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Generalized Kannan-type contraction and fixed point theorems

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Abstract. In this paper, the generalized Kannan-type contraction in cone metric spaces over Banach algebras is introduced. The fixed point theorems satisfying generalized contractive conditions are obtained, without appealing to completeness of X or normality of the cone. The continuity of the mapping is relaxed. Furthermore, we prove that the completeness in cone metric spaces over Banach algebras is necessary if the generalized Kannan-type contraction has a fixed point in X. These results greatly generalize several well-known comparable results in the literature.

§1 Introduction and preliminaries

In 1968, Kannan [13] proved the following famous fixed point theorem. The mapping satisfying the contractive condition is known as Kannan-type contraction mapping.

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that there exists $K < \frac{1}{2}$ satisfying:*

$$d(Tx, Ty) \leq K\{d(x, Tx) + d(y, Ty)\}, \tag{1.1}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z.

Afterwards, Fisher [5] and Khan [14] gave two important fixed point theorems on compact metric spaces. They proved that the continuous self-mapping on compact metric space has a unique fixed point if T satisfies

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\} \tag{1.2}$$

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or

$$d(Tx,Ty) < (d(x,Tx)d(y,Ty))^{\frac{1}{2}}$$

for all $x, y \in X$ with $x \neq y$, respectively.

The generalizations of Kannan-type contraction mapping were discussed in [7, 17]. They showed that if the inequality (1.1) was replaced by the condition either

$$d(Tx,Ty) \leq Ad(x,Tx) + Bd(y,Ty) + Cd(x,y), x, y \in X \tag{1.3}$$

where $A, B, C \geq 0$ and $A + B + C < 1$, or

$$d(Tx,Ty) < Ad(x,Tx) + Bd(y,Ty) + Cd(x,y), x, y \in X \tag{1.4}$$

where $A, B, C \geq 0$ and $A + B + C = 1$, respectively, then the conclusion of Theorem 1.1 was also true.

In the literature known by the authors, the cone metric space was reintroduced by Huang and Zhang [12] in 2007, which is a generalization of metric space. Then, many fixed point results in cone metric spaces were introduced and references mentioned therein (see [3, 20]). Later on, Liu and Xu [16] defined the notion of cone metric space over Banach algebra and obtained the existence of some fixed point results in such spaces. Moreover, they gave an example to illustrate the non-equivalence of fixed point theorems between metric spaces and cone metric spaces over Banach algebras.

Very recently, Górnicki [7] and Garai et al. [6] established some meaningful theorems when T was a Kannan-type contractive self-mapping (satisfying (1.2)) in metric spaces. Górnicki [8] showed some fixed point theorems about extensions of Kannan-type contraction in metric spaces. In this paper, we introduce the concept of generalized Kannan-type contraction in cone metric spaces over Banach algebras. By defining the notions of bounded compactness, T-orbital compactness, orbital continuity and asymptotic regularity, we establish corresponding fixed point theorems in cone metric spaces over Banach algebras. Our main results improve and generalize some important known results in the literature [1, 6, 7, 9, 13, 17]. Furthermore, we prove that the completeness in cone metric spaces over Banach algebras is necessary if the generalized Kannan-type contraction mapping has a fixed point in X. In addition, we give some examples to show that the main results are genuine improvements and generalizations of the corresponding results in the literature.

First, we recall some basic definitions of Banach algebras and cone metric spaces.

Let $\mathcal{A}$ be a real Banach algebra, i.e., $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties: for all $x, y, z \in \mathcal{A}, a \in \mathbb{R}$

(1) $x(yz) = (xy)z$;

(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;

(3) $a(xy) = (ax)y = x(ay)$;

(4) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that the Banach algebra $\mathcal{A}$ has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [19].

A subset P of $\mathcal{A}$ is called a cone if :

(i) P is non-empty, closed and $\{\theta, e\} \subset P$, where θ denotes the zero element of $\mathcal{A}$;
(ii) $\alpha P + \beta P \in P$ for all non-negative real numbers α, β;
(iii) $P^2 = PP \subset P$;
(iv) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P.

A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \| x \| \leq K \| y \| .$$

The least positive number satisfying the above inequality is called the normal constant of P.

A cone P is called regular if every increasing sequence which is bounded from above is convergent. In other words, if there is a $y \in \mathcal{A}$ such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y,$$

then there exists $x \in \mathcal{A}$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Equivalently, a cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that every regular cone is normal.

A cone P is called strongly minihedral if each subset of $\mathcal{A}$ which is bounded from above has a supremum. If P is a strongly minihedral cone, then every subset of $\mathcal{A}$ bounded below has an infimum (see [3, 20]).

Throughout this paper, we always assume that P is a cone over Banach algebra $\mathcal{A}$ with $\mathrm{int}P \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P.

Definition 1.2. ([12],[16]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to P \subset \mathcal{A}$ satisfies:

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space over a Banach algebra $\mathcal{A}$.

Definition 1.3. ([12],[16]) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x if for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a natural number N such that for all $n > N,\ d(x_n, x) \ll c$;
(ii) $\{x_n\}$ is a Cauchy sequence if for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a natural number N such that for all $n, m > N,\ d(x_n, x_m) \ll c$;
(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent in X.

Definition 1.4. ([15]) Let P be a solid cone in a Banach space $\mathcal{A}$. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

Lemma 1.5. *([11]) Let P be a solid cone in a Banach space $\mathcal{A}$ and let $\{x_n\}$ and $\{y_n\}$ be* *sequences in P. If $\{x_n\}$ and $\{y_n\}$ are c-sequences and $\alpha, \beta \in P$ then $\{\alpha x_n + \beta y_n\}$ is a c-sequence.*

Lemma 1.6. *([19]) Let $\mathcal{A}$ be a real Banach algebra with a unit e and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually, $(e-x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Definition 1.7. ([20]) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and M be a non-empty subset of X. Let P be a normal and strongly minihedral cone. The distance between the set M and the singleton $\{x\}$ is defined as follows:

$$d(x, M) = \inf\{d(x, y) : y \in M\}.$$

Lemma 1.8. *([12]) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and P be a normal cone with a normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x$ and $y_n \to y$. Then $d(x_n, y_n) \to d(x, y)(n \to \infty)$.*

§2 Bounded compactness and T-orbital compactness

The concepts of bounded compactness and T-orbital compactness were studied in usual metric spaces in [6], which were important to weaken the condition of compactness. In the following, we give the notions of generalized Kannan-type contraction, bounded compactness and T-orbital compactness in the framework of cone metric spaces over Banach algebras, which are generalizations of metric spaces.

Definition 2.1. Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ with a unit e. The mapping $T : X \to X$ is said to be a generalized Kannan-type contraction, if it satisfies

$$d(Tx, Ty) \prec \frac{e}{2}\{d(x, Tx) + d(y, Ty)\}, \tag{2.1}$$

for all $x, y \in X$ with $x \neq y$.

Definition 2.2. Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and T be a self-mapping on X. Let $x \in X$ and $O_T(x) = \{x, Tx, T^2x, T^3x, \ldots\}$.

The space (X, d) is said to be boundedly compact, if every bounded sequence in X has a convergent subsequence.

The mapping T is said to be orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O_T(x)$ (for all $x \in X$), $x_n \to z$ as $n \to \infty$ implies $Tx_n \to Tz$ as $n \to \infty$. Obviously, every continuous mapping is orbitally continuous, but the converse is not true.

The set X is said to be a T-orbitally compact set, if every sequence in $O_T(x)$ has a convergent subsequence for all $x \in X$.

Example 2.3. *Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1] \times C^1_{\mathbb{R}}[0,1]$ with the norm*

$$\|(x_1, x_2)\| = \|x_1\|_\infty + \|x_2\|_\infty + \|x_1'\|_\infty + \|x_2'\|_\infty.$$

Define the multiplication by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1),$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{A}$. *Then* $\mathcal{A}$ *is a Banach algebra with a unit* $e = (1, 0)$. *Let* $P = \{(x_1(t), x_2(t)) \in \mathcal{A} : x_1(t) \geq 0, x_2(t) \geq 0, t \in [0, 1]\}$.

(1) Let $X = [0, \infty) \times [0, \infty)$ *and define the cone metric* $d : X \times X \to \mathcal{A}$ *by* $d((x_1, x_2), (y_1, y_2))(t) = (|x_1 - y_1|, |x_2 - y_2|) \exp(t) \in P$, $\forall x = (x_1, x_2), y = (y_1, y_2) \in X$. *We make a conclusion that* (X, d) *is a complete cone metric space over Banach algebra* $\mathcal{A}$. *Put mappings* $T_1, T_2 : X \to X$ *as*

$$T_1(x_1, x_2) = (\frac{x_1}{2^{n+1}}, \frac{x_2}{2^{m+1}}), n \leq x_1 < n + 1, m \leq x_2 < m + 1,$$

and

$$T_2(x_1, x_2) = (2x_1, 3x_2)$$

for all $x \in X$ *and* $n, m \in \mathbb{N}$. *It is easy to show that* X *is* T_1*-orbitally compact but not* T_2*-orbitally compact and also* X *is boundedly compact.*

(2) Let $X = [0, 1) \times [0, 1)$. *The cone metric is defined the same as above and* $T : X \to X$ *is defined by* $T(x_1, x_2) = (\frac{x_1}{3}, \frac{x_2}{4})$. *We observe that* X *is* T*-orbitally compact but not complete.*

(3) Let $X = [0, 2] \times [0, 2]$. *The cone metric is defined the same as above and* $T : X \to X$ *is defined by*

$$T(x, y) = \begin{cases} (0, 0), & (x, y) \in [0, 1] \times [0, 1]; \\ (\frac{x}{2}, \frac{y}{2}), & \textit{otherwise}. \end{cases}$$

Then, for any $x_0 = (x, y) \in X, n \in \mathbb{N}, x_n = Tx_{n-1}, x_n \to \theta$ *implies* $Tx_n \to T\theta = \theta (\theta = (0, 0))$. *So* T *is* T*-orbitally continuous but not continuous in* X.

In the rest of this section, we assume that (X, d) is a cone metric space over Banach algebra $\mathcal{A}$ with regular cone P such that $d(x, y) \in \text{int}P$ for all $x, y \in X$ with $x \neq y$.

Theorem 2.4. *Let* (X, d) *be a boundedly compact cone metric space over Banach algebra* $\mathcal{A}$ *with a unit* e. *Suppose that the mapping* $T : X \to X$ *is orbitally continuous and satisfies*

$$d(Tx, Ty) \prec Ad(x, Tx) + Bd(y, Ty) + Cd(x, y), \tag{2.2}$$

for all $x, y \in X$ *with* $x \neq y$, *where* $A, B, C \in P$ *with* $A + B + C = e$ *and* $(e - B)^{-1}, (e - C)^{-1}$ *exist. Then* T *has a unique fixed point* $z \in X$ *and for each* $x \in X$ *the iterated sequence* $\{T^n x\}$ *converges to* z, *i.e.,* T *is a Picard operator.*

Proof. Let x_0 be an arbitrary point in X and set $x_n = Tx_{n-1} = T^n x_0, n \geq 1$. If there exists an integer $n \in \mathbb{N}$ such that $x_n = x_{n+1} = Tx_n$, then T must have a fixed point. Now, we assume that $x_n \neq x_{n+1}, \forall n \in \mathbb{N}$. Set $s_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. According to (2.2), we have

$$\begin{aligned} s_n &= d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\ &\prec Ad(x_{n-1}, x_n) + Bd(x_n, x_{n+1}) + Cd(x_{n-1}, x_n) \\ &= As_{n-1} + Bs_n + Cs_{n-1}, \end{aligned}$$

which means that $s_n \prec (e - B)^{-1}(A + C)s_{n-1}$. Let $h = (e - B)^{-1}(A + C)$, then $h = e$ by $A + B + C = e$. Therefore, $\{s_n\}$ is a strictly decreasing sequence which is bounded from below, that is

$$\theta \prec \cdots \prec s_n \prec s_{n-1} \prec \cdots \prec s_0 = d(x_0, x_1).$$

Since the cone is regular, we know there is a $b \succeq \theta$ in $\mathcal{A}$ such that $s_n \to b$ $(n \to \infty)$. Thus, for all $n, m \in \mathbb{N}$, we infer

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &= d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m) \\ &\prec d(x_n, x_{n+1}) + Ad(x_n, x_{n+1}) + Bd(x_m, x_{m+1}) + Cd(x_n, x_m) + d(x_{m+1}, x_m), \end{aligned}$$

which implies that

$$\begin{aligned} d(x_n, x_m) &\preceq (e - C)^{-1}\{(e + A)d(x_n, x_{n+1}) + (e + B)d(x_m, x_{m+1})\} \\ &\prec (e - C)^{-1}(2e + A + B)s_0 = [e + 2(e - C)^{-1}]s_0. \end{aligned}$$

So, $\{x_n\}$ is bounded. Owing to the bounded compactness property of X, there exist a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $z \in X$ such that $x_{n_i} \to z$ as $i \to \infty$. By the orbital continuity of T, we obtain $Tx_{n_i} \to Tz$. If $b \succ \theta$, by the fact that every regular cone is a normal cone, we have

$$\theta \prec b = \lim_{i\to\infty} d(x_{n_i}, Tx_{n_i}) = d(z, Tz).$$

Moreover, by the regularity of cone and (2.2), we get

$$\theta \prec b = \lim_{i\to\infty} s_{n_i} = \lim_{i\to\infty} d(Tx_{n_i}, T^2x_{n_i}) = d(Tz, T^2z) \prec Ad(z, Tz) + Bd(Tz, T^2z) + Cd(z, Tz),$$

which implies

$$b = d(Tz, T^2z) \prec d(z, Tz) = b,$$

a contradiction. Thus, $b = \theta$ and $z = Tz$. That is, z is a fixed point of T. Moreover,

$$\begin{aligned} d(x_{n+1}, z) &= d(Tx_n, Tz) \\ &\prec Ad(x_n, Tx_n) + Bd(z, Tz) + Cd(x_n, z) \\ &= Ad(x_n, x_{n+1}) + Cd(x_n, z) \\ &\preceq (A + C)d(x_n, x_{n+1}) + Cd(x_{n+1}, z). \end{aligned}$$

Therefore, $(e - C)d(x_{n+1}, z) \preceq (A + C)d(x_n, x_{n+1}) \to \theta$ and the iterated sequence $\{T^n x\}$ converges to z, i.e., T is a Picard operator.

Now we shall show the fixed point is unique. If there is another fixed point y, then from (2.2), we obtain

$$d(y, z) = d(Ty, Tz) \prec Ad(y, Ty) + Bd(z, Tz) + Cd(y, z) = Cd(y, z),$$

which is a contradiction, which yields the result. □

Theorem 2.5. *Let (X, d) be a T-orbitally compact cone metric space over Banach algebra $\mathcal{A}$ with a unit e, where $T : X \to X$ is orbitally continuous and satisfies*

$$d(Tx, Ty) \prec Ad(x, Tx) + Bd(y, Ty) + Cd(x, y),$$

for all $x, y \in X$ with $x \neq y$, where $A, B, C \in P$ with $A + B + C = e$ and $(e - B)^{-1}, (e - C)^{-1}$ exist. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z, i.e., T is a Picard operator.

Proof. Similar to Theorem 2.4, we get the sequence $x_n = Tx_{n-1} = T^n x_0, n \geq 1$. If there exists an integer $n \in \mathbb{N}$ such that $x_n = x_{n+1} = Tx_n$, then T must have a fixed point. Now, we assume that $x_n \neq x_{n+1}, \forall n \in \mathbb{N}$. It is easy to prove that $s_n \to b \succeq \theta$ $(n \to \infty)$. Since X is T-orbitally compact, there exist a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $z \in X$ such that

$x_{n_i} \to z$ as $n \to \infty$. By orbitally continuity of T, we obtain $Tx_{n_i} \to Tz$. The rest proof is similar to Theorem 2.4. □

Corollary 2.6. *Let (X, d) be a boundedly compact cone metric space over Banach algebra $\mathcal{A}$ with a unit e. Let $T : X \to X$ be a generalized Kannan-type contraction mapping which is orbitally continuous. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z, i.e., T is a Picard operator.*

Proof. Taking $A = B = \frac{e}{2}$ and $C = \theta$, we obtain the conclusion by Theorem 2.4. □

Corollary 2.7. *Let (X, d) be a T-orbitally compact cone metric space over Banach algebra $\mathcal{A}$ with a unit e, where $T : X \to X$ is a generalized Kannan-type contraction mapping and orbitally continuous. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z, i.e., T is a Picard operator.*

Proof. The proof is analogous. □

Example 2.8. *Let $\mathcal{A} = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$. The multiplication is defined by*

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1),$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{A}$. Then $\mathcal{A}$ is a Banach algebra with a unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. Then P is a normal cone with a normal constant $K = 1$. Let $X = [0, 1) \cup \{2\} \times [0, 1) \cup \{2\}$ and define the cone metric $d : X \times X \to \mathcal{A}$ by

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, k|x_2 - y_2|) \in P,$$

for all $x = (x_1, x_2), y = (y_1, y_2)$ in X, where $k > 0$ is a constant. Furthermore, define the mapping $T : X \to X$ by:

$$Tx = \begin{cases} (\frac{1}{6}x_1, \frac{1}{4}x_2), & x \neq 2; \\ (0, 0), & otherwise, \end{cases}$$

where $x \neq 2$ is equivalent to $x = (x_1, x_2) \neq (2, x_2)$ and $x = (x_1, x_2) \neq (x_1, 2)$. Obviously, T is not continuous but T-orbitally continuous. We make a conclusion that (X, d) is an incomplete cone metric space over Banach algebra $\mathcal{A}$ but T-orbitally compact. There are the following three cases.

(i) For $x \neq 2$, $y = 2$,

$$\begin{aligned} d(Tx, T2) &= (|\frac{1}{6}x_1 - 0|, k|\frac{1}{4}x_2 - 0|) \\ &= (\frac{1}{6}x_1, \frac{1}{4}kx_2) \\ &\prec (\frac{5}{12}x_1, \frac{3}{8}kx_2) \\ &= (\frac{1}{2}, 0)[(\frac{5}{6}x_1, k\frac{3}{4}x_2) + (0, 0)] \\ &= (\frac{1}{2}, 0)[(|x_1 - \frac{1}{6}x_1|, k|x_2 - \frac{1}{4}x_2|) + (0, 0)] \\ &\preceq \frac{e}{2}\{d(x, Tx) + d(2, T2)\}. \end{aligned}$$

(ii) For $x, y \in X$ with $x \neq 2, y \neq 2$ and $x \neq y$,

$$\begin{aligned} d(Tx,Ty) &= (|\frac{1}{6}x_1 - \frac{1}{6}y_1|, k|\frac{1}{4}x_2 - \frac{1}{4}y_2|) \\ &\prec [\frac{5}{12}(x_1+y_1), \frac{3}{8}k(x_2+y_2)] \\ &= (\frac{1}{2},0)[(\frac{5}{6}x_1 + \frac{5}{6}y_1), k(\frac{3}{4}x_2 + \frac{3}{4}y_2)] \\ &= (\frac{1}{2},0)[(|x_1 - \frac{1}{6}x_1|, k|x_2 - \frac{1}{4}x_2|) + (|y_1 - \frac{1}{6}y_1|, k|y_2 - \frac{1}{4}y_2|)] \\ &= \frac{e}{2}\{d(x,Tx) + d(y,Ty)\}. \end{aligned}$$

(iii) For $x = 2$ and $y = 2$,

$$d(Tx,Ty) = (0,0) \prec \frac{e}{2}\{d(x,Tx) + d(y,Ty)\}$$

is obviously true. Therefore, we can apply Corollary 2.7 to obtain that T has a unique fixed point $(0,0) \in X$.

§3 Asymptotic regularity

In the following, we obtain the fixed point theorems of generalized contractive mapping in complete cone metric spaces over Banach algebras, under the condition of asymptotic regularity. The regularity or normality of the cone is not necessary. At first, the definition of asymptotic regularity is given, which is a sharp generalization of the counterpart in metric spaces.

Definition 3.1. ([2]) Let (X,d) be a metric space. The mapping $T : X \to X$ is said to be asymptotically regular, if $\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$ for all $x \in X$.

Definition 3.2. Let (X,d) be a cone metric space over a Banach algebra $\mathcal{A}$. The mapping $T : X \to X$ is said to be asymptotically regular, if for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a natural number N such that for all $n \geq N, x \in X$, $d(T^{n+1}x, T^n x) \ll c$. That is, $\{d(T^{n+1}x, T^n x)\}$ is a c-sequence for all $x \in X$.

The continuity of the mapping is not necessary in the following theorems.

Theorem 3.3. *Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$. Let $T : X \to X$ be an asymptotically regular mapping and satisfy*

$$d(Tx,Ty) \preceq Ad(x,Tx) + Bd(y,Ty) + Cd(x,y), \tag{3.1}$$

for all $x, y \in X$, where $A, B, C \in P$ with $r(B) < 1$ and $r(C) < 1$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z, i.e., T is a Picard operator.

Proof. Let x_0 be an arbitrary point in X and set $x_n = Tx_{n-1} = T^n x_0, n \geq 1$. If there exists an integer $n \in \mathbb{N}$ such that $x_n = x_{n+1} = Tx_n$, then T must have a fixed point. Now, we assume that $x_n \neq x_{n+1}, \forall n \in \mathbb{N}$. By asymptotic regularity of T, $\{d(x_n, x_{n+1})\}$ is a c-sequence. So for all $m > n$, we get

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &= d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m) \\ &\preceq d(x_n, x_{n+1}) + Ad(x_n, x_{n+1}) + Bd(x_m, x_{m+1}) + Cd(x_n, x_m) + d(x_{m+1}, x_m). \end{aligned}$$

Since $r(C) < 1, (e - C)$ is invertible. It follows that

$$d(x_n, x_m) \preceq (e - C)^{-1}\{(e + A)d(x_n, x_{n+1}) + (e + B)d(x_m, x_{m+1})\}.$$

From Lemma 1.5, it is obvious that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Then

$$d(z, Tz) \preceq d(z, Tx_n) + d(Tx_n, Tz) \preceq d(z, x_{n+1}) + Ad(x_n, Tx_n) + Bd(z, Tz) + Cd(x_n, z),$$

which means that

$$d(z, Tz) \preceq (e - B)^{-1}[d(z, x_{n+1}) + Ad(x_n, x_{n+1}) + Cd(x_n, z)].$$

Then, the right side is a c-sequence, so $z = Tz$. Similar to Theorem 2.4, it is easy to prove that z is unique. □

Corollary 3.4. *Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$. Let the mapping $T : X \to X$ be an asymptotically regular mapping and satisfy*

$$d(Tx, Ty) \preceq M[d(x, Tx) + d(y, Ty) + d(x, y)],$$

for all $x, y \in X$, where $M \in P$ with $r(M) < 1$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z, i.e., T is a Picard operator.

Now, we show that if T is orbitally continuous, then the condition $r(B) < 1$ can be deleted.

Theorem 3.5. *Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$. Let $T : X \to X$ be an asymptotically regular mapping and satisfy*

$$d(Tx, Ty) \preceq Ad(x, Tx) + Bd(y, Ty) + Cd(x, y), \tag{3.2}$$

for all $x, y \in X$, where $A, B, C \in P$ with $r(C) < 1$. If T is orbitally continuous, then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z, i.e., T is a Picard operator.

Proof. From Theorem 3.3, we know that there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Because T is orbitally continuous, $x_{n+1} = Tx_n \to Tz$ as $n \to \infty$. Then $z = Tz$. Similar to Theorem 3.3, the conclusion is true. □

Corollary 3.6. *Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$. Let $T : X \to X$ be an asymptotically regular mapping and satisfy*

$$d(Tx, Ty) \preceq K[d(x, Tx) + d(y, Ty)] + Md(x, y)$$

for all $x, y \in X$, where $K, M \in P$ with $r(M) < 1$. If T is orbitally continuous, then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z, i.e., T is a Picard operator.

Remark 3.7. Theorem 2.4 and Theorem 2.5 greatly improve Theorem 2.3 in [7]. The assumptions of compactness and continuity considered in Theorem 2.3 of [7] are relaxed by bounded compactness or T-orbital compactness and T-orbital continuity, respectively. Corollary 2.6 and Corollary 2.7 mainly improve and generalize Theorem 2.2 in [7] and Theorem 2.1, Theorem 2.2 in [6]. Corollary 3.4 is a generalization of Theorem 3.1, 3.3 in [7]. Similarly, Corollary 3.6 is an extension of Theorem 2.6 in [9] and Theorem 2.1 in [1]. Due to the non-equivalence of cone metric spaces over Banach algebras and metric spaces, the conclusions in this paper are focused on fixed point theorems in cone metric spaces over Banach algebras instead of theorems only in usual metric spaces (see [1, 6, 7, 9, 13]) or cone metric spaces (see [12, 18]), which are more meaningful. Moreover, the completeness of (X, d) is deleted in Theorem 2.4, 2.5 and Corollary 2.6, 2.7, which is quite different from the corresponding results in [12, 16].

Example 3.8. *Let $X = [0, 1]$ and $\mathcal{A}$ be a set of all real valued function on $[0, 1]$ which also have continuous derivates on $[0, 1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and the usual multiplication. Then $\mathcal{A}$ is a Banach algebra with a unit $e = 1$. Let $P = \{x(t) \in \mathcal{A} : x(t) \geq 0, t \in [0, 1]\}$. It is clear that P is a non-normal cone. Define the cone metric $d : X \times X \to \mathcal{A}$ by $d(x, y)(t) = |x - y| \cdot 2^t \in P, \forall x, y \in X$. It is obviously seen that (X, d) is a complete cone metric space over Banach algebra $\mathcal{A}$. Take $A(t) = 2t + 3, B(t) = 3t + 4$ and $C(t) = \frac{1}{3}t + \frac{1}{3}$. We observe that*

$$C^n(t) = (\frac{1}{3}t + \frac{1}{3})^n, \quad (C^n(t))' = \frac{n}{3}(\frac{1}{3}t + \frac{1}{3})^{n-1},$$

we have (t=1)

$$\|C^n\| = \|C^n\|_\infty + \|(C^n)'\|_\infty = (\frac{2}{3})^n + \frac{n}{3}(\frac{2}{3})^{n-1} = \frac{n}{3}(\frac{2}{3})^{n-1}(\frac{3}{n} \cdot \frac{2}{3} + 1) = \frac{n}{3}(\frac{2}{3})^{n-1}(1 + \frac{2}{n}).$$

Further, we get

$$r(C) = \lim_{n\to\infty} \|C^n\|^{\frac{1}{n}} = \lim_{n\to\infty} (\frac{n}{3})^{\frac{1}{n}}(\frac{2}{3})^{\frac{n-1}{n}}(1 + \frac{2}{n})^{\frac{1}{n}} = \frac{2}{3}.$$

Thus, it is easy to obtain

$$r(C) = \frac{2}{3} < 1.$$

Furthermore, define the mapping $T : X \to X$ by:

$$Tx = \begin{cases} \frac{x}{4}\sin\frac{x}{4}, & x \in \mathbb{Q} \cap X; \\ \frac{x}{5}, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap X. \end{cases}$$

Obviously, T is asymptotically regular but not continuous. Next, we will prove that (3.2) is satisfied. There are the following several cases.

(1) If $x, y \in \mathbb{Q} \cap X$, then we have

$$\begin{aligned} d(Tx, Ty)(t) &= |\frac{x}{4}\sin\frac{x}{4} - \frac{y}{4}\sin\frac{y}{4}| \cdot 2^t \\ &\preceq (\frac{1}{16}x + \frac{1}{4})|x - y| \cdot 2^t \\ &\preceq \frac{5}{16}|x - y| \cdot 2^t \\ &\preceq (2t + 3)|x - \frac{x}{4}\sin\frac{x}{4}| \cdot 2^t + (3t + 4)|y - \frac{y}{4}\sin\frac{y}{4}| \cdot 2^t + (\frac{1}{3}t + \frac{1}{3})|x - y| \cdot 2^t \\ &= Ad(x, Tx)(t) + Bd(y, Ty)(t) + Cd(x, y)(t). \end{aligned}$$

(2) If $x, y \in (\mathbb{R} \setminus \mathbb{Q}) \cap X$, then we have

$$d(Tx, Ty)(t) = |\frac{x}{5} - \frac{y}{5}| \cdot 2^t$$

$$\preceq (2t+3)|x-\frac{x}{5}|\cdot 2^t+(3t+4)|y-\frac{y}{5}|\cdot 2^t+(\frac{1}{3}t+\frac{1}{3})|x-y|\cdot 2^t$$
$$= Ad(x,Tx)(t)+Bd(y,Ty)(t)+Cd(x,y)(t).$$

(3) If $x\in\mathbb{Q}\cap X$, $y\in(\mathbb{R}\setminus\mathbb{Q})\cap X$, then we have

$$\begin{aligned}d(Tx,Ty)(t) &= |\frac{x}{4}\sin\frac{x}{4}-\frac{y}{5}|\cdot 2^t\\ &= |\frac{y}{5}-\frac{x}{4}\sin\frac{x}{4}|\cdot 2^t\\ &\preceq |\frac{y}{5}|\cdot 2^t\\ &\preceq (2t+3)|x-\frac{x}{4}\sin\frac{x}{4}|\cdot 2^t+(3t+4)|y-\frac{y}{5}|\cdot 2^t+(\frac{1}{3}t+\frac{1}{3})|x-y|\cdot 2^t\\ &= Ad(x,Tx)(t)+Bd(y,Ty)(t)+Cd(x,y)(t).\end{aligned}$$

Similarly, it is not difficult to prove that $d(Tx,Ty)(t)\preceq Ad(x,Tx)(t)+Bd(y,Ty)(t)+Cd(x,y)(t)$ when $x\in(\mathbb{R}\setminus\mathbb{Q})\cap X$, $y\in\mathbb{Q}\cap X$. Therefore, all conditions of Theorem 3.5 are satisfied and consequently T has a unique fixed point 0 in X.

§4 Completeness and fixed point

Now, we prove an important theorem that the completeness in cone metric spaces over Banach algebras is necessary if the generalized Kannan-type contraction has a fixed point in X.

Theorem 4.1. *Let (X,d) be a cone metric space over Banach algebra $\mathcal{A}$ with a unit e. Let P be a normal and strongly minihedral cone. If every self-mapping T satisfying*

$$d(Tx,Ty)\prec\frac{e}{2}\{d(x,Tx)+d(y,Ty)\} \tag{4.1}$$

for all $x,y\in X$ with $x\neq y$, has a unique fixed point, then (X,d) must be a complete cone metric space over Banach algebra $\mathcal{A}$.

Proof. We prove it by contradiction. Assume that (X,d) is not complete. Then, there must be a Cauchy sequence $\{x_n\}$ in X, which is not convergent. If there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to z\in X$, we have

$$d(x_m,z)\preceq d(x_m,x_{n_k})+d(x_{n_k},z)$$

for all $m\geq n_k$. It is a c-sequence while $\{d(x_m,x_{n_k})\}$ and $\{d(x_{n_k},z)\}$ are c-sequences. Thus, we can suppose that all terms of the sequence $\{x_n\}$ are distinct. Let $M=\{x_n:n\in\mathbb{N}\}$. Since the sequence $\{x_n\}$ does not converge in X, we obtain $d(x,M)\succ\theta$ for all $x\in X-M$. Let $x\in X$ be an arbitrary point. If $x\in X-M$, then there is an integer $n_x\in\mathbb{N}$ such that

$$d(x_m,x_{n_x})\prec\frac{e}{2}d(x,M)\preceq\frac{e}{2}d(x,x_n),$$

for all $m\geq n_x$ and arbitrary $n\in\mathbb{N}$. That is

$$d(x_m,x_{n_x})\prec\frac{e}{2}d(x,x_n),\forall m\geq n_x \text{ and } \forall n\in\mathbb{N}. \tag{4.2}$$

Suppose $x'\in M$, then $x'=x_{n_0}$ for some $n_0\in\mathbb{N}$. As $\{x_n\}$ is a Cauchy sequence, we can find some $n_0'\in\mathbb{N}$ such that

$$d(x_m,x_{n_0'})\prec\frac{e}{2}d(x_{n_0'},x_{n_0}),\forall m\geq n_0'>n_0. \tag{4.3}$$

Now, we define $T : X \to X$ by

$$Tx = \begin{cases} x_{n_x}, & \text{if} \quad x \in X - M; \\ x_{n_0'}, & \text{if} \quad x \in M \quad \text{and} \quad x = x_{n_0}. \end{cases}$$

Set $x, y \in X$ be arbitrary elements with $x \neq y$. There are three cases for the proof.

Case 1. If $x, y \in X - M$, then $Tx = x_{n_x}$ and $Ty = x_{n_y}$. Without loss of generality, we assume that $n_y \geq n_x$. By (4.2), we get

$$d(Tx, Ty) = d(x_{n_x}, x_{n_y}) \prec \frac{e}{2} d(x, x_{n_x}) = \frac{e}{2} d(x, Tx).$$

That is, $d(Tx, Ty) \prec \frac{e}{2}\{d(x, Tx) + d(y, Ty)\}$.

Case 2. If $x, y \in M$, then $x = x_{n_0}$ and $y = x_{m_0}$ for some $n_0, m_0 \in \mathbb{N}$. Then $Tx = x_{n_0'}$ and $Ty = x_{m_0'}$. Without loss of generality, we assume that $m_0' \geq n_0'$. Then, by (4.3), we have

$$d(Tx, Ty) = d(x_{m_0'}, x_{n_0'}) \prec \frac{e}{2} d(x_{n_0'}, x_{n_0}) = \frac{e}{2} d(Tx, x),$$

which implies that $d(Tx, Ty) \prec \frac{e}{2}\{d(x, Tx) + d(y, Ty)\}$.

Case 3. If $x \in X - M, y \in M$, then $y = x_{n_0}$ for some $n_0 \in \mathbb{N}$. Therefore $Tx = x_{n_x}$ and $Ty = x_{n_0'}$. If $n_0' \geq n_x$, from (4.2), we obtain

$$d(Tx, Ty) = d(x_{n_x}, x_{n_0'}) = d(x_{n_0'}, x_{n_x}) \prec \frac{e}{2} d(x, x_{n_x}) = \frac{e}{2} d(x, Tx),$$

which implies that $d(Tx, Ty) \prec \frac{e}{2}\{d(x, Tx) + d(y, Ty)\}$. If $n_x \geq n_0'$, from (4.3), we deduce

$$d(Tx, Ty) = d(x_{n_x}, x_{n_0'}) \prec \frac{e}{2} d(x_{n_0'}, x_{n_0}) = \frac{e}{2} d(y, Ty),$$

which also gives that $d(Tx, Ty) \prec \frac{e}{2}\{d(x, Tx) + d(y, Ty)\}$. Thus, for all $x, y \in X$ with $x \neq y$, we have $d(Tx, Ty) \prec \frac{e}{2}\{d(x, Tx) + d(y, Ty)\}$. Therefore, T is a generalized Kannan-type contraction mapping which has no fixed point in X. It is a contradiction. So the assumption does not hold and (X, d) must be a complete cone metric space over Banach algebra $\mathcal{A}$. Then, the conclusion is true. □

Remark 4.2. From the proof of Theorem 4.1, it is clear that the inequality (4.1) can be replaced by the following condition: $d(Tx, Ty) \preceq k\{d(x, Tx) + d(y, Ty)\}$, for all $x, y \in X$ and a fixed $k \in P$.

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Conflict of interest The authors declare no conflict of interest.

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