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A novel three-step implicit iteration process for three finite family of asymptotically generalized Φ-hemicontractive mapping in the intermediate sense

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Abstract. In this paper, we introduce a three-step composite implicit iteration process for approximating the common fixed point of three uniformly continuous and asymptotically generalized Φ-hemicontractive mappings in the intermediate sense. We prove that our proposed iteration process converges to the common fixed point of three finite family of asymptotically generalized Φ-hemicontractive mappings in the intermediate sense. Our results extends, improves and complements several known results in literature.

§1 Introduction and Preliminaries

Let E be an arbitrary real Banach space with dual E^*. We denote by J the *normalized duality* mapping from E into 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},\ \forall\, x \in X, \tag{1.1}$$

where $\langle .,. \rangle$ denotes the generalized duality pairing. Let j denote the single-valued-normalized duality mapping.

In the sequel, we give the following definitions which will be useful in this study.

Definition 1.1. Let K be a nonempty subset of real Banach space E. A mapping $T : K \to K$ is said to be:

(1) *strongly pseudocontractive* (Kim et al. [13]) if for all $x, y \in K$, there exists a constant $k \in (0, 1)$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \leq k\|x-y\|^2; \tag{1.2}$$

(2) *ϕ-strongly pseudocontractive* (Kim et al. [13]) if for all $x, y \in K$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|; \tag{1.3}$$

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In [23], it was proved that the class of strongly pseudocontractive mappings is a proper subclass of ϕ-strongly pseudocontractive mappings. By taking $\Phi(t) = t\phi(t)$, where $\phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function with $\phi(0) = 0$. However, the converse is not true.

(3) *generalized* Φ*-pseudocontractive* (Albert et al. [1] , Chidume and Chidume [3]) if for all $x, y \in K$, there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|). \tag{1.4}$$

The class of generalized Φ-pseudocontractive mappings is also called uniformly pseudocontractive mappings (see [3]). Clearly, the class of generalized Φ-pseudocontractive mappings properly contains the class of ϕ-pseudocontractive mappings;

(4) *generalized* Φ*-hemicontractive* if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$, and there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$, such that for all $x \in K$, $p \in F(T)$, there exists $j(x-p) \in J(x-p)$ such that the following inequality holds:

$$\langle Tx - p, j(x-p)\rangle \leq \|x-p\|^2 - \Phi(\|x-p\|). \tag{1.5}$$

Clearly, the class of generalized Φ-hemicontractive mappings includes the class of generalized Φ-pseudocontractive mappings in which the fixed points set $F(T)$ is nonempty;

(5) *asymptotically generalized* Φ*-pseudocontractive* (Kim et al. [13]) with sequence $\{h_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} h_n = 1$, if for each $x, y \in K$, there exist a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ satisfying

$$\langle T^n x - T^n y, j(x-y)\rangle \leq h_n\|x-y\|^2 - \Phi(\|x-y\|). \tag{1.6}$$

The class of asymptotically generalized Φ-pseudocontractive mappings is a generalization of the class of strongly pseudocontractive maps and the class of ϕ-strongly peudocontractive maps. The class of asymptotically generalized Φ-pseudocontractive mappings was introduced by Kim et al. [13] in 2009;

(6) *asymptotically generalized* Φ*–hemicontractive* with sequence $\{h_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} h_n = 1$ if there exist a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$, such that for each $x \in K$, $p \in F(T)$, there exists $j(x-p) \in J(x-p)$ such that the following inequality holds:

$$\langle T^n - p, j(x-p)\rangle \leq h_n\|x-p\|^2 - \Phi(\|x-p\|). \tag{1.7}$$

Clearly, every asymptotically generalized Φ–pseudocontractive mapping with a nonempty fixed point set is an asymptotically generalized Φ–hemicontractive mapping.

(7) *asymptotically generalized* Φ*-hemicontractive in the intermediate sense* with sequence $\{h_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} h_n = 1$ if $F(T) \neq \emptyset$ and for each $n \in \mathbb{N}$, $x \in K$ and $p \in F(T)$, there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and $j(x-p) \in J(x-p)$ satisfying

$$\limsup_{n\to\infty} \sup_{(x,p)\in K\times F(T)} (\langle T^n x - p, j(x-p)\rangle - h_n\|x-p\|^2 + \Phi(\|x-p\|)) \leq 0. \tag{1.8}$$

Set

$$\tau_n = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle T^n x - p, j(x-p)\rangle - h_n\|x-p\|^2 + \Phi(\|x-p\|))\right\}$$

It follows that $\tau_n \geq 0$, $\tau_n \to 0$ as $n \to \infty$. Hence, (1.8) yields the following inequality:

$$\langle T^n x - p, j(x-p)\rangle \leq h_n\|x-p\|^2 + \tau_n - \Phi(\|x-p\|). \tag{1.9}$$

This class of mapping was first introduced and studied by Okeke et al. [22]. Clearly, the class of asymptotically generalized Φ-hemicontractive mapping in the intermediate sense is more general than the class of asymptotically generalized Φ-hemicontractive mappings. Hence, the class of asymptotically generalized Φ-hemicontractive mappings in the intermediate sense is the most general so far introduced in literature since it includes the class of asymptotically generalized Φ-hemicontractive maps.

For recent results on the approximation of fixed points of mappings which are asymptotically generalized Φ-hemicontractive mappings (see for example, [2–4,9,25] and the references there in) and for recent results on the approximation of fixed points of mappings which are asymptotically generalized Φ-hemicontractive mappings in the intermediate sense, (see for example, Chidume et al. [7], Okeke et al. [22], Olaleru and Okeke [20], Kaczor et al. [10], Qin et al. [24], and the references contained therein). Also, there exist several other recent papers relating to this class of mappings (see for example, [11, 12, 15–18, 21, 30, 32] and the references therein).

Recently, Okeke and Olaleru [19] introduced the following three-step explicit iterative scheme with errors for the approximation of the unique common fixed point of a family of strongly pseudocontractive maps as follows:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n-\beta_n-e_n)x_n + \alpha_n T y_n + \beta_n T z_n + e_n u_n, \\ y_n = (1-a_n-b_n-e'_n)x_n + a_n S z_n + b_n S x_n + e'_n v_n, \\ z_n = (1-c_n-e''_n)x_n + c_n H x_n + e''_n w_n, \end{cases} \quad \forall n \geq 1, \tag{1.10}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$, $\{a_n\}$, $\{b_n\}$, $\{e'_n\}$, $\{c_n\}$, $\{e''_n\}$ are real sequences in $[0,1]$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K.

Motivated and inspired by Okeke and Olaleru [19] as well as the above results, we introduce a modified three-step composite implicit iteration process for finite family of 3 asymptotically generalized Φ-hemicontractive mappings in the intermediate sense defined as follows:

$$\begin{cases} x_0 \in K, \\ x_n = (1-a_n-b_n)x_{n-1} + a_n T_{i(n)}^{k(n)} y_n + b_n T_{i(n)}^{k(n)} z_n, \\ y_n = (1-a'_n-b'_n)x_{n-1} + a'_n S_{i(n)}^{k(n)} z_n + b'_n S_{i(n)}^{k(n)} x_n, \\ z_n = (1-a''_n)x_n + a''_n H_{i(n)}^{k(n)} x_n \end{cases} \quad \forall n \geq 1, \tag{1.11}$$

where $\{a_n\}$, $\{b_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{a''_n\}$ are real sequences in $[0,1]$ satisfying $a_n+b_n \leq 1$, $a'_n+b'_n \leq 1$ and $n = (k-1)N+i$, $i = i(n) \in \{1,2,...,N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

In the sequel, we will need the following Lemmas.

Lemma 1.2. *([2]). Let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any*

$x, y \in E$, *one has*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \ \forall j(x+y) \in J(x+y). \tag{1.12}$$

Lemma 1.3. *(see [2]). Let* $\Phi : [0,\infty] \to [0,\infty)$ *be a strictly increasing function with* $\Phi(0) = 0$ *and let* $\{\rho_n\}, \{\lambda_n\}, \{\mu_n\}$ *be nonnegative real sequences such that* $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$, $\lim\limits_{n\to\infty} \mu_n = 0$. *Suppose that*

$$\rho_{n+1}^2 \leq \rho_n^2 - \lambda_n \Phi(\rho_{n+1}) + \lambda_n \mu_n, \ n \geq 1. \tag{1.13}$$

Then

$$\lim_{n\to\infty} \rho_n = 0. \tag{1.14}$$

§2 Main results

Theorem 2.1. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $N \geq 1$ *be a positive integer and* $I = \{1, 2, 3, ..., N\}$. *Let* $T_i : K \to K$ *be an asymptotically generalized* Φ*-hemicontractive mapping in the intermediate sense with sequence* $\{\eta_n^i\} \subset [1,\infty)$, *where* $\eta_n^i \to 1$ *as* $n \to \infty$. *Let* $S_i : K \to K$ *be an asymptotically generalized* Φ*-hemicontractive mapping in the intermediate sense with sequence* $\{\zeta_n^i\} \subset [1,\infty)$, *where* $\zeta_n^i \to 1$ *as* $n \to \infty$ *and* $H_i : K \to K$ *be an asymptotically generalized* Φ*-hemicontractive mapping in the intermediate sense with sequence* $\{t_n^i\} \subset [1,\infty)$, *where* $t_n^i \to 1$ *as* $n \to \infty$, *for each* $i \in I$. *Furthermore, let* $T_i(K)$ *be bounded,* T_i, S_i *and* H_i *be uniformly continuous for each* $i \in I$. *Let* $h_n = \max\{\eta_n, \zeta_n, t_n\}$, *where* $\eta_n = \max\{\eta_n^i : i \in I\}$, $\zeta_n = \max\{\zeta_n^i : i \in I\}$ *and* $t_n = \max\{t_n^i : i \in I\}$. *Assume that* $\boldsymbol{F} = (\bigcap_{i=1}^N F(T_i)) \bigcap (\bigcap_{i=1}^N F(S_i)) \bigcap (\bigcap_{i=1}^N F(H_i)) \neq \emptyset$. *Let* $\{a_n\}$, $\{b_n\}$, $\{a_n'\}$, $\{b_n'\}$ *and* $\{a_n''\}$ *be sequences in [0,1] such that* $a_n + b_n \leq 1$ *and* $a_n' + b_n' \leq 1$, *for each* $n \geq 1$. *Let* $\{x_n\}$ *be a sequence generated in* (1.11). *Put*

$$\varpi_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle T_i^n x - p, j(x-p)\rangle - \eta_n^i \|x-p\|^2 + \Phi_i(\|x-p\|))\right\},$$

$$\ell_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle S_i^n x - p, j(x-p)\rangle - \zeta_n^i \|x-p\|^2 + \Phi_i(\|x-p\|))\right\} \text{ and}$$

$$\vartheta_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle H_i^n x - p, j(x-p)\rangle - t_n^i \|x-p\|^2 + \Phi_i(\|x-p\|))\right\}.$$

Let $\tau_n = \max\{\varpi_n, \ell_n, \vartheta_n\}$, *where* $\varpi_n = \max\{\varpi_n^i : i \in I\}$, $\ell_n = \max\{\ell_n^i : i \in I\}$ *and* $\vartheta_n = \max\{\vartheta_n^i : i \in I\}$. *Let* $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, *for each* $\wp \geq 0$. *Assume that the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty}(a_n + b_n) = 0 = \lim\limits_{n\to\infty} a_n' = \lim\limits_{n\to\infty} b_n' = \lim\limits_{n\to\infty} a_n''$;

(ii) $\sum\limits_{n=1}^{\infty}(a_n + b_n) = \infty$;

Then the sequence $\{x_n\}$ *defined by* (1.11) *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Fixing $p \in \mathbf{F}$, since T_i has a bounded range, we let

$$M_1 = \|x_0 - p\| + \sup_{n\geq 1} \|T_{i(n)}^{k(n)} y_n - p\| + \sup_{n\geq 1} \|T_{i(n)}^{k(n)} z_n - p\|. \tag{2.1}$$

Obviously, $M_1 < \infty$. It is clear that $\|x_0 - p\| \leq M_1$. Let $\|x_{n-1} - p\| \leq M_1$. Next we prove that $\|x_n - p\| \leq M_1$.

Using (1.11) we obtain that

$$\begin{aligned}
\|x_n - p\| &= \|(1 - a_n - b_n)x_{n-1} + a_n T_{i(n)}^{k(n)} y_n + b_n T_{i(n)}^{k(n)} z_n - p\| \\
&= \|(1 - a_n - b_n)(x_{n-1} - p) + a_n (T_{i(n)}^{k(n)} y_n - p) \\
&\quad + b_n (T_{i(n)}^{k(n)} z_n - p)\| \\
&\leq (1 - a_n - b_n)\|x_{n-1} - p\| + a_n \|T_{i(n)}^{k(n)} y_n - p\| \\
&\quad + b_n \|T_{i(n)}^{k(n)} z_n - p\| \\
&\leq (1 - a_n - b_n) M_1 + a_n M_1 + b_n M_1 \\
&= M_1.
\end{aligned}$$

So, from the the demonstration above, it follows that the sequence $\{\|x_n - p\|\}$ is bounded. Since H_i is uniformly continuous, it follows also that $\{\|H_{i(n)}^{k(n)} x_n - p\|\}$ is also bounded. Setting

$$M_2 = \max\{M_1, \sup_{n\geq 1}\{\|H_{i(n)}^{k(n)} x_n - p\|\}\}$$

thus, we obtain

$$\begin{aligned}
\|z_n - p\| &= \|(1 - a'')x_n + a'' H_{i(n)}^{k(n)} x_n - p\| \\
&= \|(1 - a'')(x_n - p) + a'' (H_{i(n)}^{k(n)} x_n - p)\| \\
&\leq (1 - a'')\|x_n - p\| + a'' \|H_{i(n)}^{k(n)} x_n - p\| \\
&\leq (1 - a'') M_1 + a'' M_2 \\
&\leq (1 - a'') M_2 + a'' M_2 \\
&= M_2.
\end{aligned}$$

Again, from the above demonstration, we can conclude that the sequence $\{\|z_n - p\|\}$ is bounded. Since S_i is uniformly continuous, it follows that $\{\|S_{i(n)}^{k(n)} x_n - p\|\}$ and $\{\|S_{i(n)}^{k(n)} z_n - p\|\}$ are bounded.

Denote

$$M = \sup_{n\geq 1} \|S_{i(n)}^{k(n)} z_n - p\| + \sup_{n\geq 1} \|S_{i(n)}^{k(n)} x_n - p\| + M_2.$$

Using (1.11), Lemma 1.2 and (1.9) we obtain

$$\begin{aligned}
\|x_n - p\|^2 &= \|(1 - a_n - b_n)x_{n-1} + a_n T_{i(n)}^{k(n)} y_n + b_n T_{i(n)}^{k(n)} z_n - p\| \\
&= \|(1 - a_n - b_n)(x_{n-1} - p) + a_n (T_{i(n)}^{k(n)} y_n - p) \\
&\quad + b_n (T_{i(n)}^{k(n)} z_n - p)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-a_n-b_n)^2\|x_{n-1}-p\|^2+2\langle a_n(T_{i(n)}^{k(n)}y_n-p) \\
&\quad +b_n(T_{i(n)}^{k(n)}z_n-p), j(x_n-p)\rangle \\
&= (1-a_n-b_n)^2\|x_{n-1}-p\|^2 \\
&\quad +2a_n\langle T_{i(n)}^{k(n)}y_n-p, j(x_n-p)\rangle \\
&\quad +2b_n\langle T_{i(n)}^{k(n)}z_n-p, j(x_n-p)\rangle \\
&= (1-a_n-b_n)^2\|x_{n-1}-p\|^2 \\
&\quad +2a_n\langle T_{i(n)}^{k(n)}y_n-T_{i(n)}^{k(n)}x_n+T_{i(n)}^{k(n)}x_n-p, j(x_n-p)\rangle \\
&\quad +2b_n\langle T_{i(n)}^{k(n)}z_n-T_{i(n)}^{k(n)}x_n+T_{i(n)}^{k(n)}x_n-p, j(x_n-p)\rangle \\
&= (1-a_n-b_n)^2\|x_{n-1}-p\|^2 \\
&\quad +2a_n\langle T_{i(n)}^{k(n)}y_n-T_{i(n)}^{k(n)}x_n, j(x_n-p)\rangle \\
&\quad +2a_n\langle T_{i(n)}^{k(n)}x_n-p, j(x_n-p)\rangle \\
&\quad +2b_n\langle T_{i(n)}^{k(n)}z_n-T_{i(n)}^{k(n)}x_n, j(x_n-p)\rangle \\
&\quad +2b_n\langle T_{i(n)}^{k(n)}x_n-p, j(x_n-p)\rangle \\
&\leq (1-a_n-b_n)^2\|x_{n-1}-p\|^2 \\
&\quad +2a_n\|T_{i(n)}^{k(n)}y_n-T_{i(n)}^{k(n)}x_n\|\|x_n-p\| \\
&\quad +2b_n\|T_{i(n)}^{k(n)}z_n-T_{i(n)}^{k(n)}x_n\|\|x_n-p\| \\
&\quad +2a_n\{h_{k(n)}\|x_n-p\|^2+\tau_{k(n)}-\Phi(\|x_n-p\|)\} \\
&\quad +2b_n\{h_{k(n)}\|x_n-p\|^2+\tau_{k(n)}-\Phi(\|x_n-p\|)\} \\
&\leq (1-a_n-b_n)^2\|x_{n-1}-p\|^2+2a_n\xi_n^i+2b_n\delta_n^i \\
&\quad +2a_nh_{k(n)}\|x_n-p\|^2+2a_n\tau_{k(n)}-2a_n\Phi(\|x_n-p\|) \\
&\quad +2b_nh_{k(n)}\|x_n-p\|^2+2b_n\tau_{k(n)}-2b_n\Phi(\|x_n-p\|) \\
&= (1-a_n-b_n)^2\|x_{n-1}-p\|^2-2(a_n+b_n)\Phi(\|x_n-p\|) \\
&\quad +2(a_n+b_n)h_{k(n)}\|x_n-p\|^2+2(a_n\xi_n^i+b_n\delta_n^i) \\
&\quad +2(a_n+b_n)\tau_{k(n)} \\
&\leq (1-a_n-b_n)^2\|x_{n-1}-p\|^2-2(a_n+b_n)\Phi(\|x_n-p\|) \\
&\quad +2(a_n+b_n)h_{k(n)}\|x_n-p\|^2+2(a_n+b_n)\max\{\xi_n^i,\delta_n^i\} \\
&\quad +2(a_n+b_n)\tau_{k(n)} \qquad (2.2)
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_n^i &= M\|T_{i(n)}^{k(n)}y_n-T_{i(n)}^{k(n)}x_n\| \\
\delta_n^i &= M\|T_{i(n)}^{k(n)}z_n-T_{i(n)}^{k(n)}x_n\|.
\end{aligned}
$$

From (1.11) we have

$$\begin{aligned}
\|y_n - x_n\| &= \|y_n - x_{n-1} + x_{n-1} - x_n\| \\
&\leq \|y_n - x_{n-1}\| + \|x_{n-1} - x_n\| \\
&= \|(1 - a'_n - b'_n)x_{n-1} + a'_n S^{k(n)}_{i(n)} z_n \\
&\quad + b'_n S^{k(n)}_{i(n)} x_n - x_{n-1}\| \\
&\quad + \|x_{n-1} - [(1 - a_n - b_n)x_{n-1} \\
&\quad + a_n T^{k(n)}_{i(n)} y_n + b_n T^{k(n)}_{i(n)} z_n]\| \\
&= \|a'_n(S^{k(n)}_{i(n)} z_n - x_{n-1}) + b'_n(S^{k(n)}_{i(n)} x_n - x_{n-1})\| \\
&\quad + \|a_n(x_{n-1} - T^{k(n)}_{i(n)} y_n) + b_n(x_{n-1} - T^{k(n)}_{i(n)} z_n)\| \\
&\leq a'_n(\|S^{k(n)}_{i(n)} z_n - p\| + \|x_{n-1} - p\|) + b'_n(\|S^{k(n)}_{i(n)} x_n - p\| + \|x_{n-1} - p\|) \\
&\quad + \|x_{n-1} - p\|) + a_n(\|T^{k(n)}_{i(n)} y_n - p\| + \|x_{n-1} - p\|) \\
&\quad + b_n(\|T^{k(n)}_{i(n)} z_n - p\| + \|x_{n-1} - p\|) + \|x_{n-1} - p\|) \\
&\leq 2a'_n M + 2b'_n M + 2a_n M + 2b_n M \\
&= 2M(a'_n + b'_n + a_n + b_n). \qquad (2.3)
\end{aligned}$$

From the condition (i) and (2.3), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0, \qquad (2.4)$$

and the uniform continuity of T_i leads to

$$\lim_{n\to\infty} M\|T^{k(n)}_{i(n)} y_n - T^{k(n)}_{i(n)} x_n\| = 0,$$

thus, we have

$$\lim_{n\to\infty} \xi^i_n = 0. \qquad (2.5)$$

Again from (1.11) we have

$$\begin{aligned}
\|z_n - x_n\| &= \|(1 - a'')x_n + a''_n H^{k(n)}_{i(n)} x_n - x_n\| \\
&= \|a''_n(H^{k(n)}_{i(n)} x_n - x_n)\| \\
&= \|a''_n(H^{k(n)}_{i(n)} x_n - p + p - x_n)\| \\
&\leq a''_n(\|H^{k(n)}_{i(n)} x_n - p\| + \|x_n - p\|) \\
&\leq a''_n(M + M) \\
&= 2a''_n M. \qquad (2.6)
\end{aligned}$$

From the condition (i) and (2.6), we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0, \qquad (2.7)$$

and the uniform continuity of T_i leads to

$$\lim_{n\to\infty} M\|T^{k(n)}_{i(n)} z_n - T^{k(n)}_{i(n)} x_n\| = 0, \qquad (2.8)$$

thus, we have

$$\lim_{n\to\infty} \delta^i_n = 0. \qquad (2.9)$$

From (2.2) we obtain

$$\begin{aligned}
\|x_n - p\|^2 &\leq \frac{(1-a_n-b_n)^2}{1-2(a_n+b_n)h_{k(n)}}\|x_{n-1}-p\|^2 - \frac{2(a_n+b_n)}{1-2(a_n+b_n)h_{k(n)}}\Phi(\|x_n-p\|) \\
&\quad + \frac{2(a_n+b_n)\tau_{k(n)}}{1-2(a_n+b_n)h_{k(n)}} + \frac{2(a_n+b_n)\max\{\xi_n^i, \delta_n^i\}}{1-2(a_n+b_n)h_{k(n)}} \\
&= \left[1 + \frac{(1-a_n-b_n)^2 - \{1-2(a_n+b_n)h_{k(n)}\}}{1-2(a_n+b_n)h_{k(n)}}\right]\|x_{n-1}-p\|^2 \\
&\quad - \frac{2(a_n+b_n)}{1-2(a_n+b_n)h_{k(n)}}\Phi(\|x_n-p\|) + \frac{2(a_n+b_n)\tau_{k(n)}}{1-2(a_n+b_n)h_{k(n)}} \\
&\quad + \frac{2(a_n+b_n)\max\{\xi_n^i, \delta_n^i\}}{1-2(a_n+b_n)h_{k(n)}} \\
&= \left[1 + \frac{1-2(a_n+b_n)+(a_n+b_n)^2 - 1 + 2(a_n+b_n)h_n}{1-2(a_n+b_n)h_{k(n)}}\right]\|x_{n-1}-p\|^2 \\
&\quad - \frac{2(a_n+b_n)}{1-2(a_n+b_n)h_{k(n)}}\Phi(\|x_n-p\|) + \frac{2(a_n+b_n)\tau_{k(n)}}{1-2(a_n+b_n)h_{k(n)}} \\
&\quad + \frac{2(a_n+b_n)\max\{\xi_n^i, \delta_n^i\}}{1-2(a_n+b_n)h_{k(n)}} \\
&= \left[1 + \frac{(a_n+b_n)^2 + 2(a_n+b_n)(h_n-1)}{1-2(a_n+b_n)h_{k(n)}}\right]\|x_{n-1}-p\|^2 \\
&\quad - \frac{2(a_n+b_n)}{1-2(a_n+b_n)h_{k(n)}}\Phi(\|x_n-p\|) + \frac{2(a_n+b_n)\tau_{k(n)}}{1-2(a_n+b_n)h_n} + \\
&\quad \frac{2(a_n+b_n)\max\{\xi_n^i, \delta_n^i\}}{1-2(a_n+b_n)h_{k(n)}}. \qquad (2.10)
\end{aligned}$$

Since $(a_n + b_n) \to 0$, $h_{k(n)} \to 1$ as $n \to \infty$, there exists a positive integer n_0 such that

$$\frac{1}{2} < 1 - 2(a_n+b_n)h_{k(n)} \leq 1 \ \forall n \geq n_0. \qquad (2.11)$$

Therefore, it follows from (2.10) that

$$\begin{aligned}
\|x_n - p\|^2 &\leq [1 + 2(a_n+b_n)\{(a_n+b_n) + 2(h_{k(n)}-1)\}]\|x_{n-1}-p\|^2 \\
&\quad -2(a_n+b_n)\Phi(\|x_n-p\|) + 4(a_n+b_n)\tau_{k(n)} \\
&\quad +4(a_n+b_n)\max\{\xi_n^i, \delta_n^i\} \\
&\leq \|x_{n-1}-p\|^2 - 2(a_n+b_n)\Phi(\|x_n-p\|) \\
&\quad +2(a_n+b_n)\{(a_n+b_n) + 2(h_{k(n)}-1)\}M^2 + 4(a_n+b_n)\tau_{k(n)} \\
&\quad +4(a_n+b_n)\max\{\xi_n^i, \delta_n^i\} \\
&= \|x_{n-1}-p\|^2 - 2(a_n+b_n)\Phi(\|x_n-p\|) + 2(a_n+b_n)[M^2\{(a_n+b_n) \\
&\quad +2(h_{k(n)}-1)\} + 2\tau_{k(n)} + 2\max\{\xi_n^i, \delta_n^i\}] \\
&= \|x_{n-1}-p\|^2 - 2(a_n+b_n)\Phi(\|x_n-p\|) + 2(a_n+b_n)\jmath_n, \qquad (2.12)
\end{aligned}$$

where

$$\jmath_n = [M^2\{(a_n+b_n) + 2(h_{k(n)}-1)\} + 2\tau_{k(n)} + 2\max\{\xi_n^i, \delta_n^i\}] \to 0 \qquad (2.13)$$

as $n \to \infty$.

For all $n \geq 1$, put

$$\begin{aligned} \rho_n &= \|x_{n-1} - p\| \\ \lambda_n &= 2(a_n + b_n) \\ \mu_n &= 2(a_n + b_n)\jmath_n. \end{aligned}$$

Now, with the help of (i)-(ii), $\lim_{n\to\infty} \tau_{k(n)} = 0$, (2.5), (2.9), (2.13) and Lemma 1.3, we obtain from (2.12) that

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

This completes the proof of Theorem 2.1. □

Remark 2.2. Theorem 2.1 extends and improves the corresponding results of Ofoedu [14], Kim [13], Rafiq [25], Tan and Xu [29], Zeng [33], Chang [2], Cho et al. [8], Chidume [5]- [6], Schu [27], Saluja [26], Gu [9], Thakur [31], Sun [28].

Using the method of proof in Theorem 2.1, we have the following results.

Corollary 2.3. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $T_i : K \to K$ be an asymptotically generalized Φ-hemicontractive mapping in the intermediate sense with sequence $\{\eta_n^i\} \subset [1, \infty)$, where $\eta_n^i \to 1$ as $n \to \infty$. Let $S_i : K \to K$ be an asymptotically generalized Φ-hemicontractive mapping in the intermediate sense with sequence $\{\zeta_n^i\} \subset [1, \infty)$, where $\zeta_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $T_i(K)$ be bounded, T_i and S_i be uniformly continuous for each $i \in I$. Let $h_n = \max\{\eta_n, \zeta_n\}$, where $\eta_n = \max\{\eta_n^i : i \in I\}$ and $\zeta_n = \max\{\zeta_n^i : i \in I\}$. Assume that $\boldsymbol{F} = (\bigcap_{i=1}^N F(T_i)) \bigcap (\bigcap_{i=1}^N F(S_i)) \neq \emptyset$. Let $\{a_n\}$, $\{a'_n\}$ be sequences in [0,1], for each $n \geq 1$. Put*

$$\varpi_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle T_i^n x - p, j(x-p)\rangle - k_n^i \|x-p\|^2 + \Phi_i(\|x-p\|))\right\} \text{ and}$$

$$\ell_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle S_i^n x - p, j(x-p)\rangle - \zeta_n^i \|x-p\|^2 + \Phi_i(\|x-p\|))\right\}.$$

Let $\tau_n = \max\{\varpi_n, \ell_n\}$, where $\varpi_n = \max\{\varpi_n^i : i \in I\}$, $\ell_n = \max\{\ell_n^i : i \in I\}$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a'_n = 0$;

(ii) $\sum_{n=1}^{\infty} a_n = \infty$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in K, \\ x_n = (1-a_n)x_{n-1} + a_n T_{i(n)}^{k(n)} y_n, & \forall n \geq 1, \\ y_n = (1-a'_n)x_{n-1} + a'_n S_{i(n)}^{k(n)} x_n \end{cases} \tag{2.14}$$

Then the sequence $\{x_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Set $b_n = b'_n = a''_n = 0$ in Theorem 2.1. □

Corollary 2.4. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $T_i : K \to K$ be an asymptotically generalized Φ-hemicontractive mapping in the intermediate sense with sequence $\{\eta_n^i\} \subset [1, \infty)$, where $\eta_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $T_i(K)$ be bounded and T_i be uniformly continuous for each $i \in I$. Let $h_n = \max\{\eta_n^i : i \in I\}$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{a_n\}$ and $\{a'_n\}$ be sequences in [0,1] for each $n \geq 1$. Put*

$$\varpi_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle T_i^n x - p, j(x-p)\rangle - k_n^i\|x-p\|^2 + \Phi_i(\|x-p\|))\right\}.$$

Let $\tau_n = \max\{\varpi_n^i : i \in I\}$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a'_n = 0$;

(ii) $\sum_{n=1}^{\infty} a_n = \infty$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in K, \\ x_n = (1-a_n)x_{n-1} + a_n T_{i(n)}^{k(n)} y_n, & \forall n \geq 1, \\ y_n = (1-a'_n)x_{n-1} + a'_n T_{i(n)}^{k(n)} x_n \end{cases} \tag{2.15}$$

Then the sequence $\{x_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Set $T_i = S_i$ in Corollary 2.3. □

Corollary 2.5. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $T_i : K \to K$ be an asymptotically generalized Φ-hemicontractive mapping in the intermediate sense with sequence $\{\eta_n^i\} \subset [1, \infty)$, where $\eta_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $T_i(K)$ be bounded and T_i be uniformly continuous for each $i \in I$. Let $h_n = \max\{\eta_n^i : i \in I\}$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{u_n\}$ be bounded in K and $\{a_n\}$ be a sequence in [0,1], for each $n \geq 1$. Put*

$$\varpi_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle T_i^n x - p, j(x-p)\rangle - k_n^i\|x-p\|^2 + \Phi_i(\|x-p\|))\right\}.$$

Let $\tau_n = \max\{\varpi_n^i : i \in I\}$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = 0$;

(ii) $\sum_{n=1}^{\infty} a_n = \infty$;.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in K, \\ x_n = (1-a_n)x_{n-1} + a_n T_{i(n)}^{k(n)} x_{n-1} \end{cases} \quad \forall n \geq 1. \tag{2.16}$$

Then the sequence $\{x_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Set $a'_n = 0$ in Corollary 2.4. □

Corollary 2.6. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $T_i : K \to K$ be an asymptotically generalized Φ-hemicontractive mapping in the intermediate sense with sequence $\{\eta_n^i\} \subset [1, \infty)$, where $\eta_n^i \to 1$ as $n \to \infty$ and $H_i : K \to K$ an asymptotically generalized Φ-hemicontractive mapping in the intermediate sense with sequence $\{t_n^i\} \subset [1, \infty)$, where $t_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $T_i(K)$ be bounded, T_i and H_i be uniformly continuous for each $i \in I$. Let $h_n = \max\{\eta_n, t_n\}$, where $\eta_n = \max\{\eta_n^i : i \in I\}$ and $t_n = \max\{t_n^i : i \in I\}$. Assume that $\boldsymbol{F} = (\bigcap_{i=1}^N F(T_i)) \bigcap (\bigcap_{i=1}^N F(H_i)) \neq \emptyset$. Let $\{b_n\}$ and $\{a''_n\}$ be sequences in [0,1] for each $n \geq 1$. Put*

$$\varpi_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle T_i^n x - p, j(x-p)\rangle - k_n^i\|x-p\|^2 + \Phi_i(\|x-p\|))\right\} \text{ and}$$

$$\vartheta_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle H_i^n x - p, j(x-p)\rangle - t_n^i\|x-p\|^2 + \Phi_i(\|x-p\|))\right\}.$$

Let $\tau_n = \max\{\varpi_n, \vartheta_n\}$, where $\varpi_n = \max\{\varpi_n^i : i \in I\}$ and $\vartheta_n = \max\{\vartheta_n^i : i \in I\}$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} b_n = a''_n = 0$;

(ii) $\sum_{n=1}^{\infty} b_n = \infty$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in K, \\ x_n = (1-b_n)x_{n-1} + b_n T_{i(n)}^{k(n)} z_n, & \forall n \geq 1, \\ z_n = (1-a''_n)x_n + a''_n H_{i(n)}^{k(n)} x_n \end{cases} \tag{2.17}$$

Then the sequence $\{x_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Set $a_n = b'_n = c'_n = 0$ in Theorem 2.1. □

Corollary 2.7. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $T_i : K \to K$ be an asymptotically generalized Φ-hemicontractive mapping in the intermediate sense with sequence $\{\eta_n^i\} \subset [1, \infty)$, where $\eta_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $T_i(K)$ be bounded and T_i be uniformly continuous for each $i \in I$. Let $h_n = \max\{\eta_n^i : i \in I\}$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{b_n\}$ and $\{a''_n\}$ be sequences in [0,1], for each $n \geq 1$. Put*

$$\varpi_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle T_i^n x - p, j(x-p)\rangle - k_n^i\|x-p\|^2 + \Phi_i(\|x-p\|))\right\}.$$

Let $\tau_n = \max\{\varpi_n^i : i \in I\}$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a''_n = 0$;

(ii) $\sum_{n=1}^{\infty} b_n = \infty$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in K, \\ x_n = (1-b_n)x_{n-1} + b_n T_{i(n)}^{k(n)} z_n, & \forall n \geq 1, \\ z_n = (1-a''_n)x_n + a''_n T_{i(n)}^{k(n)} x_n \end{cases} \tag{2.18}$$

Then the sequence $\{x_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Set $T_i = H_i$ in Corollary 2.6. □

Corollary 2.8. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1,2,3,...,N\}$. Let $T_i : K \to K$ be an asymptotically generalized Φ-hemicontractive mapping in the intermediate sense with sequence $\{\eta_n^i\} \subset [1,\infty)$, where $\eta_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $T_i(K)$ be bounded and T_i be uniformly continuous for each $i \in I$. Let $h_n = \max\{\eta_n^i : i \in I\}$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{b_n\}$ be a sequence in [0,1] ,for each $n \geq 1$. Put*

$$\varpi_n^i = \max\left\{0, \sup_{(x,p)\in K\times F(T)} (\langle T_i^n x - p, j(x-p)\rangle - k_n^i\|x-p\|^2 + \Phi_i(\|x-p\|))\right\}.$$

Let $\tau_n = \max\{\varpi_n^i : i \in I\}$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} b_n = 0$;

(ii) $\sum_{n=1}^{\infty} b_n = \infty$

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in K, \\ x_n = (1-b_n)x_{n-1} + b_n T_{i(n)}^{k(n)} x_n \end{cases} \quad \forall n \geq 1. \tag{2.19}$$

Then the sequence $\{x_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Set $a''_n = 0$ in Corollary 2.7. □

Remark 2.9. Under suitable conditions, the sequence $\{x_n\}$ defined by (1.11) can also be generalized to the iterative sequences with errors. Thus all the results proved in this paper can also be proved for the iterative process with errors. In this case our main iterative process (1.11) looks like

$$\begin{cases} x_0 \in K, \\ x_n = (1-a_n-b_n-c_n)x_{n-1} + a_n T_{i(n)}^{k(n)} y_n + b_n T_{i(n)}^{k(n)} z_n + c_n u_n, \\ y_n = (1-a'_n-b'_n-c'_n)x_{n-1} + a'_n S_{i(n)}^{k(n)} z_n + b'_n S_{i(n)}^{k(n)} x_n + c'_n v_n, \\ z_n = (1-a''_n-b''_n)x_n + a''_n H_{i(n)}^{k(n)} x_n + b''_n w_n \end{cases} \forall n \geq 1, \tag{2.20}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$ are real sequences in $[0,1]$ satisfying $a_n+b_n+c_n \leq 1$, $a'_n+b'_n+c'_n \leq 1$ and $a''_n+b''_n \leq 1$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K and $n=(k-1)N+i$, $i=i(n)\in\{1,2,...,N\}$, $k=k(n)\geq 1$ is some positive integers and $k(n)\to\infty$ as $n\to\infty$.

Next, we give the following example to support our results.

Example 2.10. Let $E=(-\infty,+\infty)$ with the usual norm and $K=[0,+\infty)$. Let $\Phi:[0,\infty)\to[0,\infty)$ be a strictly increasing function with $\Phi(0)=0$. For $N=2$, let $\{T_i\}_{i=1}^2,\{S_i\}_{i=1}^2,\{H_i\}_{i=1}^2:K\to K$ be defined by:

$$\begin{aligned}
T_1x &= \frac{x}{2(1+x)},\ x\in[0,\infty)\ \text{ and }\ \Phi(s)=\frac{s^3}{1+s},\\
T_2x &= \frac{x}{(1+x)},\ x\in[0,\infty)\ \text{ and }\ \Phi(s)=\frac{s^3}{1+s},\\
S_1x &= \begin{cases}\frac{2x^2}{1+2x}, & \text{for }\ x\in[0,\infty)\\ x, & \text{for }\ x\in(-\infty,0)\end{cases} \quad \text{and } \Phi(s)=\frac{s^2}{1+2s},\\
S_2x &= \frac{x}{1+\alpha x},\ x\in[0,\infty)\ \text{ and } \alpha \text{ is closing to zero, } \forall n\in\mathbb{N} \text{ and } \Phi(s)=\frac{s^3}{1+s},\\
H_1x &= \frac{x^3}{1+x^2},\ x\in[0,\infty)\ \text{ and }\ \Phi(s)=\frac{s^2}{1+s^2},\\
H_2x &= \frac{x}{4},\ x\in[0,\infty)\ \text{ and }\ \Phi(s)=\frac{s^2}{4}.
\end{aligned}$$

Set $a_n=\frac{1}{n+1}, b_n=\frac{1}{n}, a'_n=\frac{1}{(n+1)^2}, b'_n=\frac{1}{n^3}, a''_n=\frac{1}{(n+1)+(n+1)^2}$, for all $n\geq 1$.

Clearly, $\{T_i\}_{i=1}^2,\{S_i\}_{i=1}^2$ and $\{H_i\}_{i=1}^2$ are asymptotically generalized Φ-hemicontractive mappings with constant sequence $\{k_n\}=\{1\}$ for all $n\geq 1$ and also uniformly continuous mappings on $[0,+\infty)$ and hence, they are asymptotically generalized Φ-hemicontractive mappings in the intermediate sense with $\tau_n=0$. Furthermore, $R(T_1)=[0,\frac{1}{2})$ and $R(T_2)=[0,1)$. This follows that T_1 and T_2 have bounded ranges. Obviously, $\mathbf{F}=(\bigcap_{i=1}^2 F(T_i))\bigcap(\bigcap_{i=1}^2 F(S_i))\bigcap(\bigcap_{i=1}^2 F(H_i))=\{0\}=p\neq\emptyset$. For arbitrary $x_0\in K$, the sequence $\{x_n\}_{n=1}^\infty\in K$ defined by (1.11) converges strongly to the common fixed point of T_i and S_i and H_i $(i=1,2)$ which is $\{0\}$, satisfying Theorem 2.1. This means that Theorem 2.1 is applicable.

Remark 2.11. All of the above results are also valid for Lipschitz asymptotically generalized Φ-hemicontractive mappings in the intermediate sense.

§3 Conclusion

In this paper, we proposed a three-step composite implicit iteration process for approximating the common fixed point of three uniformly continuous and asymptotically generalized Φ-hemicontractive mappings in the intermediate. Our new three-step composite implicit iteration process (1.11) properly includes several iteration processes in literature and also the class of asymptotically generalized Φ-hemicontractive mappings in the intermediate sense is the most general of those mentioned the literature. Hence, our result extends, generalizes and improves the corresponding results of Ofoedu [14], Kim [13], Rafiq [25], Tan and Xu [29], Zeng [33], Chang [2], Cho et al. [8], Chidume [5]- [6], Schu [27], Saluja [26], Gu [9], Thahur [31], Sun [28] since their results are special cases of our result.

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Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] Y I Albert, C E Chidume, H Zegeye. *Regularization of nonlinear ill-posed equations with accretive operators*, Fixed Point Theory and Applications, 2005, 1: 11-33.

[2] S S Chang. *Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings*, Proc Amer Math Soc, 2001, 129: 845-853.

[3] C E Chidume, C O Chidume. *Convergence theorems for fixed points of uniformly continuous generalized Φ-hemi-contractive mappings*, J Math Anal Appl, 2005, 303: 545-554.

[4] C E Chidume, A U Bello, M E Okpala, P Ndambomve. *Strong convergence theorem for fixed points of nearly nniformly L–Lipschitzian Asymptotically Generalized Φ-hemicontractive mappings*, *International Journal of Mathematical Analysis*, 2015, 9(52): 2555-2569.

[5] C E Chidume, C O Chidume. *Convergence theorems for fixed points of uniformly continuous generalized Φ–hemi-contractive mappings*, J Math Anal Appl, 2005, 303: 545-554.

[6] C E Chidume, C O Chidume. *Convergence theorem for zeros of generalized Lipschitz generalized phi-quasi-accretive operators*, Proc Amer Math Soc, 2006, 134: 243-251.

[7] C E Chidume, N Shahzad, H Zegeye. *Convergence theorems for mappings which are asymptoyically nonexpansive in the intermediate sense*, Numer Funct Anal Optim, 2005, 25: 239-257.

[8] Y J Cho, H Y Zhou, G Guo. *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings*, Comput Math Appl, 2004, 47: 707-717.

[9] F Gu, *Convergence theorems for ϕ-pseudo-contractive type mappings in normed linear spaces*, Northeast Math J, 2001, 17(3): 340-346.

[10] W Kaczor, T Kuczumow, S Reich. *A mean ergodic theory for mappings which are asymptotically nonexpansive in the intermediate sense*, Nonlinear Analysis: Theory, Methods and Applications, 2006, 47(4): 2731-2742.

[11] S H Khan, B Gunduz, S Akbulut. *Solving nonlinear ϕ-strongly accretive operatorequations by a one-step-two-mappings iterative scheme*, J Nonlinear Sci Appl, 2015, 8: 837-846.

[12] J K Kim, A Rafiq, H G Hyun. *Almost stability of the Ishikawa iteration method with error terms involving strictly hemicontractive mappings in smooth Banach spaces*, Nonlinear Funct Anal Appl, 2012, 17(2): 225-234.

[13] J K Kim, D R Sahu, Y M Nam. *Convergence theorems for fixed points of nearly uniformly L-Lipschitzian asymptotically generalized Φ- hemicontractive mappings*, Nonlinear Anal, 2009, 71: 2833-2838.

[14] E U Ofoedu. *Strong convergence theorem for uniformly L -Lipschitzian asymptotically pseudocontractive mapping in a real Banach space*, J Math Anal Appl, 2006, 321: 722-728.

[15] A E Ofem, D I Igbokwe. *An efficient iterative method and its applications to a nonlinear integral equation and a delay differential equation in Banach spaces*, Turkish J Ineq, 2020, 4(2): 79-107.

[16] A E Ofem. *Strong convergence of a multi-step implicit iterative scheme with errors for common fixed points of uniformly L-Lipschitzian total asymptotically strict pseudocontractive mappings*, Results in Nonlinear Analysis, 2020, 3(3): 100-116.

[17] A E Ofem. *Strong convergence of modified implicit hybrid S-iteration scheme for finite family of nonexpansive and asymptotically generalized Φ-hemicontractive mappings*, Malaya J Matematik, 2020, 8(4): 1643-1649

[18] A E Ofem, D I Igbokwe, X A Udo-utun. *Implicit iteration process for Lipschitzian α-hemicontraction semigroups*, MathLAB J, 2020, 7: 43-52.

[19] G A Okeke, J O Olareru. *Modifed Noor iterations with errors for nonlinear equations in Banach spaces*, J Nonlinear Sci Appl, 2014, 7: 180-187.

[20] J O Olaleru, G A Okeke. *Strong convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense*, British Journal of Mathematics and Computer Science, 2012, 2(3): 151-162.

[21] J O Olaleru, G A Okeke, *Convergence theorems on asymptotically demicontractive and hemicontractive mappings in the intermediate sense*, Fixed Point Theory and Applications, 2013, 2013: 352.

[22] G A Okeke, J O Olaleru, H Akewe. *Convergence theorems on asymptotically generalized Φ-hemicontractive mappings in the intermediate sense*, International Journal of Mathematical Analysis, 2013, 7(40): 1991-2003.

[23] M O Osilike. *Iterative solution of nonlinear equations of the ϕ-strongly accretive type*, J Math Anal Appl, 1996, 200: 259-271.

[24] X Qin, S Y Cho, J K Kim. *Convergence Theorems on asymptotically pseudocontractive mappings in the intermediate sense*, Fixed Point Theory and Applications, 2010, 2010: 186874.

[25] A Rafiq, M Imdad. *Implicit Mann Iteration Scheme for hemicontractive mapping*, Journal of the Indian Math Soc, 2014, 81(1-2): 147-153.

[26] G S Saluja. *Convergence of the explicit iteration method for strictly asymptotically pseudocontractive mappings in the intermediate sense*, Novi Sad J Math, 2014, 44(1): 75-90.

[27] J Schu. *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull Austr Math Soc, 1991, 43: 153-159.

[28] Z H Sun. *Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings*, J Math Anal Appl, 2003, 286: 351-358.

[29] K K Tan, H K Xu. *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J Math Anal Appl, 1993, 178 (2): 301-308.

[30] J Tang, S S Chang, J Dong. *Split equality fixed point problem for two quasi-asymptotically pseudo-contractive mappings*, J Nonlinear Funct Anal, 2017, 2017: 26.

[31] B S Thakur. *Weak and strong convergence of composite implicit iteration process*, Appl Math Comput, 2007, 190: 965-973.

[32] C Yang, S He. *The successive contraction method for fixed point of pseudocontractive mappings*, Appl Set-Valued Anal Optim, 2019, 1: 337-347.

[33] L C Zeng. *On the iterative approximation for asymptotically pseudocontractive mappings in uniformly smooth Banach spaces*, Chinese Math Ann, 2005, 26: 283-290.

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