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On Boolean elements and derivations in 2-dimension linguistic lattice implication algebras

ZHU Hua[1,3] ZHAO Jian-bin[2,4,*]

Abstract. A 2-dimension linguistic lattice implication algebra (2DL-LIA) can build a bridge between logical algebra and 2-dimension fuzzy linguistic information. In this paper, the notion of a Boolean element is proposed in a 2DL-LIA and some properties of Boolean elements are discussed. Then derivations on 2DL-LIAs are introduced and the related properties of derivations are investigated. Moreover, it proves that the derivations on 2DL-LIAs can be constructed by Boolean elements.

§1 Introduction

In real life, human intelligent activities are often associated with fuzziness and incomparability. As two kinds of uncertainty [37], fuzziness and incomparability exist not only in the processed object itself, but also in the course of the object being dealt with. Lattice-valued logic, as an important non-classical logic, has been extensively studied to establish the logical foundation for uncertainty inference[33, 34]. Accordingly, in order to provide algebraic semantics with lattice-valued logic, Xu et al. [35] proposed the concept of lattice implication algebras (LIAs). By use of the algebraic structures of LIAs, we can describe the relationships between uncertain information, especially for incomparable relationships. A lot of literatures [10, 15, 16, 47] have researched algebraic structures and properties of LIAs. Meanwhile, LIAs have been extended to lattice implication ordered semigroups [24], residuated lattices [14], linguistic truth-valued intuitionistic fuzzy lattices [49], linguistic truth-valued lattice implication algebras (L-LIAs) [36] and 2-dimension linguistic lattice implication algebras (2DL-LIAs) [46].

Zadeh [39] put forward the notion of fuzzy linguistic information, which is important for describing qualitative attributes such as low, medium and high. For precisely representing fuzzy

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*Corresponding author.

linguistic information, Zhu et al. [48] proposed the concept of 2-dimension fuzzy linguistic information. The 2-dimension fuzzy linguistic information includes two common linguistic labels: one describes the evaluation result of alternatives, the other describes the self-assessment of the decision maker on the reliability of the given evaluation result. Further, aiming to precisely describe the relationships between 2-dimension fuzzy linguistic information, especially for incomparable relationships, Zhu et al. [46] gave the notion of a 2-dimension linguistic lattice implication algebra (2DL-LIA). Under the structure of 2DL-LIA, some important decision making methods are proposed to deal with 2-dimension linguistic information [41, 42, 43, 45]. A 2DL-LIA has not only the features of logical algebra but also the features of evaluation sets between fuzzy linguistic information. Therefore, it can build a bridge for logical algebras and 2-dimension fuzzy linguistic information.

The notion of derivation, which comes from the analytic theory, is also helpful to investigate algebraic structures and properties of various kinds of algebras. The derivation in a prime ring $(R;+,\cdot)$ has been proposed by Posner [25], which is a mapping $d: R\to R$ such that two conditions (1) $d(x+y)=d(x)+d(y)$ and (2) $d(x\cdot y)=d(x)\cdot y+x\cdot d(y)$ for all $x,y\in R$. After that, derivations on rings and near rings have been investigated by many researchers [2, 23]. In 2004, derivations on BCI-algebras have been introduced by Jun et al. [11] and further studied in [5, 9, 18, 19, 20, 21, 22]. Besides, derivations on regular algebras [3], derivations on CSL-algebras [40], derivations on f-algebras [12], derivations on basic algebras [13], and derivations on L-LIAs [44] have been studied by different researchers. Moreover, derivations on lattices have been discussed in [6, 30, 31, 32]. Furthermore, derivations on MV-algebras and GMV-algebras have been investigated in [1, 7, 26, 38]. Especially, a derivation on a residuated lattice $(L,\wedge,\vee,\odot,\to,0,1)$ is proposed by He et al. [8], which is a mapping $d: L\to L$ satisfying the conditions $d(x\odot y)=(d(x)\odot y)\vee(x\odot d(y))$ for all $x,y\in L$.

Inspired by the above-mentioned work, especially by derivations on rings [25] and derivations on residuated lattices [8], derivations on 2DL-LIAs are proposed in this paper. This paper is organized as follows: Section 2 reviews some basic concepts about LIAs and 2DL-LIAs. In Section 3, a Boolean element is proposed in a 2DL-LIA, and some properties of Boolean elements are investigated. Section 4 introduces derivations on 2DL-LIAs and discusses some properties of derivations. The conclusions are drawn in Section 5.

§2 Preliminaries

This section gives some results about lattice implication algebras and 2-dimension linguistic lattice implication algebras.

2.1 Lattice implication algebras

For a LIA [35], we mean a bounded lattice $(L,\vee,\wedge,0,1)$ with order-reversing involution $'$, in which 0 and 1 are the smallest and the greatest element of L respectively, and a binary operation $\to$ satisfying the following axioms:

(I_1) $x\to(y\to z)=y\to(x\to z)$;

(I_2) $x \to x = 1$;
(I_3) $x \to y = y' \to x'$;
(I_4) if $x \to y = y \to x = 1$, then $x = y$;
(I_5) $(x \to y) \to y = (y \to x) \to x$;
(L_1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
(L_2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$;
for all $x, y, z \in L$.

A LIA L is called a lattice H implication algebra (LHIA), if for all $x, y, z \in L$,

$$x \vee y \vee ((x \wedge y) \to z) = 1.$$

A lattice implication homomorphism is a mapping $f : L_1 \to L_2$ from LIAs L_1 to L_2, such that for any $x, y \in L_1$,

$$\begin{aligned} f(x \to y) &= f(x) \to f(y), \\ f(x \vee y) &= f(x) \vee f(y), \\ f(x \wedge y) &= f(x) \wedge f(y), \\ f(x') &= (f(x))'. \end{aligned}$$

Let L be a LIA, the binary operators $\otimes$ and $\oplus$ are defined as follows: for all $x, y \in L$,

$$x \otimes y = (x \to y')', x \oplus y = x' \to y.$$

***Theorem* 2.1.** [35] Let L be a LIA. Then L is a LHIA if and only if for all $x \in L$, $x \oplus x = x$, $x \otimes x = x$.

***Example* 2.1.** *(Lukasiewicz implication algebra on a finite chain L_n)* [35] Let L be a finite chain, $L = \{a_i | i = 1, 2, \cdots, n\}$ and $0 = a_1 \le a_2 \le \cdots \le a_n = 1$, where $n \in N$. For any $a_i, a_j \in L$, where $0 \le i, j \le n$ and $i, j \in N$, define operations $\vee, \wedge, \to$ and $'$ as follows:

$$\begin{aligned} a_i \vee a_j &= a_{max\{i,j\}}, \\ a_i \wedge a_j &= a_{min\{i,j\}}, \\ a_i \to a_j &= a_{min\{n-i+j,n\}}, \\ (a_i)' &= a_{n-i+1}. \end{aligned}$$

Then $(L, \vee, \wedge, ', \to, a_1, a_n)$ is a LIA, denoted by L_n.

Definition 2.1. [4, 35] Let L_{m+1}, L_{n+1} be two Łukasiewicz implication algebras, $m, n \in N$ and $L_{m+1} = \{a_0, a_1, \cdots, a_m\} : a_0 \le a_1 \le \cdots \le a_m$, $L_{n+1} = \{b_0, b_1, \cdots, b_n\} : b_0 \le b_1 \le \cdots \le b_n$. Define the direct product of L_{m+1} and L_{n+1} as follows: $L_{m+1} \times L_{n+1} = \{(a, b) | a \in L_{m+1}, b \in L_{n+1}\}$. The operations on $L_{m+1} \times L_{n+1}$ are defined respectively as follows: for any (a_i, b_k), $(a_j, b_l) \in L_{m+1} \times L_{n+1}$,

$$\begin{aligned} (a_i, b_k) \vee (a_j, b_l) &= (a_i \vee a_j, b_k \vee b_l) = (a_{\max\{i,\ j\}}, b_{\max\{k,\ l\}}), \\ (a_i, b_k) \wedge (a_j, b_l) &= (a_i \wedge a_j, b_k \wedge b_l) = (a_{\min\{i,\ j\}}, b_{\min\{k,\ l\}}), \\ (a_i, b_k) \to (a_j, b_l) &= (a_i \to a_j, b_k \to b_l) = (a_{\min\{m-i+j,\ m\}}, b_{\min\{n-k+l,\ n\}}), \\ (a_i, b_k)' &= (a_i', b_k') = (a_{m-i}, b_{n-k}). \end{aligned}$$

Then $(L_{m+1} \times L_{n+1}, \vee, \wedge, ', \rightarrow, (a_0, b_0), (a_m, b_n))$ is a LIA, denoted by $L_{(m+1)\times(n+1)}$.

Let L be a LIA, for all $x, y, z \in L$, define the partial relation $\leq$ in L as $x \leq y \Longleftrightarrow x \rightarrow y = 1$, then the following hold [35]:

(1) $x \rightarrow 0 = x'$;

(2) $x \vee y = (x \rightarrow y) \rightarrow y$;

(3) $If\ (x \otimes y) \leq z, then\ y \leq (x \rightarrow z)$;

(4) $x \otimes y \leq x \wedge y \leq x \vee y \leq x \oplus y$;

(5) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z),\ x \otimes (y \wedge z) = (x \otimes y) \wedge (x \otimes z)$;

(6) $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z),\ x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.

For more details of LIAs, we refer to the monograph [35].

2.2 2-dimension linguistic lattice implication algebras

Firstly, linguistic label sets and 2-dimension fuzzy linguistic information are reviewed as follows:

Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic label set with the cardinality $g + 1$, where $g \in N$. For any $s_i, s_j \in S$, where $i, j \in \{0, 1, \cdots, g\}$, the following properties should hold [17]:

(1) if $i \leq j$, then $s_i \leq s_j$;

(2) $(s_i)' = s_{g-i}$;

(3) if $s_i \leq s_j$, then $\max(s_i, s_j) = s_j$;

(4) if $s_i \leq s_j$, then $\min(s_i, s_j) = s_i$.

In some real decision making environments, a decision maker provides the evaluation result of alternatives by use of linguistic labels as well as his (or her) self-appraisal. For example, when an expert is invited to express his (or her) opinions on a submitted journal paper, there are always two linguistic label sets provided, where one linguistic label set is given to evaluate the submitted paper, the other is supplied to evaluate the familiar degree of the expert with the contents of the submitted paper. Aiming to describe such phenomena, Zhu et al. [48] introduced the notion of 2-dimension fuzzy linguistic information, which is reviewed as follows.

Definition 2.2. [48] Let $S = \{s_0, s_1, \cdots, s_g\}$ and $H = \{h_0, h_1, \cdots, h_t\}$ be two linguistic label sets, where $g + 1$ is the cardinality of S and $t + 1$ is the cardinality of H, $g, t \in N$. $\hat{r}$=(s_i, h_j) is called a 2-dimension linguistic label (2DLL), in which $h_j \in H$ represents the assessment information about the alternative given by the decision maker, and $s_i \in S$ represents the self-assessment of the decision maker.

In order to precisely describe the relationships between 2-dimension fuzzy linguistic information, a 2DL-LIA is constructed by combining two linguistic label sets with a LIA structure, which is reviewed as follows.

Definition 2.3. [46] Let $S = \{s_0, s_1, \cdots, s_g\}$: $s_0 \leq s_1 \leq \cdots \leq s_g$, $H = \{h_0, h_1, \cdots, h_t\}$: $h_0 \leq h_1 \leq \cdots \leq h_t$ be two linguistic label sets, $g, t \in N$ and $L_{(g+1)\times(t+1)}$ be a LIA as defined in Definition 2.1. Let a mapping $f : S \times H \rightarrow L_{(g+1)\times(t+1)}$ be defined such that

$f((s_i, h_j)) = (a_i, b_j)$, where $i \in \{0, 1, \cdots, g\}$, $j \in \{0, 1, \cdots, t\}$. Then f is a bijection, denoted its inverse mapping as f^{-1}. For any $(s_i, h_k), (s_j, h_l) \in S \times H$, define

$$\begin{aligned} (s_i, h_k) \vee (s_j, h_l) &= f^{-1}(f((s_i, h_k)) \vee f((s_j, h_l))), \\ (s_i, h_k) \wedge (s_j, h_l) &= f^{-1}(f((s_i, h_k)) \wedge f((s_j, h_l))), \\ (s_i, h_k) \to (s_j, h_l) &= f^{-1}(f((s_i, h_k)) \to f((s_j, h_l))), \\ (s_i, h_k)' &= f^{-1}((f(s_i, h_k))'). \end{aligned}$$

Then it is obvious to verify that $(S \times H, \vee, \wedge, \to, ', (s_0, h_0), (s_g, h_t))$ is a LIA, which is called a 2-dimension linguistic lattice implication algebra (2DL-LIA), whose Hasse Diagram is shown in Figure 1.

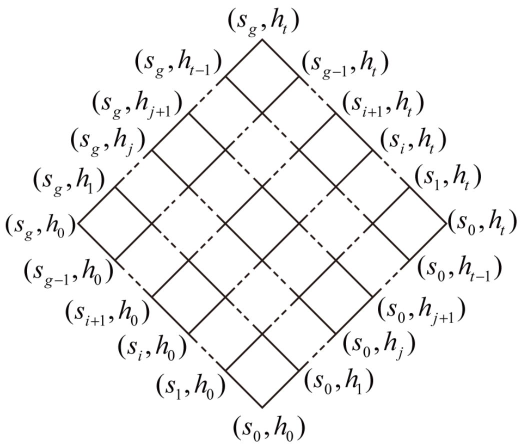

Figure 1. Hasse Diagram of 2DL-LIA.

In the following, $S \times H$ is always denoted as a 2DL-LIA, where $S = \{s_0, s_1, \cdots, s_g\}$, $H = \{h_0, h_1, \cdots, h_t\}$ be two linguistic label sets, $g, t \in N$.

By use of the indexes of linguistic labels in $S \times H$, some operations including $\vee, \wedge, \to, '$ can make direct computations in the following theorem.

Theorem 2.2. [45] Let $(S \times H, \vee, \wedge, \to, ')$ be a 2DL-LIA, (s_g, h_t) and (s_0, h_0) are the maximal element and minimal element of $S \times H$, $(s_{i_1}, h_{j_1}), (s_{i_2}, h_{j_2}) \in S \times H$. Then

$$\begin{aligned} (s_{i_1}, h_{j_1}) \vee (s_{i_2}, h_{j_2}) &= (s_{\max\{i_1, i_2\}}, h_{\max\{j_1, j_2\}}), \\ (s_{i_1}, h_{j_1}) \wedge (s_{i_2}, h_{j_2}) &= (s_{\min\{i_1, i_2\}}, h_{\min\{j_1, j_2\}}), \\ (s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2}) &= (s_{\min\{g-i_1+i_2,\ g\}}, h_{\min\{t-j_1+j_2,\ t\}}), \\ (s_{i_1}, h_{j_1})' &= (s_{g-i_1}, h_{t-j_1}). \end{aligned}$$

Now, mainly for aggregation of 2-dimension fuzzy linguistic information, two logical operators $\oplus$ and $\otimes$ can be defined in a 2DL-LIA $S \times H$ as follows: for all $(s_{i_1}, h_{j_1}), (s_{i_2}, h_{j_2}) \in S \times H$,

$$\begin{aligned} (s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2}) &= (s_{i_1}, h_{j_1})' \to (s_{i_2}, h_{j_2}), \\ (s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2}) &= ((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})')'. \end{aligned}$$

Similarly, the computational methods of $\oplus$ and $\otimes$ are provided by using of the indexes of linguistic labels in a 2DL-LIA.

***Theorem* 2.3.** *Let $S \times H$ be a 2DL-LIA. Then for all $(s_{i_1}, h_{j_1}), (s_{i_2}, h_{j_2}) \in S \times H$, where $i_1, i_2 \in \{0, 1, \cdots, g\}, j_1, j_2 \in \{0, 1, \cdots, t\}$, we have:*
(1) $(s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2}) = (s_{\min\{i_1+i_2,g\}}, h_{\min\{j_1+j_2,t\}})$;
(2) $(s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2}) = (s_{\max\{i_1+i_2-g,0\}}, h_{\max\{j_1+j_2-t,0\}})$.

Proof. (1) Since $(s_{i_1}, h_{j_1})' \to (s_{i_2}, h_{j_2}) = (s_{g-i_1}, h_{t-j_1}) \to (s_{i_2}, h_{j_2}) = (s_{\min\{i_1+i_2,g\}}, (h_{\min\{j_1+j_2,t\}})$ by Theorem 2.2, we obtain that $(s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2}) = (s_{\min\{i_1+i_2,g\}}, h_{\min\{j_1+j_2,t\}})$;
(2) Because $(s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})' = (s_{i_1}, h_{j_1}) \to (s_{g-i_2}, h_{t-j_2}) = (s_{\min\{2g-i_1-i_2,g\}}, h_{\min\{2t-j_1-j_2,t\}})$ and $(s_{\min\{2g-i_1-i_2,g\}}, h_{\min\{2t-j_1-j_2,t\}})' = (s_{\max\{i_1+i_2-g,0\}}, h_{\max\{j_1+j_2-t,0\}})$ by Theorem 2.2, we have $(s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2}) = (s_{\max\{i_1+i_2-g,0\}}, h_{\max\{j_1+j_2-t,0\}})$.

Next, the relationships among $\vee$, $\oplus$, $\otimes$ and $\wedge$ are discussed in a 2DL-LIA.

***Theorem* 2.4.** *Let $S \times H$ be a 2DL-LIA. Then for all $(s_{i_1}, h_{j_1}), (s_{i_2}, h_{j_2}) \in S \times H$, we have:*
(1) $(s_{i_1}, h_{j_1}) \vee (s_{i_2}, h_{j_2}) = ((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})') \oplus (s_{i_2}, h_{j_2})$;
(2) $(s_{i_1}, h_{j_1}) \wedge (s_{i_2}, h_{j_2}) = ((s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2})') \otimes (s_{i_2}, h_{j_2})$.

Proof. (1) Since $(s_{i_1}, h_{j_1}) \vee (s_{i_2}, h_{j_2}) = ((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) \to (s_{i_2}, h_{j_2})$, then we have $(s_{i_1}, h_{j_1}) \vee (s_{i_2}, h_{j_2}) = ((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})') \oplus (s_{i_2}, h_{j_2})$.
(2) Since $((s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2})') \otimes (s_{i_2}, h_{j_2}) = (((s_{i_1}, h_{j_1})' \to (s_{i_2}, h_{j_2})') \to (s_{i_2}, h_{j_2})')'$, then we have $((s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2})') \otimes (s_{i_2}, h_{j_2}) = ((s_{i_1}, h_{j_1})' \vee (s_{i_2}, h_{j_2})')' = (s_{i_1}, h_{j_1}) \wedge (s_{i_2}, h_{j_2})$.

Finally, we give the notion of a 2-dimension linguistic lattice H implication algebra (2DL-LHIA), which will be mentioned in next section.

Definition 2.4. *Let $S \times H$ be a 2DL-LIA. If for all $(s_{i_1}, h_{j_1}), (s_{i_2}, h_{j_2}), (s_{i_3}, h_{j_3}) \in S \times H$,*

$$(s_{i_1}, h_{j_1}) \vee (s_{i_2}, h_{j_2}) \vee (((s_{i_1}, h_{j_1}) \wedge (s_{i_2}, h_{j_2})) \to (s_{i_3}, h_{j_3})) = (s_g, h_t),$$

where $i_1, i_2, i_3 \in \{0, 1, \cdots g\}, j_1, j_2, j_3 \in \{0, 1, \cdots, t\}$, then $S \times H$ is called 2-dimension linguistic lattice H implication algebra (2DL-LHIA).

***Example* 2.2.** *Let $S \times H$ be a 2DL-LIA, whose Hasse Diagram is shown in Figure 2, where $S = \{s_0, s_1\}$, $H = \{h_0, h_1\}$.*

The operations $'$ and $\to$ can be computed by Theorem 2.2 as follows: $(s_0, h_0)' = (s_1, h_1)$, $(s_0, h_1)' = (s_1, h_0), (s_1, h_0)' = (s_0, h_1)$, $(s_1, h_1)' = (s_0, h_0)$ and

$\to$	(s_0, h_0)	(s_0, h_1)	(s_1, h_0)	(s_1, h_1)
(s_0, h_0)	(s_1, h_1)	(s_1, h_1)	(s_1, h_1)	(s_1, h_1)
(s_0, h_1)	(s_1, h_0)	(s_1, h_1)	(s_1, h_0)	(s_1, h_1)
(s_1, h_0)	(s_0, h_1)	(s_0, h_1)	(s_1, h_1)	(s_1, h_1)
(s_1, h_1)	(s_0, h_0)	(s_0, h_1)	(s_1, h_0)	(s_1, h_1)

According to Definition 2.4, it can check that $S \times H$ is a 2DL-LHIA.

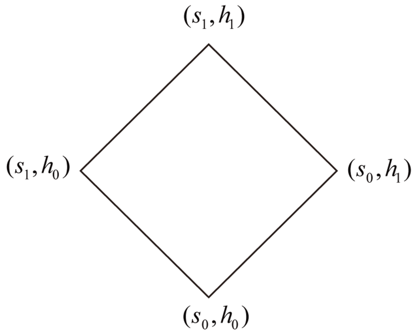

Figure 2. Hasse Diagram of $S \times H$.

§3 Boolean elements of 2DL-LIAs

In this section, a Boolean element is defined in a 2DL-LIA, then some properties of Boolean elements are investigated. Finally, logical operator $\oplus$ is discussed in a 2DL-LIA, which can build a bridge between 2-dimension fuzzy linguistic information and logical algebras.

Definition 3.1. *Let $S \times H$ be a 2DL-LIA, (s_g, h_t) and (s_0, h_0) are the maximal element and minimal element of $S \times H$ respectively, $(s_i, h_j) \in S \times H$. If $(s_i, h_j) \vee (s_i, h_j)' = (s_g, h_t)$, (or equivalently $(s_i, h_j) \wedge (s_i, h_j)' = (s_0, h_0)$), then (s_i, h_j) is called a boolean element of $S \times H$.*

In our further discussion, denote $B(S \times H)$ be the set of boolean elements in $S \times H$.

***Example* 3.1.** *As in Example 2.2, it can check that $(s_0, h_0), (s_0, h_1), (s_1, h_0), (s_1, h_1)$ are all boolean elements according to Definition 3.1, that is $(s_0, h_0), (s_0, h_1), (s_1, h_0), (s_1, h_1) \in B(S \times H)$.*

Remark 1. In order to show that some elements in a 2DL-LIA may be not boolean elements, Appendix gives some examples.

Now, some properties of boolean elements are investigated in a 2DL-LIA.

***Proposition* 3.1.** *Let $S \times H$ be a 2DL-LIA. Then $(s_i, h_j) \in B(S \times H)$ if and only if $(s_i, h_j)' \in B(S \times H)$.*

Proof. It is obvious by Definition 3.1.

***Proposition* 3.2.** *Let $S \times H$ be a 2DL-LIA, $(s_i, h_j) \in S \times H$. Then we have:*
(1) *$(s_i, h_j) \in B(S \times H)$ if and only if $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$;*
(2) *$(s_i, h_j) \in B(S \times H)$ if and only if $(s_i, h_j) \otimes (s_i, h_j) = (s_i, h_j)$.*

Proof. (1) Suppose $(s_i, h_j) \in B(S \times H)$. Then $(s_i, h_j) \vee (s_i, h_j)' = (s_g, h_t)$. Because $((s_i, h_j) \oplus (s_i, h_j)) \to (s_i, h_j) = ((s_i, h_j)' \to (s_i, h_j)) \to (s_i, h_j) = (s_i, h_j)' \vee (s_i, h_j) = (s_g, h_t)$, we have $(s_i, h_j) \oplus (s_i, h_j) \leq (s_i, h_j)$. It is obvious that $(s_i, h_j) \leq (s_i, h_j) \oplus (s_i, h_j)$. Therefore $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$.

On the other hand, assume $(s_i,h_j)\oplus(s_i,h_j)=(s_i,h_j)$. Then we have $(s_i,h_j)'\vee(s_i,h_j)$
$=((s_i,h_j)'\to(s_i,h_j))\to(s_i,h_j)=((s_i,h_j)\oplus(s_i,h_j))\to(s_i,h_j)$
$=(s_i,h_j)\to(s_i,h_j)=(s_g,h_t)$.

(2) The conclusion can be obtained analogously.

Proposition 3.3. *Let $S\times H$ be a 2DL-LIA, (s_g,h_t) and (s_0,h_0) be the maximal element and minimal element of $S\times H$ respectively. If $(s_i,h_j)\in B(S\times H)$, then for all $(s_{i_1},h_{j_1})\in S\times H$, we have:*

(1) $(s_{i_1},h_{j_1})\otimes(s_i,h_j)=(s_{i_1},h_{j_1})\wedge(s_i,h_j)$;

(2) $(s_{i_1},h_{j_1})\oplus(s_i,h_j)=(s_{i_1},h_{j_1})\vee(s_i,h_j)$.

Proof. (1) We only need to prove $(s_{i_1},h_{j_1})\wedge(s_i,h_j)\le(s_{i_1},h_{j_1})\otimes(s_i,h_j)$.

Suppose $(s_i,h_j)\in B(S\times H)$. Then we have $(s_i,h_j)\vee(s_i,h_j)'=(s_g,h_t)$ by Definition 3.1.

Since $(s_{i_1},h_{j_1})\wedge(s_i,h_j)\to(s_{i_1},h_{j_1})\otimes(s_i,h_j)=((s_{i_1},h_{j_1})\to(s_{i_1},h_{j_1})\otimes(s_i,h_j))\vee((s_i,h_j)\to(s_{i_1},h_{j_1})\otimes(s_i,h_j))=(((s_{i_1},h_{j_1})\to(s_i,h_j)')\to(s_{i_1},h_{j_1})')\vee(((s_{i_1},h_{j_1})\to(s_i,h_j)')\to(s_i,h_j)')$ $=(s_i,h_j)\vee(s_{i_1},h_{j_1})'\vee(s_{i_1},h_{j_1})\vee(s_i,h_j)'\ge(s_i,h_j)\vee(s_i,h_j)'=(s_g,h_t)$, that is $(s_{i_1},h_{j_1})\wedge(s_i,h_j)\to(s_{i_1},h_{j_1})\otimes(s_i,h_j)=(s_g,h_t)$, then we have $(s_{i_1},h_{j_1})\wedge(s_i,h_j)\le(s_{i_1},h_{j_1})\otimes(s_i,h_j)$.

(2) We only need to prove that $(s_{i_1},h_{j_1})\oplus(s_i,h_j)\le(s_{i_1},h_{j_1})\vee(s_i,h_j)$.

Suppose $(s_i,h_j)\in B(S\times H)$. Then we have $(s_i,h_j)\vee(s_i,h_j)'=(s_g,h_t)$ by Definition 3.1.

Since $(s_{i_1},h_{j_1})\oplus(s_i,h_j)\to(s_{i_1},h_{j_1})\vee(s_i,h_j)=((s_{i_1},h_{j_1})\oplus(s_i,h_j)\to(s_{i_1},h_{j_1}))\vee((s_{i_1},h_{j_1})\oplus(s_i,h_j)\to(s_i,h_j))=(((s_i,h_j)'\to(s_{i_1},h_{j_1}))\to(s_{i_1},h_{j_1}))\vee(((s_{i_1},h_{j_1})'\to(s_i,h_j))\to(s_i,h_j))$ $=(s_i,h_j)'\vee(s_{i_1},h_{j_1})\vee(s_{i_1},h_{j_1})'\vee(s_i,h_j)\ge(s_i,h_j)'\vee(s_i,h_j)=(s_g,h_t)$, that is $(s_{i_1},h_{j_1})\oplus(s_i,h_j)\to(s_{i_1},h_{j_1})\vee(s_i,h_j)=(s_g,h_t)$, then we have $(s_{i_1},h_{j_1})\oplus(s_i,h_j)\le(s_{i_1},h_{j_1})\vee(s_i,h_j)$.

Proposition 3.4. *Let $S\times H$ be a 2DL-LIA. If $(s_i,h_j)\in B(S\times H)$, then for all (s_{i_1},h_{j_1}), $(s_{i_2},h_{j_2})\in S\times H$, we have:*

(1) $((s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2}))\wedge(s_i,h_j)=((s_{i_1},h_{j_1})\wedge(s_i,h_j))\oplus((s_{i_2},h_{j_2})\wedge(s_i,h_j))$;

(2) $((s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2}))\vee(s_i,h_j)=((s_{i_1},h_{j_1})\vee(s_i,h_j))\oplus((s_{i_2},h_{j_2})\vee(s_i,h_j))$;

(3) $((s_{i_1},h_{j_1})\otimes(s_{i_2},h_{j_2}))\wedge(s_i,h_j)=((s_{i_1},h_{j_1})\wedge(s_i,h_j))\otimes((s_{i_2},h_{j_2})\wedge(s_i,h_j))$;

(4) $((s_{i_1},h_{j_1})\otimes(s_{i_2},h_{j_2}))\vee(s_i,h_j)=((s_{i_1},h_{j_1})\vee(s_i,h_j))\otimes((s_{i_2},h_{j_2})\vee(s_i,h_j))$;

(5) $((s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2}))\otimes(s_i,h_j)=((s_{i_1},h_{j_1})\otimes(s_i,h_j))\oplus((s_{i_2},h_{j_2})\otimes(s_i,h_j))$;

(6) $((s_{i_1},h_{j_1})\otimes(s_{i_2},h_{j_2}))\oplus(s_i,h_j)=((s_{i_1},h_{j_1})\oplus(s_i,h_j))\otimes((s_{i_2},h_{j_2})\oplus(s_i,h_j))$.

Proof. We only prove (1) and (2).

(1) Suppose $(s_i,h_j)\in B(S\times H)$. Then we have $(s_i,h_j)\oplus(s_i,h_j)=(s_i,h_j)$ by Proposition 3.2(1).

Since $((s_{i_1},h_{j_1})\wedge(s_i,h_j))\oplus((s_{i_2},h_{j_2})\wedge(s_i,h_j))=((s_{i_1},h_{j_1})\wedge(s_i,h_j))'\to((s_{i_2},h_{j_2})\wedge(s_i,h_j))$ $=(s_{i_1},h_{j_1})'\vee(s_i,h_j)'\to(s_{i_2},h_{j_2})\wedge(s_i,h_j)=((s_{i_1},h_{j_1})'\to(s_{i_2},h_{j_2}))\wedge((s_{i_1},h_{j_1})'\to(s_i,h_j))\wedge((s_i,h_j)'\to(s_{i_2},h_{j_2}))\wedge((s_i,h_j)'\to(s_i,h_j))=((s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2}))\wedge((s_{i_1},h_{j_1})\oplus(s_i,h_j))\wedge((s_i,h_j)\oplus(s_{i_2},h_{j_2}))\wedge((s_i,h_j)\oplus(s_i,h_j))$ and $((s_{i_1},h_{j_1})\oplus(s_i,h_j))\wedge((s_i,h_j)\oplus(s_{i_2},h_{j_2}))\wedge((s_i,h_j)\oplus(s_i,h_j))=(s_i,h_j)$, then we have $((s_{i_1},h_{j_1})\wedge(s_i,h_j))\oplus((s_{i_2},h_{j_2})\wedge(s_i,h_j))=((s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2}))\wedge(s_i,h_j)$.

(2) Suppose $(s_i,h_j)\in B(S\times H)$. Then we have $(s_i,h_j)\oplus(s_i,h_j)=(s_i,h_j)$ by Proposition 3.2(1).

Since $((s_{i_1},h_{j_1})\vee(s_i,h_j))\oplus((s_{i_2},h_{j_2})\vee(s_i,h_j))=((s_{i_1},h_{j_1})\vee(s_i,h_j))'\to(s_{i_2},h_{j_2})\vee(s_i,h_j)$ $=((s_{i_1},h_{j_1})'\to(s_{i_2},h_{j_2}))\vee((s_{i_1},h_{j_1})'\to(s_i,h_j))\vee((s_i,h_j)'\to(s_{i_2},h_{j_2}))\vee((s_i,h_j)'\to(s_i,h_j))$ $=((s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2}))\vee((s_{i_1},h_{j_1})\oplus(s_i,h_j))\vee((s_i,h_j)\oplus(s_{i_2},h_{j_2}))\vee((s_i,h_j)\oplus(s_i,h_j))$ and $((s_{i_1},h_{j_1})\oplus(s_i,h_j))\vee((s_i,h_j)\oplus(s_{i_2},h_{j_2}))\vee((s_i,h_j)\oplus(s_i,h_j))=(s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2})\vee(s_i,h_j)$ by Proposition 3.3(2), then we have $((s_{i_1},h_{j_1})\vee(s_i,h_j))\oplus((s_{i_2},h_{j_2})\vee(s_i,h_j))=((s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2}))\vee(s_i,h_j)$.

***Proposition* 3.5.** *Let $S\times H$ be a 2DL-LIA, (s_g,h_t) and (s_0,h_0) are the maximal element and minimal element of $S\times H$ respectively. If $(s_{i_1},h_{j_1}),(s_{i_2},h_{j_2})\in B(S\times H)$, then we have $(s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2}),(s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2}),(s_{i_1},h_{j_1})\otimes(s_{i_2},h_{j_2}),(s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2})\in B(S\times H)$.*

Proof. Firstly, we prove $(s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2})\in B(S\times H)$ and $(s_{i_1},h_{j_1})\otimes(s_{i_2},h_{j_2})\in B(S\times H)$.

Suppose $(s_{i_1},h_{j_1}),(s_{i_2},h_{j_2})\in B(S\times H)$. Then we have $(s_{i_1},h_{j_1})\wedge(s_{i_1},h_{j_1})'=(s_0,h_0)$ and $(s_{i_2},h_{j_2})\wedge(s_{i_2},h_{j_2})'=(s_0,h_0)$.

Because $((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2}))\wedge((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2}))'=((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2}))\wedge((s_{i_1},h_{j_1})'\vee(s_{i_2},h_{j_2})')=((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2})\wedge(s_{i_1},h_{j_1})')\vee((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2})\wedge(s_{i_2},h_{j_2})')=(s_0,h_0)$, then we have $(s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2})\in B(S\times H)$ by Definition 3.1, which implies $(s_{i_1},h_{j_1})\otimes(s_{i_2},h_{j_2})\in B(S\times H)$ by Proposition 3.3(1).

Next, we prove $(s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2})\in B(S\times H)$ and $(s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2})\in B(S\times H)$.

Suppose $(s_{i_1},h_{j_1}),(s_{i_2},h_{j_2})\in B(S\times H)$. Then we have $(s_{i_1},h_{j_1})\vee(s_{i_1},h_{j_1})'=(s_g,h_t)$ and $(s_{i_2},h_{j_2})\vee(s_{i_2},h_{j_2})'=(s_g,h_t)$.

Because $((s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2}))\vee((s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2}))'=((s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2}))\vee((s_{i_1},h_{j_1})'\wedge(s_{i_2},h_{j_2})')=((s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2})\vee(s_{i_1},h_{j_1})')\wedge((s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2})\vee(s_{i_2},h_{j_2})')=(s_g,h_t)$, then we have $(s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2})\in B(S\times H)$ by Definition 3.1, which implies $(s_{i_1},h_{j_1})\oplus(s_{i_2},h_{j_2})\in B(S\times H)$ by Proposition 3.3(2).

***Theorem* 3.1.** *Let $S\times H$ be a 2DL-LIA. If for all $(s_i,h_j)\in S\times H$, $(s_i,h_j)\in B(S\times H)$, then $S\times H$ is a 2DL-LHIA.*

Proof. Suppose $\forall(s_i,h_j)\in S\times H$, $(s_i,h_j)\in B(S\times H)$. Then we have $(s_i,h_j)\otimes(s_i,h_j)=(s_i,h_j)$ and $(s_i,h_j)\oplus(s_i,h_j)=(s_i,h_j)$ by Proposition 3.2. Therefore $S\times H$ is a 2DL-LHIA by Theorem 2.1.

Finally, we focus on logical operator $\oplus$ and its application in a 2DL-LIA, which can build a bridge between 2-dimension fuzzy linguistic information and logical algebras.

As we know, in real decision making environment, the weights of 2-dimension fuzzy linguistic information are critical for decision makers to aggregate 2-dimension fuzzy linguistic information. Therefore we define operations between constants and 2DLLs in a 2DL-LIA as follows.

Definition 3.2. *Let $S\times H$ be a 2DL-LIA, $S=\{s_0,s_1,\cdots,s_g\}$, $H=\{h_0,h_1,\cdots,t\}$. Then for all $(s_i,h_j)\in S\times H,\lambda\in R^+$, $\lambda(s_i,h_j)=(s_{\min\{\lambda i,g\}},h_{\min\{\lambda j,t\}})$.*

***Example* 3.2.** *Suppose there are two reviewers who are invited to evaluate the same submitted manuscript by using of 2DLLs in a* $S \times H$. *One reviewer's opinion is* (s_2, h_3), *and the other is* (s_2, h_2).

If the weight vector of two reviewers is $(0.6, 0.4)$, *then we can obtain the final opinion of this manuscript is*

$$0.6(s_2, h_3) \oplus 0.4(s_2, h_2) = (s_2, h_{2.6}).$$

Remark 2. By use of the operation $\oplus$ and the operations between constants and 2DLLs, we can aggregate 2DLLs provided by decision makers to obtain the collective one. Therefore the operation $\oplus$ builds a bridge between logical operators and aggregation operators in a certain sense.

§4 Derivations on 2DL-LIAs

In this section, derivations on 2DL-LIAs are introduced to investigate algebraic structures of 2DL-LIAs, then the related properties of derivations are discussed. Finally, by use of Boolean elements in a 2DL-LIAs, some derivations on 2DL-LIAs can be constructed.

Definition 4.1. *Let* $S \times H$ *be a 2DL-LIA,* $d : S \times H \to S \times H$ *be a mapping. For any* (s_{i_1}, h_{j_1}), $(s_{i_2}, h_{j_2}) \in S \times H$, *where* $i_1, i_2 \in \{0, 1, \cdots, g\}, j_1, j_2 \in \{0, 1, \cdots, t\}$, *if*

(1) $d((s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2})) = d((s_{i_1}, h_{j_1})) \oplus d((s_{i_2}, h_{j_2}))$

(2) $d((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})) = (d((s_{i_1}, h_{j_1})) \otimes (s_{i_2}, h_{j_2})) \vee ((s_{i_1}, h_{j_1}) \otimes d((s_{i_2}, h_{j_2})))$,

then d *is called a derivation on* $S \times H$.

Now, some examples are given to indicate that there exist some derivations on 2DL-LIAs.

***Example* 4.1.** *Let* $S \times H$ *be a 2DL-LIA, where* (s_0, h_0) *is the minimal element of* $S \times H$. *For all* $(s_i, h_j) \in S \times H$, *define a mapping* d *on* $S \times H$ *as* $d((s_i, h_j)) = (s_0, h_0)$. *Then* d *is a derivation on 2DL-LIA, which is called a zero derivation.*

***Example* 4.2.** *Let* $S \times H$ *be a 2DL-LIA. For all* $(s_i, h_j) \in S \times H$, *define a mapping* d *on* $S \times H$ *as* $d((s_i, h_j)) = (s_i, h_j)$. *Then* d *is a derivation on 2DL-LIA, which is called an identity derivation.*

***Example* 4.3.** *As in Example 2.2, define a mapping* $d_1 : S \times H \to S \times H$ *such that* $d_1((s_0, h_0)) = (s_0, h_0)$, $d_1((s_0, h_1)) = (s_0, h_0)$, $d_1((s_1, h_0)) = (s_1, h_0)$, $d_1((s_1, h_1)) = (s_1, h_0)$ *and a mapping* $d_2 : S \times H \to S \times H$ *such that* $d_2((s_0, h_0)) = (s_0, h_0)$, $d_2((s_0, h_1)) = (s_0, h_0)$, $d_2((s_1, h_0)) = (s_0, h_1)$, $d_2((s_1, h_1)) = (s_0, h_1)$.

Because $d_1((s_1, h_0)) = (s_1, h_0)$ and $d_2((s_1, h_0)) = (s_0, h_1)$, we get that d_1 and d_2 are different mapping. Then by Definition 4.1, it can check that d_1 and d_2 are two derivations on $S \times H$.

We only prove that d_1 is a derivation on $S \times H$ as follows. Similarly, it can prove that d_2 is a derivation on $S \times H$.

According to Example 3.1 and Proposition 3.3, we have for all $(s_{i_1}, h_{j_1}), (s_{i_2}, h_{j_2}) \in S \times H$, $(s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2}) = (s_{i_1}, h_{j_1}) \wedge (s_{i_2}, h_{j_2})$ and $(s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2}) = (s_{i_1}, h_{j_1}) \vee (s_{i_2}, h_{j_2})$. Then the operations $\otimes$ and $\oplus$ are listed as follows:

$\otimes$	(s_0, h_0)	(s_0, h_1)	(s_1, h_0)	(s_1, h_1)
(s_0, h_0)	(s_0, h_0)	(s_0, h_0)	(s_0, h_0)	(s_0, h_0)
(s_0, h_1)	(s_0, h_0)	(s_0, h_1)	(s_0, h_0)	(s_0, h_1)
(s_1, h_0)	(s_0, h_0)	(s_0, h_0)	(s_1, h_0)	(s_1, h_0)
(s_1, h_1)	(s_0, h_0)	(s_0, h_1)	(s_1, h_0)	(s_1, h_1)

$\oplus$	(s_0, h_0)	(s_0, h_1)	(s_1, h_0)	(s_1, h_1)
(s_0, h_0)	(s_0, h_0)	(s_0, h_1)	(s_1, h_0)	(s_1, h_1)
(s_0, h_1)	(s_0, h_1)	(s_0, h_1)	(s_1, h_1)	(s_1, h_1)
(s_1, h_0)	(s_1, h_0)	(s_1, h_1)	(s_1, h_0)	(s_1, h_1)
(s_1, h_1)	(s_1, h_1)	(s_1, h_1)	(s_1, h_1)	(s_1, h_1)

Take the elements (s_0, h_1) and (s_1, h_0) for example. We can compute that the left side of Definition 4.1(1) is $d_1((s_0, h_1) \oplus (s_1, h_0)) = d_1((s_1, h_1)) = (s_1, h_0)$. On the other hand, the right side of Definition 4.1(1) is $d_1((s_0, h_1)) \oplus d_1((s_1, h_0)) = (s_0, h_0) \oplus (s_1, h_0) = (s_1, h_0)$. Therefore it concludes that the elements (s_0, h_1) and (s_1, h_0) satisfy the condition (1) of Definition 4.1.

Next, we can compute that the left side of Definition 4.1(2) is $d_1((s_0, h_1) \otimes (s_1, h_0)) = d_1((s_0, h_0)) = (s_0, h_0)$. On the other hand, the right side of Definition 4.1(2) can be divided into two parts. One is $d_1((s_0, h_1)) \otimes (s_1, h_0) = (s_0, h_0) \otimes (s_1, h_0) = (s_0, h_0)$, the other is $(s_0, h_1) \otimes d_1((s_1, h_0)) = (s_0, h_1) \otimes (s_1, h_0) = (s_0, h_0)$. Hence we get that $(d_1((s_0, h_1)) \otimes (s_1, h_0)) \vee ((s_0, h_1) \otimes d_1((s_1, h_0))) = (s_0, h_0) \vee (s_0, h_0) = (s_0, h_0)$, that is the elements (s_0, h_1) and (s_1, h_0) satisfy the condition (2) of Definition 4.1.

Similarly, according to the definition of the mapping d_1, other elements of $S \times H$ can be validated to satisfy Definition 4.1(1) and (2).

Then some properties of derivations are investigated in a 2DL-LIA.

Proposition 4.1. *Let d be a derivation on $S \times H$, (s_g, h_t) and (s_0, h_0) be the maximal element and minimal element of $S \times H$ respectively. Then for all $(s_{i_1}, h_{j_1}), (s_{i_2}, h_{j_2}) \in S \times H$, we have:*
(1) *$d((s_0, h_0)) = (s_0, h_0)$;*
(2) *if $(s_{i_1}, h_{j_1}) \leq (s_{i_2}, h_{j_2})$, then $d((s_{i_1}, h_{j_1})) \leq d((s_{i_2}, h_{j_2}))$; (i.e. d is istone)*
(3) *$d((s_g, h_t)) \in B(S \times H)$;*
(4) *$d((s_{i_1}, h_{j_1})) \leq (s_{i_1}, h_{j_1})$.*

Proof. (1) Suppose d be a derivation on $S \times H$. Then $d((s_0, h_0)) = d((s_0, h_0) \otimes (s_0, h_0)) = (d((s_0, h_0)) \otimes (s_0, h_0)) \vee ((s_0, h_0) \otimes d((s_0, h_0))) = (s_0, h_0)$ by Definition 4.1(2).
(2) Suppose d be a derivation on $S \times H$. If $(s_{i_1}, h_{j_1}) \leq (s_{i_2}, h_{j_2})$, then we have $(s_{i_2}, h_{j_2}) = (s_{i_2}, h_{j_2}) \vee (s_{i_1}, h_{j_1}) = ((s_{i_2}, h_{j_2}) \otimes (s_{i_1}, h_{j_1})') \oplus (s_{i_1}, h_{j_1})$ by Theorem 2.4(1). Then $d((s_{i_2}, h_{j_2})) = d((s_{i_2}, h_{j_2}) \vee (s_{i_1}, h_{j_1})) = d(((s_{i_2}, h_{j_2}) \otimes (s_{i_1}, h_{j_1})') \oplus (s_{i_1}, h_{j_1})) = d((s_{i_2}, h_{j_2}) \otimes (s_{i_1}, h_{j_1})') \oplus d((s_{i_1}, h_{j_1}))$ by Definition 4.1(1), thus $d((s_{i_1}, h_{j_1})) \leq d((s_{i_2}, h_{j_2}))$. Therefore every derivation

on 2DL-LIA is istone.
(3) Suppose d be a derivation on $S\times H$. Because $d((s_g,h_t)) = d((s_g,h_t)\oplus(s_g,h_t)) = d((s_g,h_t))\oplus d((s_g,h_t))$ by Definition 4.1(1), then we have $d((s_g,h_t)) \in B(S\times H)$ by Proposition 3.2(1).
(4) Suppose d be a derivation on $S\times H$. Then $\forall (s_{i_1},h_{j_1}) \in S\times H$, $(s_0,h_0) = d((s_0,h_0)) = d((s_{i_1},h_{j_1})\otimes(s_{i_1},h_{j_1})') = (d((s_{i_1},h_{j_1}))\otimes(s_{i_1},h_{j_1})')\vee((s_{i_1},h_{j_1})\otimes d((s_{i_1},h_{j_1})'))$, thus $d((s_{i_1},h_{j_1}))\otimes(s_{i_1},h_{j_1})' = (s_0,h_0)$. Because $d((s_{i_1},h_{j_1}))\otimes(s_{i_1},h_{j_1})' = (d((s_{i_1},h_{j_1}))\to(s_{i_1},h_{j_1}))' = (s_0,h_0)$, we have $d((s_{i_1},h_{j_1}))\to(s_{i_1},h_{j_1}) = (s_g,h_t)$. Therefore $d((s_{i_1},h_{j_1}))\le(s_{i_1},h_{j_1})$.

***Proposition* 4.2.** *Let d be a derivation on $S\times H$, (s_g,h_t) be the maximal element of $S\times H$. Then for all $(s_{i_1},h_{j_1}),(s_{i_2},h_{j_2})\in S\times H$,*
(1) $d((s_{i_1},h_{j_1})) = (s_{i_1},h_{j_1})\wedge d((s_g,h_t)) = (s_{i_1},h_{j_1})\otimes d((s_g,h_t))$;
(2) $d((s_{i_1},h_{j_1})^n) = (s_{i_1},h_{j_1})^{n-1}\otimes d((s_{i_1},h_{j_1}))$, *where* $(s_{i_1},h_{j_1})^n = (s_{i_1},h_{j_1})^{n-1}\otimes(s_{i_1},h_{j_1}), n\in N$;
(3) $d((s_{i_1},h_{j_1})') \le (d((s_{i_1},h_{j_1})))'$;
(4) *if* $d((s_g,h_t)) = (s_g,h_t)$, *then d is an identity derivation.*

Proof. (1) Let d be a derivation on $S\times H$. Then $d((s_{i_1},h_{j_1})) = d((s_{i_1},h_{j_1})\otimes(s_g,h_t)) = (d((s_{i_1},h_{j_1}))\otimes(s_g,h_t))\vee((s_{i_1},h_{j_1})\otimes d((s_g,h_t))) = d((s_{i_1},h_{j_1}))\vee((s_{i_1},h_{j_1})\otimes d((s_g,h_t)))$, thus $(s_{i_1},h_{j_1})\otimes d((s_g,h_t))\le d((s_{i_1},h_{j_1}))$.

On the other hand, we have $d((s_{i_1},h_{j_1}))\le(s_{i_1},h_{j_1})$ by Proposition 4.1(4), and $d((s_{i_1},h_{j_1}))\le d((s_g,h_t))$ by Proposition 4.1(2), thus $d((s_{i_1},h_{j_1}))\le(s_{i_1},h_{j_1})\wedge d((s_g,h_t))$. Because $(s_{i_1},h_{j_1})\wedge d((s_g,h_t)) = (s_{i_1},h_{j_1})\otimes d((s_g,h_t))$ by Proposition 4.1(3) and Proposition 3.3(1), then we have $d((s_{i_1},h_{j_1})) = (s_{i_1},h_{j_1})\wedge d((s_g,h_t)) = (s_{i_1},h_{j_1})\otimes d((s_g,h_t))$.
(2) Let d be a derivation on $S\times H$. Then we have $d((s_{i_1},h_{j_1})^2) = d((s_{i_1},h_{j_1})\otimes(s_{i_1},h_{j_1})) = (d((s_{i_1},h_{j_1}))\otimes(s_{i_1},h_{j_1}))\vee((s_{i_1},h_{j_1})\otimes d((s_{i_1},h_{j_1})) = (s_{i_1},h_{j_1})\otimes d((s_{i_1},h_{j_1}))$. Thus we can obtain $d((s_{i_1},h_{j_1})^n) = (s_{i_1},h_{j_1})^{n-1}\otimes d((s_{i_1},h_{j_1}))$ by induction for all $n\ge 2$.
(3) Let d be a derivation on $S\times H$. Then $d((s_{i_1},h_{j_1})')\le(s_{i_1},h_{j_1})'$ by Proposition 4.1(4). Thus $d((s_{i_1},h_{j_1})')\le(s_{i_1},h_{j_1})'\le(s_{i_1},h_{j_1})'\vee d((s_g,h_t))' = ((s_{i_1},h_{j_1})\wedge d((s_g,h_t)))'$. Because $d((s_{i_1},h_{j_1})) = (s_{i_1},h_{j_1})\wedge d((s_g,h_t))$ by (1), then we have $((s_{i_1},h_{j_1})\wedge d((s_g,h_t)))'\le(d((s_{i_1},h_{j_1})))'$. Hence $d((s_{i_1},h_{j_1})')\le(d((s_{i_1},h_{j_1})))'$.
(4) If $d((s_g,h_t)) = (s_g,h_t)$, then we have $(s_{i_1},h_{j_1}) = (s_{i_1},h_{j_1})\otimes d((s_g,h_t)) = d((s_{i_1},h_{j_1}))$ by (1), which implies d is an identity derivation.

***Proposition* 4.3.** *Let d be a derivation on $S\times H$. Then for all $(s_{i_1},h_{j_1}),(s_{i_2},h_{j_2})\in S\times H$:*
(1) $d((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2})) = d((s_{i_1},h_{j_1}))\wedge d((s_{i_2},h_{j_2}))$;
(2) $d((s_{i_1},h_{j_1})\vee(s_{i_2},h_{j_2})) = d((s_{i_1},h_{j_1}))\vee d((s_{i_2},h_{j_2}))$;
(3) $d((s_{i_1},h_{j_1})\otimes(s_{i_2},h_{j_2})) = d((s_{i_1},h_{j_1}))\otimes d((s_{i_2},h_{j_2}))$.

Proof. (1) Suppose d be a derivation on $S\times H$. Then we have $d((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2})) = ((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2}))\otimes d((s_g,h_t))$ by Proposition 4.2(1). Since $d((s_g,h_t))\in B(S\times H)$ by Proposition 4.1(3), we get $((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2}))\otimes d((s_g,h_t)) = ((s_{i_1},h_{j_1})\otimes d((s_g,h_t)))\wedge((s_{i_2},h_{j_2})\otimes d((s_g,h_t))) = d((s_{i_1},h_{j_1}))\wedge d((s_{i_2},h_{j_2}))$ by Proposition 4.2(1), that is $d((s_{i_1},h_{j_1})\wedge(s_{i_2},h_{j_2})) = d((s_{i_1},h_{j_1}))\wedge d((s_{i_2},h_{j_2}))$.

(2) and (3) can be proved analogously.

***Proposition* 4.4.** *Let d be a derivation on $S \times H$. Then for all $(s_{i_1}, h_{j_1}), (s_{i_2}, h_{j_2}) \in S \times H$:*
(1) $d((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) \leq d((s_{i_1}, h_{j_1})) \to d((s_{i_2}, h_{j_2}))$;
(2) $d((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) \leq (s_{i_1}, h_{j_1}) \to d((s_{i_2}, h_{j_2}))$;
(3) $(s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2}) \leq d((s_{i_1}, h_{j_1})) \to d((s_{i_2}, h_{j_2}))$.

Proof. (1) Let d be a derivation on $S \times H$. Then we have $d((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) = d((s_{i_1}, h_{j_1})' \oplus (s_{i_2}, h_{j_2})) = d((s_{i_1}, h_{j_1})') \oplus d((s_{i_2}, h_{j_2}))$ by Definition 4.1(1). Since $d((s_{i_1}, h_{j_1})') \oplus d((s_{i_2}, h_{j_2})) \leq (d((s_{i_1}, h_{j_1})))' \oplus d((s_{i_2}, h_{j_2})) = d((s_{i_1}, h_{j_1})) \to d((s_{i_2}, h_{j_2}))$ by Proposition 4.2(3), then we have $d((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) \leq d((s_{i_1}, h_{j_1})) \to d((s_{i_2}, h_{j_2}))$.

(2) Let d be a derivation on $S \times H$. Since $(s_{i_1}, h_{j_1}) \otimes ((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) \leq (s_{i_2}, h_{j_2})$, we have $d((s_{i_1}, h_{j_1}) \otimes ((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2}))) \leq d((s_{i_2}, h_{j_2}))$ by Proposition 4.1(2). Then $d((s_{i_1}, h_{j_1}) \otimes ((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2}))) = (d((s_{i_1}, h_{j_1})) \otimes ((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2}))) \vee ((s_{i_1}, h_{j_1}) \otimes d((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})))$ by Definition 4.1(2), that is $(s_{i_1}, h_{j_1}) \otimes d((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) \leq d((s_{i_2}, h_{j_2}))$ and $d((s_{i_1}, h_{j_1})) \otimes ((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) \leq d((s_{i_2}, h_{j_2}))$. Hence $d((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) \leq (s_{i_1}, h_{j_1}) \to d((s_{i_2}, h_{j_2}))$. And $(s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2}) \leq d((s_{i_1}, h_{j_1})) \to d((s_{i_2}, h_{j_2}))$, which implies (3) holds.

The following theorem shows the relationships between derivations and lattice implication homomorphisms in 2DLIAs.

***Theorem* 4.1.** *Let d be a derivation on $S \times H$. If for all $(s_i, h_j) \in S \times H$, $(d((s_i, h_j)))' \leq d((s_i, h_j)')$, then d is a lattice implication homomorphism.*

Proof. Let d be a derivation on $S \times H$. Suppose $\forall (s_i, h_j) \in S \times H$, $(d((s_i, h_j)))' \leq d((s_i, h_j)')$, then we have $(d((s_i, h_j)))' = d((s_i, h_j)')$ by Proposition 4.2(3). Next by Definition 4.1(1), we have $d((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) = d((s_{i_1}, h_{j_1})' \oplus (s_{i_2}, h_{j_2})) = d((s_{i_1}, h_{j_1})') \oplus d((s_{i_2}, h_{j_2}))$ $= (d((s_{i_1}, h_{j_1})))' \oplus d((s_{i_2}, h_{j_2})) = d((s_{i_1}, h_{j_1})) \to d((s_{i_2}, h_{j_2}))$, that is $d((s_{i_1}, h_{j_1}) \to (s_{i_2}, h_{j_2})) = d((s_{i_1}, h_{j_1})) \to d((s_{i_2}, h_{j_2}))$. Combining with Proposition 4.3(1) and (2), we get that d is a lattice implication homomorphism.

Now, some special mappings are defined by Boolean elements and such mappings are verified to be derivations on 2DL-LIAs.

Let $S \times H$ be a 2DL-LIA, $(s_i, h_j) \in B(S \times H)$. Define $f_1 : S \times H \to S \times H$ and $f_2 : S \times H \to S \times H$ be the mappings such that $f_1 : x \to x \wedge (s_i, h_j)$, $f_2 : x \to x \otimes (s_i, h_j)$.

***Proposition* 4.5.** *Let f_1, f_2 be defined as above. Then $f_1 = f_2$.*

Proof. The conclusions are obvious by Proposition 3.3(1).

Next we investigate some properties of the mapping f_1 on 2DL-LIAs.

***Proposition* 4.6.** *Let f_1 be defined as above. Then for all $(s_{i_1}, h_{j_1}), (s_{i_2}, h_{j_2}) \in S \times H$, we have:*
(1) $f_1((s_{i_1}, h_{j_1}) \vee (s_{i_2}, h_{j_2})) = f_1((s_{i_1}, h_{j_1})) \vee f_1((s_{i_2}, h_{j_2}))$;
(2) $f_1((s_{i_1}, h_{j_1}) \wedge (s_{i_2}, h_{j_2})) = f_1((s_{i_1}, h_{j_1})) \wedge f_1((s_{i_2}, h_{j_2}))$;

(3) $f_1((s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2})) = f_1((s_{i_1}, h_{j_1})) \oplus f_1((s_{i_2}, h_{j_2}))$;
(4) $f_1((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})) = f_1((s_{i_1}, h_{j_1})) \otimes f_1((s_{i_2}, h_{j_2}))$;
(5) $f_1((s_{i_1}, h_{j_1})') = f_1((f_1((s_{i_1}, h_{j_1})))')$.

Proof. (1) Let $f_1 : x \to x \wedge (s_i, h_j)$. Then we have $f_1((s_{i_1}, h_{j_1}) \vee (s_{i_2}, h_{j_2})) = ((s_{i_1}, h_{j_1}) \vee (s_{i_2}, h_{j_2})) \wedge (s_i, h_j) = ((s_{i_1}, h_{j_1}) \wedge (s_i, h_j)) \vee ((s_{i_2}, h_{j_2})) \wedge (s_i, h_j)) = f_1((s_{i_1}, h_{j_1})) \vee f_1((s_{i_2}, h_{j_2}))$. Analogously for (2).

(3) Let $f_1 : x \to x \wedge (s_i, h_j)$. Then we have $f_1((s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2})) = ((s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2})) \wedge (s_i, h_j) = ((s_{i_1}, h_{j_1}) \wedge (s_i, h_j)) \oplus ((s_{i_2}, h_{j_2})) \wedge (s_i, h_j)) = f_1((s_{i_1}, h_{j_1})) \oplus f_1((s_{i_2}, h_{j_2}))$ by Proposition 3.4(1). Analogously for (4).

(5) Let $f_1 : x \to x \wedge (s_i, h_j)$. Then we have $f_1((f_1((s_{i_1}, h_{j_1})))') = f_1(((s_{i_1}, h_{j_1}) \wedge (s_i, h_j))') = ((s_{i_1}, h_{j_1}) \wedge (s_i, h_j))' \wedge (s_i, h_j) = ((s_{i_1}, h_{j_1})' \vee (s_i, h_j)') \wedge (s_i, h_j)$. Since $(s_i, h_j) \in B(S \times H)$, we have $((s_{i_1}, h_{j_1})' \vee (s_i, h_j)') \wedge (s_i, h_j) = (s_{i_1}, h_{j_1})' \wedge (s_i, h_j) = f_1((s_{i_1}, h_{j_1})')$, that is $f_1((s_{i_1}, h_{j_1})') = f_1((f_1((s_{i_1}, h_{j_1})))')$.

Finally, we prove that the mapping f_1 is a derivation on $S \times H$.

***Theorem* 4.2.** *Let f_1 be defined as above. Then f_1 is a derivation on $S \times H$.*

Proof. Let $f_1 : x \to x \wedge (s_i, h_j)$. Then we have $f_1((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})) = ((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})) \wedge (s_i, h_j)$, $f_1((s_{i_1}, h_{j_1})) \otimes (s_{i_2}, h_{j_2}) = ((s_{i_1}, h_{j_1}) \wedge (s_i, h_j)) \otimes (s_{i_2}, h_{j_2})$, and $(s_{i_1}, h_{j_1}) \otimes f_1((s_{i_2}, h_{j_2})) = (s_{i_1}, h_{j_1}) \otimes ((s_{i_2}, h_{j_2}) \wedge (s_i, h_j))$. Since $(s_i, h_j) \in B(S \times H)$, we get $((s_{i_1}, h_{j_1}) \wedge (s_i, h_j)) \otimes (s_{i_2}, h_{j_2}) = ((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})) \otimes (s_i, h_j) = ((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})) \wedge (s_i, h_j)$ and $(s_{i_1}, h_{j_1}) \otimes ((s_{i_2}, h_{j_2}) \wedge (s_i, h_j)) = ((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})) \wedge (s_i, h_j)$ by Proposition 3.3(1), which implies that $f_1((s_{i_1}, h_{j_1}) \otimes (s_{i_2}, h_{j_2})) = (f_1((s_{i_1}, h_{j_1})) \otimes (s_{i_2}, h_{j_2})) \vee ((s_{i_1}, h_{j_1}) \otimes f_1((s_{i_2}, h_{j_2})))$.

By Proposition 4.6(3), we have $f_1((s_{i_1}, h_{j_1}) \oplus (s_{i_2}, h_{j_2})) = f_1((s_{i_1}, h_{j_1})) \oplus f_1((s_{i_2}, h_{j_2}))$. Hence f_1 is a derivation on $S \times H$ by Definition 4.1.

§5 Conclusions

This paper firstly introduced the concept of Boolean elements in 2DL-LIAs, then investigated some properties of Boolean elements. Next, we proposed the notion of derivations on 2DL-LIAs and studied some properties of derivations. Finally, some special mappings defined by Boolean elements are proved to be derivations on 2DL-LIAs. The above work not only enriches algebraic structures and properties of 2DL-LIAs, but also provides theoretical foundations for lattice-valued logic systems based on 2DL-LIAs.

Since logical operator $\oplus$ defined in 2DL-LIA can build a bridge between logical algebras and 2-dimension fuzzy linguistic information aggregations, we hope that the results of this manuscript can supply theoretical supports for 2-dimension linguistic multiple attribute decision making. In future, we consider to extend the notion of derivations on various algebraic structures which may have some applications in various fields of computer sciences, decision making [27, 29, 43], medicine [28], etc.

Appendix

The following examples are given to show that some elements in a 2DL-LIA may be not boolean elements.

***Example* 5.1.** *Let $S \times H$ be a 2DL-LIA, whose Hasse Diagram is shown in Figure 3, where $S = \{s_0, s_1\}$, $H = \{h_0, h_1, h_2\}$.*

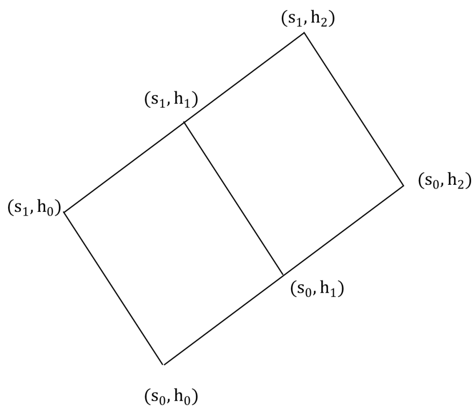

Figure 3. Hasse Diagram of $S \times H$.

The operations $'$ *and* $\rightarrow$ *can be defined as follows:* $(s_0, h_0)' = (s_1, h_2), (s_0, h_1)' = (s_1, h_1), (s_0, h_2)' = (s_1, h_0), (s_1, h_1)' = (s_0, h_1), (s_1, h_0)' = (s_0, h_2)$, $(s_1, h_2)' = (s_0, h_0)$ *and*

$\rightarrow$	(s_0,h_0)	(s_0,h_1)	(s_0,h_2)	(s_1,h_0)	(s_1,h_1)	(s_1,h_2)
(s_0,h_0)	(s_1,h_2)	(s_1,h_2)	(s_1,h_2)	(s_1,h_2)	(s_1,h_2)	(s_1,h_2)
(s_0,h_1)	(s_1,h_1)	(s_1,h_2)	(s_1,h_2)	(s_1,h_1)	(s_1,h_2)	(s_1,h_2)
(s_0,h_2)	(s_1,h_0)	(s_1,h_1)	(s_1,h_2)	(s_1,h_0)	(s_1,h_1)	(s_1,h_2)
(s_1,h_0)	(s_0,h_2)	(s_0,h_2)	(s_0,h_2)	(s_1,h_2)	(s_1,h_2)	(s_1,h_2)
(s_1,h_1)	(s_0,h_1)	(s_0,h_2)	(s_0,h_2)	(s_1,h_1)	(s_1,h_2)	(s_1,h_2)
(s_1,h_2)	(s_0,h_0)	(s_0,h_1)	(s_0,h_2)	(s_1,h_0)	(s_1,h_1)	(s_1,h_2)

According to Definition 2.3, it can check that $S \times H$ is a 2DL-LIA.

***Example* 5.2.** *As in Example 5.1, it can check that* $(s_0, h_0), (s_1, h_0), (s_0, h_2), (s_1, h_2)$ *are boolean elements according to Definition 3.1, that is* $(s_0, h_0), (s_1, h_0), (s_0, h_2), (s_1, h_2) \in B(S \times H)$.

However, $(s_1, h_1) \vee (s_1, h_1)' = (s_1, h_1) \neq (s_1, h_2)$ *and* $(s_0, h_1) \vee (s_0, h_1)' = (s_1, h_1) \neq (s_1, h_2)$, *hence it can get that* (s_1, h_1) *and* (s_0, h_1) *are not boolean elements according to Definition 3.1.*

Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China.
[2]Henan Academy of Big Data, Zhengzhou University, Zhengzhou 450001, China.
[3]Henan Key Laboratory of Financial Engineering, Zhengzhou 450001, China.
[4]Key Laboratory of Big Data Analysis and Application,Zhengzhou 450001, China.
Email: zhaojianbin@zzu.edu.cn