\alpha x_3^* - \mu_d^\alpha x_2^*,$$

$$D^\alpha x_3^* = (a^\alpha + v^\alpha) x_2^* - \delta^\alpha x_3^* - \mu_d^\alpha x_3^*.$$

To detail extra about optimal fractional control ([33]-[36]).

§4 Applications

In this section, two analytical methods are inserted to resolve FFMA (1) using we STM ([7]-[14], [19], [20]) and RDTM ([23]-[25]) with initial condition:

$$x_1(0) = 300, x_2(0) = 600, x_3(0) = 100, \tag{26}$$

with $N = 1000, a = 0.02, \delta = 0.2, u = 0.01, v = 0.05, k = 0.01$.

4.1 By using SDM

By taking the ST on both sides of Eq.(1) at $\mu_b = \mu_d = 0$ we get

$$\begin{aligned} S\left[D^\alpha x_1(t)\right] &= S\left[-u^\alpha x_1 - \frac{k^\alpha}{x_1+x_2+x_3}(x_1x_2+x_1x_3)\right], \\ S\left[D^\alpha x_2(t)\right] &= S\left[u^\alpha x_1 + \frac{k^\alpha}{x_1+x_2+x_3}(x_1x_2+x_1x_3) - (a^\alpha+v^\alpha)x_2 + \delta^\alpha x_3\right], \\ S\left[D^\alpha x_3(t)\right] &= S\left[(a^\alpha+v^\alpha)x_2 - \delta^\alpha x_3\right], \end{aligned} \tag{27}$$

using the property of the ST and the I.C. in Eq.(27),

$$\begin{aligned} S\left[x_1(t)\right] &= x_1(0) + w^\alpha S\left[-u^\alpha x_1 - \frac{k^\alpha}{x_1+x_2+x_3}(x_1x_2+x_1x_3)\right], \\ S\left[x_2(t)\right] &= x_2(0) + w^\alpha S\left[u^\alpha x_1 + \frac{k^\alpha}{x_1+x_2+x_3}(x_1x_2+x_1x_3) - (a^\alpha+v^\alpha)x_2 + \delta^\alpha x_3\right], \\ S\left[x_3(t)\right] &= x_3(0) + w^\alpha S\left[(a^\alpha+v^\alpha)x_2 - \delta^\alpha x_3\right], \end{aligned}$$

$$\begin{aligned} S\left[x_1(t)\right] &= 300 + w^\alpha S\left[-u^\alpha x_1 - \frac{k^\alpha}{x_1+x_2+x_3}(x_1x_2+x_1x_3)\right], \\ S\left[x_2(t)\right] &= 600 + w^\alpha S\left[u^\alpha x_1 + \frac{k^\alpha}{x_1+x_2+x_3}(x_1x_2+x_1x_3) - (a^\alpha+v^\alpha)x_2 + \delta^\alpha x_3\right], \\ S\left[x_3(t)\right] &= 100 + w^\alpha S\left[(a^\alpha+v^\alpha)x_2 - \delta^\alpha x_3\right]. \end{aligned} \tag{28}$$

Taking the Sumudu inverse of Eq.(28), we obtain

$$\begin{aligned} x_1(t) &= 300 + S^{-1}\left[w^\alpha S\left[-u^\alpha x_1 - \frac{k^\alpha}{x_1+x_2+x_3}(x_1x_2+x_1x_3)\right]\right], \\ x_2(t) &= 600 + S^{-1}\left[w^\alpha S\left[u^\alpha x_1 + \frac{k^\alpha}{x_1+x_2+x_3}(x_1x_2+x_1x_3) - (a^\alpha+v^\alpha)x_2 + \delta^\alpha x_3\right]\right], \\ x_3(t) &= 100 + S^{-1}\left[w^\alpha S\left[(a^\alpha+v^\alpha)x_2 - \delta^\alpha x_3\right]\right]. \end{aligned} \tag{29}$$

By assuming that:

$$x_1(t) = \sum_{n=0}^{\infty} x_{1n}(t), x_2(t) = \sum_{n=0}^{\infty} x_{2n}(t), x_3(t) = \sum_{n=0}^{\infty} x_{3n}(t). \tag{30}$$

By putting Eq.(30) in Eq.(29) we given

$$\sum_{n=0}^{\infty} x_{1n} = 300$$

$$+S^{-1}\left[w^\alpha S\left[-u^\alpha \sum_{n=0}^{\infty} x_{1n} - \frac{k^\alpha}{\sum_{n=0}^{\infty} x_{1n} + \sum_{n=0}^{\infty} x_{2n} + \sum_{n=0}^{\infty} x_{3n}}(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n)\right]\right],$$

$$\sum_{n=0}^{\infty} x_{2n} = 600$$

$$+S^{-1}\left[w^\alpha S\left[\begin{array}{c} u^\alpha \sum_{n=0}^{\infty} x_{1n}(t) + \frac{k^\alpha}{\sum_{n=0}^{\infty} x_{1n}+\sum_{n=0}^{\infty} x_{2n}+\sum_{n=0}^{\infty} x_{3n}}(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n) \\ -(a^\alpha+v^\alpha)\sum_{n=0}^{\infty} x_{2n} + \delta^\alpha \sum_{n=0}^{\infty} x_{3n} \end{array}\right]\right]$$

$$\sum_{n=0}^{\infty} x_{3n} = 100$$
$$+S^{-1}\left[w^\alpha S\left[(a^\alpha+v^\alpha)\sum_{n=0}^{\infty} x_{2n} - \delta^\alpha \sum_{n=0}^{\infty} x_{3n}\right]\right], \tag{31}$$

where A_n, B_n are Adomian polynomials which explain nonlinear term. So Adomian polynomials are presented as next:

$$A_n(t) = x_1(t)x_2(t), B_n(t) = x_1(t)x_3(t)$$

The Adomian polynomial's small components are presented.

$$\begin{aligned} A_0(t) &= x_{10}(t)x_{20}(t) \\ A_1(t) &= x_{10}(t)x_{21}(t) + x_{11}(t)x_{20}(t) \\ A_2(t) &= x_{10}(t)x_{22}(t) + x_{11}(t)x_{21}(t) + x_{12}(t)x_{20}(t) \\ &\vdots \end{aligned}$$

$$\begin{aligned} B_0(t) &= x_{10}(t)x_{30}(t) \\ B_1(t) &= x_{10}(t)x_{31}(t) + x_{11}(t)x_{30}(t) \\ B_2(t) &= x_{10}(t)x_{32}(t) + x_{11}(t)x_{31}(t) + x_{12}(t)x_{30}(t) \\ &\vdots \end{aligned}$$

Then we have

$$x_{10} = 300, x_{20} = 600, x_{30} = 100, A_0 = 180000, B_0 = 30000$$

$$\begin{aligned} x_{1k+1} &= S^{-1}\left[w^\alpha S\left[-u^\alpha x_{1k} - \frac{k^\alpha}{x_{1k}+x_{2k}+x_{3k}}(A_k+B_k)\right]\right] \\ x_{2k+1} &= S^{-1}\left[w^\alpha S\left[u^\alpha x_{1k} + \frac{k^\alpha}{x_{1k}+x_{2k}+x_{3k}}(A_k+B_k) - (a^\alpha+v^\alpha)x_{2k} + \delta^\alpha x_{3k}\right]\right] \\ x_{3k+1} &= 100 + S^{-1}\left[w^\alpha S\left[((a^\alpha+v^\alpha)x_{2k} - \delta^\alpha x_{3k}\right]\right] \end{aligned}$$

$$\begin{aligned} x_{11} &= S^{-1}\left[w^\alpha S\left[-u^\alpha x_{10} - \frac{k^\alpha}{x0+x_{20}+x_{30}}(A_0+B_0)\right]\right] \\ x_{21} &= S^{-1}\left[w^\alpha S\left[u^\alpha x_{10} + \frac{k^\alpha}{x_{10}+x_{20}+x_{30}}(A_0+B_0) - (a^\alpha+v^\alpha)x_{20} + \delta^\alpha x_{30}\right]\right] \\ x_{31} &= 100 + S^{-1}\left[w^\alpha S\left[((a^\alpha+v^\alpha)x_{20} - \delta^\alpha x_{30}\right]\right] \end{aligned}$$

$$x_{11} = \frac{-300u^\alpha - 210k^\alpha}{\Gamma(\alpha+1)}t^\alpha$$
$$x_{21} = \frac{-300u^\alpha - 210k^\alpha - 600(a^\alpha+v^\alpha) + 100\delta^\alpha}{\Gamma(\alpha+1)}t^\alpha$$
$$x_{31} = \frac{600(a^\alpha+v^\alpha) - 100\delta^\alpha}{\Gamma(\alpha+1)}t^\alpha$$

If we put

$$\begin{aligned} A &= -300u^{\alpha} - 210k^{\alpha} \\ B &= -300u^{\alpha} - 210k^{\alpha} - 600(a^{\alpha} + v^{\alpha}) + 100\delta^{\alpha} \\ C &= 600(a^{\alpha} + v^{\alpha}) - 100\delta^{\alpha} \end{aligned}$$

Since

$$\begin{aligned} x_1(t) &= x_{10} + x_{11} + x_{12} + \cdots \\ x_1(t) &= x_{20} + x_{21} + x_{22} + \cdots \\ x_3(t) &= x_{30} + x_{31} + x_{32} + \cdots , \end{aligned} \tag{32}$$

$$\begin{aligned} x_1 &= 300 + \left[\frac{A}{\Gamma(\alpha+1)} - \frac{k^{\alpha} \times Y_4}{(A+B+C) \times \Gamma(\alpha+1)}\right] t^{\alpha} - \frac{u^{\alpha} A}{\Gamma(2\alpha+1)} t^{2\alpha} + \cdots \\ x_2 &= 600 + \left[\frac{Bt^{\alpha}}{\Gamma(\alpha+1)} + \frac{k^{\alpha} t^{\alpha} Y_4}{(A+B+C) \times \Gamma(\alpha+1)}\right] + \frac{(u^{\alpha} A - B(a^{\alpha} + v^{\alpha} t^{2\alpha}) + C\delta^{\alpha})}{\Gamma(2\alpha+1)} \\ x_3 &= 100 + \frac{C}{\Gamma(\alpha+1)} t^{\alpha} + \frac{B(a^{\alpha} + v^{\alpha}) - C\delta^{\alpha}}{\Gamma(2\alpha+1)} t^{2\alpha} + \cdots . \end{aligned}$$

4.2 By using RDTM

Putting RDTM to Eq.(1), we get repetition relations as:

$$\begin{aligned} x_{1k+1} &= \frac{\Gamma(k\alpha+1)}{\Gamma\left[\alpha(k+1)+1\right]} \left[-u^{\alpha} x_{1k} - Y_1\right] \\ x_{2k+1} &= \frac{\Gamma(k\alpha+1)}{\Gamma\left[\alpha(k+1)+1\right]} \left[u^{\alpha} x_{1k} + Y_2\right] \\ x_{3k+1} &= \frac{\Gamma(k\alpha+1)}{\Gamma\left[\alpha(k+1)+1\right]} \left[((a^{\alpha} + v^{\alpha}) x_{2k} - \delta^{\alpha} x_{3k}\right]. \end{aligned}$$

where

$$Y_1 = \left[\frac{k^{\alpha}}{x_{1k} + x_{2k} + x_{3k}} (\sum_{r=0}^{k} x_{1r} x_{2(k-r)} + \sum_{r=0}^{k} x_{1r} x_{3(k-r)})\right]$$

$$Y_2 = \frac{k^{\alpha}}{x_{1k} + x_{2k} + x_{3k}} (\sum_{r=0}^{k} x_{1r} x_{2(k-r)} + \sum_{r=0}^{k} x_{1r} x_{3(k-r)}) - Y_3.$$

By substituting Eq.(26), we have:

$$\begin{aligned} x_{11} &= \frac{1}{\Gamma\left[\alpha+1\right]} \left[-u^{\alpha} x_{10} - \frac{k^{\alpha}}{x_{10} + x_{20} + x_{30}} (x_{10} x_{20} + x_{10} x_{30})\right] \\ x_{21} &= \frac{1}{\Gamma\left[\alpha+1\right]} \left[u^{\alpha} x_{10} + \frac{k^{\alpha}}{x_{10} + x_{20} + x_{30}} (x_{10} x_{20} + x_{10} x_{30}) - Y_3\right] \\ x_{31} &= \frac{1}{\Gamma\left[\alpha+1\right]} \left[((a^{\alpha} + v^{\alpha}) x_{20} - \delta^{\alpha} x_{30}\right] \end{aligned}$$

$$Y_3 = (a^{\alpha} + v^{\alpha}) x_{20} + \delta^{\alpha} x_{30}.$$

In saw of the differential inverse transform, the differential transform series solution

$$x_1(t)=\sum_{n=0}^{\infty}x_{1n}t^{\alpha n},x_2(t)=\sum_{n=0}^{\infty}x_{2n}t^{\alpha n},x_3(t)=\sum_{n=0}^{\infty}x_{3n}t^{\alpha n}$$

We obtain the solution as

$$\begin{aligned}
x_1(t) &= 300+\frac{A}{\Gamma(\alpha+1)}t^{\alpha}-\left[\frac{u^{\alpha}A}{\Gamma(2\alpha+1)}+\frac{k^{\alpha}\times Y_4\times\Gamma(\alpha+1)}{(A+B+C)\times\Gamma(2\alpha+1)}\right]t^{2\alpha}\\
x_2(t) &= 600+\frac{B}{\Gamma(\alpha+1)}t^{\alpha}+\left[\frac{(u^{\alpha}A-Y_6+C\delta^{\alpha})}{\Gamma(2\alpha+1)}+\frac{k^{\alpha}Y_4\times\Gamma(\alpha+1)}{Y_5\times\Gamma(2\alpha+1)}\right]t^{2\alpha}\\
x_3(t) &= 100+\frac{C}{\Gamma(\alpha+1)}t^{\alpha}+\frac{B(a^{\alpha}+v^{\alpha})-C\delta^{\alpha}}{\Gamma(2\alpha+1)}t^{2\alpha}+\cdots.
\end{aligned}$$

$$Y_4=(700A+300B+300C),Y_5=(A+B+C),Y_6=B(a^{\alpha}+v^{\alpha}).$$

4.3 Clarify the optimal control through drawing before and after control

Clarifying the fractional optimal control through drawing before and after control. Where in Figures 11-13, the fractional financial awareness solution is shown before control at $\alpha=0.85$ using STM and RDTM. Figures 14-16 show the dynamics of solutions of the convergent solution of order fractional financial awareness model after control at $\alpha=0.85$ using STM and RDTM. It is no doubt that the activity in this way is greatly increased by the calculation of further terms x_1, x_2 *and* x_3 by using STM and RDTM and show illustrate the phase spaces.

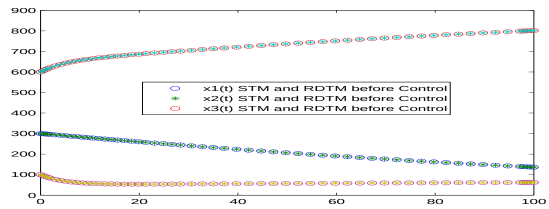

Figure 11. The solutions of x_1, x_2, x_3 before control using STM and RDTM at $\alpha=0.85$.

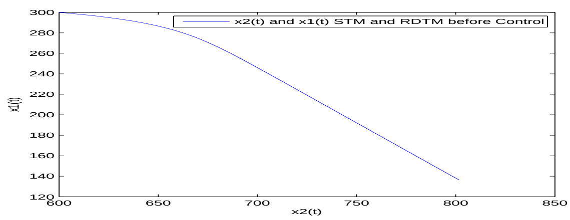

Figure 12. The solutions of x_1 and x_2 before control using STM and RDTM at $\alpha=0.85$.

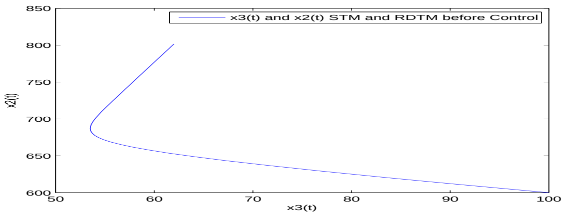

Figure 13. The solutions of x_2 and x_3 before control using STM and RDTM at $\alpha = 0.85$.

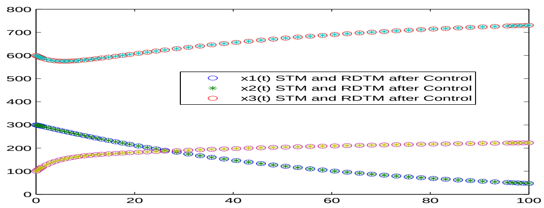

Figure 14. The solutions of x_1, x_2, x_3 after control at $\alpha = 0.85$.

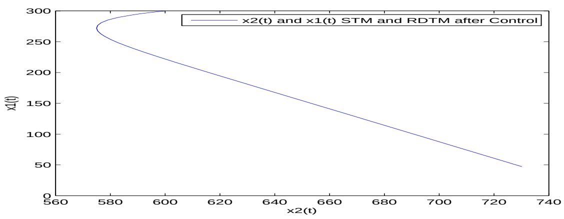

Figure 15. The solutions of x_1 and x_2 after control using STM and RDTM at $\alpha = 0.85$.

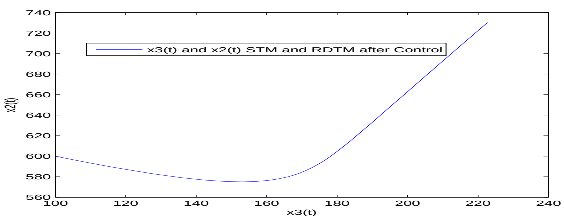

Figure 16. The solutions of x_2 and x_3 after control using STM and RDTM at $\alpha = 0.85$.

§5 Conclusions

In this paper, the graphical of the model is suggested. The (**DFE**) and the stability of the equilibrium point has clarified. The stability of the model has been satisfied by drawing the Lyapunov exponents and Poincare map. The existence of uniformly stable solutions is represented. The Caputo is described as the fractional derivative. Fractional optimal control for **NFFMA** has discussed, through clarifying the fractional optimal control through drawing before and after control. **RDTM** and **SDM** are using to take the resolution of an **NFFMA**. We are displaying that **SDM** and **RDTM** are highly identical. Finally, novel research has been carried out on past studies of one of the leading mathematical models that dub the global economic movement, and that is described as an **NFFMA**, where the researched at the upper.

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Data Availability

The information applied in this research is ready from the author at request.

Declarations

Conflict of interest The authors declare no conflict of interest.

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