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Notes concerning Codazzi pairs on almost anti-Hermitian manifolds

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Abstract. Let ∇ be a linear connection on a $2n$-dimensional almost anti-Hermitian manifold M equipped with an almost complex structure J, a pseudo-Riemannian metric g and the twin metric $G = g \circ J$. In this paper, we first introduce three types of conjugate connections of linear connections relative to g, G and J. We obtain a simple relation among curvature tensors of these conjugate connections. To clarify the relations of these conjugate connections, we prove a result stating that conjugations along with an identity operation together act as a Klein group, which is analogue to the known result for the Hermitian case in [2]. Secondly, we give some results exhibiting occurrences of Codazzi pairs which generalize parallelism relative to ∇. Under the assumption that (∇, J) being a Codazzi pair, we derive a necessary and sufficient condition the almost anti-Hermitian manifold (M, J, g, G) is an anti-Kähler relative to a torsion-free linear connection ∇. Finally, we investigate statistical structures on M under ∇ (∇ is a $J-$parallel torsion-free connection).

§1 Introduction

A pseudo-Riemannian metric g on a smooth $2n-$manifold M is called neutral if it has signature (n, n). The pair (M, g) is called a pseudo-Riemannian manifold. An anti-Kähler structure on a manifold M consists of an almost complex structure J and a neutral metric g satisfying the followings:

• algebraic conditions

(a) J is an almost complex structure: $J^2 = -id$.

(b) The neutral metric g is anti-Hermittian relative to J:

$$g(JX, JY) = -g(X, Y)$$

or equivalently

$$g(JX, Y) = g(X, JY), \forall X, Y \in TM. \tag{1.1}$$

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• analytic condition

(c) J is parallel relative to the Levi-Civita connection ∇^g ($\nabla^g J = 0$). This condition is equivalent to the C-holomorphicity (analyticity) of the anti-Hermitian metric g, that is, $\Phi_J g = 0$, where Φ_J is the Tachibana operator [5].

The C-holomorphicity (analyticity) of the anti-Hermitian metric g on anti-Kähler manifolds means that there exists a one-to-one correspondence between anti-Kähler manifolds and complex Riemannian manifolds with a holomorphic metric. This fact gives us some topological obstructions to an anti-Kähler manifold, for instance, all its odd Chern numbers vanish because its holomorphic metric gives us a complex isomorphism between the complex tangent bundle and its dual; and a compact simply connected Kähler manifold cannot be anti-Kähler because it does not admit a holomorphic metric. Hence, an anti-Kähler manifold is slightly a different family of almost complex manifolds. This kind of manifolds have been also studied under the names: almost complex manifolds with Norden (or B-) metric, Kähler-Norden manifolds [3, 8, 14].

Obviously, by algebraic conditions, the triple (M, J, g) is an almost anti-Hermitian manifold. Given the anti-Hermitian structure (J, g) on a manifold M, we can immediately recover the other anti-Hermitian metric, called the twin metric, by the formula:

$$G(X, Y) = (g \circ J)(X, Y) = g(JX, Y).$$

Thus, the triple (M, J, G) is another almost anti-Hermitian manifold. Note that the condition (1.1) also refers to the purity of g relative to J. From now on, by manifold we understand a smooth $2n-$manifold and will use the notations J, g and G for the almost complex structure, the pseudo-Riemannian metric and the twin metric, respectively. In addition, we shall assign the quadruple (M, J, g, G) as almost anti-Hermitian manifolds.

Our paper aims to study Codazzi pairs on an almost anti-Hermitian manifold (M, J, g, G). The analogous case with almost Hermitian case has been worked out earlier by Fei and Zhang [2]. The structure of the paper is as follows. In Sect. 2, we start by the $g-$conjugation, $G-$conjugation and $J-$conjugation of arbitrary linear connections. Then we state the relations among the $(0, 4)$-curvature tensors of these conjugate connections and also show that the set which has $g-$conjugation, $G-$conjugation, $J-$conjugation and an identity operation is a Klein group on the space of linear connections. In Sect. 3, we obtain some remarkable results under the assumption that (∇, G) or (∇, J) being a Codazzi pair, where ∇ is a linear connection. One of them is a necessary and sufficient condition under which the almost anti-Hermitian manifold (M, J, g, G) is an anti-Kähler relative to a torsion-free linear connection ∇. Sect. 4 closes our paper with statistical structures under the assumption that ∇ being $J-$parallel relative to a torsion-free linear connection ∇.

§2 Conjugate connections

In the following let (M, J, g, G) be an almost anti-Hermitian manifold and ∇ be a linear connection. We define respectively the conjugate connections of ∇ relative to g and G as the linear connections determined by the equations:

$$Zg\left(X,Y\right)=g\left(\nabla_{Z}X,Y\right)+g\left(X,\nabla_{Z}^{*}Y\right)$$

and

$$ZG\left(X,Y\right)=G\left(\nabla_{Z}X,Y\right)+G\left(X,\nabla_{Z}^{\dagger}Y\right)$$

for all vector fields X,Y,Z on M. We are calling these connections $g-$conjugate connection and $G-$conjugate connection, respectively. Conjugate connections with respect to the metric were studied in [1,9,10]. Note that both $g-$conjugate connection and $G-$conjugate connection of a linear connection are involutive: $(\nabla^{*})^{*}=\nabla$ and $\left(\nabla^{\dagger}\right)^{\dagger}=\nabla$. Conjugate connections are a natural generalization of Levi-Civita connections from Riemannian manifolds theory. Especially, ∇^{*} (or $\nabla^{\dagger}$) coincides with ∇ if and only if ∇ is the Levi-Civita connection of g (or G).

Given a linear connection ∇ of (M,J,g,G), the $J-$conjugate connection of ∇, denoted ∇^{J}, is a new linear connection given by

$$\nabla^{J}(X,Y)=J^{-1}(\nabla_{X}JY)$$

for any vector fields X and Y on M [12]. Conjugate connections with respect to J were studied in [2,4,12,13].

Through relationships among the $g-$conjugate connection ∇^{*}, $G-$conjugate connection $\nabla^{\dagger}$ and $J-$conjugate connection ∇^{J} of ∇, we give the following theorem which is analogue to the known result given by Fei and Zhang [2] for Hermitian setting. Also, in our setting, we present detailed proof by using different arguments.

Theorem 1. *Let (M,J,g,G) be an almost anti-Hermitian manifold. ∇^{*}, $\nabla^{\dagger}$ and ∇^{J} denote respectively $g-$conjugation, $G-$conjugation and $J-$conjugation of a linear connection ∇. Then $(id,*,\dagger,J)$ acts as the 4-element Klein group on the space of linear connections:*

$$\begin{aligned}
i)(\nabla^{*})^{*} &= \left(\nabla^{\dagger}\right)^{\dagger}=\left(\nabla^{J}\right)^{J}=\nabla,\\
ii)\left(\nabla^{\dagger}\right)^{J} &= \left(\nabla^{J}\right)^{\dagger}=\nabla^{*},\\
iii)(\nabla^{*})^{J} &= \left(\nabla^{J}\right)^{*}=\nabla^{\dagger},\\
iv)(\nabla^{*})^{\dagger} &= \left(\nabla^{\dagger}\right)^{*}=\nabla^{J}.
\end{aligned}$$

Proof. i) The statement is a direct consequence of definitions of conjugate connections.

ii) We compute

$$\begin{aligned}
G\left(\left(\nabla^{\dagger}\right)_{Z}^{J}X,Y\right) &= G\left(J^{-1}\nabla_{Z}^{\dagger}\left(JX\right),Y\right)\\
&= G\left(\nabla_{Z}^{\dagger}\left(JX\right),J^{-1}Y\right)\\
&= ZG\left(JX,J^{-1}Y\right)-G(JX,\nabla_{Z}\left(J^{-1}Y\right))\\
&= Zg\left(J^{2}X,\ J^{-1}Y\right)-g(J^{2}X,\nabla_{Z}\left(J^{-1}Y\right))\\
&= -Zg\left(X,J^{-1}Y\right)+g(X,\nabla_{Z}\left(J^{-1}Y\right))\\
&= -g\left(\nabla_{Z}^{*}X,J^{-1}Y\right)=G(\nabla_{Z}^{*}X,Y)
\end{aligned}$$

which gives $\left(\nabla^{\dagger}\right)^{J} = \nabla^{*}$. Similarly

$$\begin{aligned} ZG(X,Y) &= G\left(\nabla^{J}_{Z}X,Y\right) + G\left(X,\left(\nabla^{J}\right)^{\dagger}_{Z}Y\right), \\ Zg(JX,Y) &= g\left(JJ^{-1}\nabla_{Z}(JX),Y\right) + g\left(JX,\left(\nabla^{J}\right)^{\dagger}_{Z}Y\right), \\ Zg(JX,Y) &= g\left(\nabla_{Z}(JX),Y\right) + g\left(JX,\left(\nabla^{J}\right)^{\dagger}_{Z}Y\right), \\ g(JX,\nabla^{*}_{Z}Y) &= g\left(JX,\left(\nabla^{J}\right)^{\dagger}_{Z}Y\right) \end{aligned}$$

which establishes $\left(\nabla^{J}\right)^{\dagger} = \nabla^{*}$. Hence, we get $\left(\nabla^{\dagger}\right)^{J} = \left(\nabla^{J}\right)^{\dagger} = \nabla$.

iii) On applying the $J-$conjugation to both sides of ii), $\nabla^{\dagger} = (\nabla^{*})^{J}$ and also,

$$\begin{aligned} g\left(JX,\left(\nabla^{J}\right)^{*}_{Z}Y\right) &= Zg(JX,Y) - g\left(\nabla^{J}_{Z}(JX),Y\right) \\ &= ZG(X,Y) - G\left(J^{-1}\nabla^{J}_{Z}(JX),Y\right) \\ &= ZG(X,Y) - G\left(J^{-1}J^{-1}\nabla_{Z}\left(J^{2}X\right),Y\right) \\ &= ZG(X,Y) - G\left(\nabla_{Z}X,Y\right) \\ &= G\left(X,\nabla^{\dagger}_{Z}Y\right) = g\left(JX,\nabla^{\dagger}_{Z}Y\right). \end{aligned}$$

These show that $\nabla^{\dagger} = (\nabla^{*})^{J} = \left(\nabla^{J}\right)^{*}$.

iv) On applying the $G-$conjugation to both sides of ii), $\nabla^{J} = (\nabla^{*})^{\dagger}$ and on applying the $g-$conjugation to both sides of iii), $\nabla^{J} = \left(\nabla^{\dagger}\right)^{*}$. Thus, the proof completes. □

Recall that the curvature tensor field R of a linear connection ∇ is the tensor field, for all vector fields X, Y, Z,

$$R(X,Y)Z = \nabla_{X}\nabla_{Y}Z - \nabla_{Y}\nabla_{X}Z - \nabla_{[X,Y]}Z.$$

If (M, g) is a (pseudo-)Riemannian manifold, it is sometimes convenient to view the curvature tensor field as a $(0, 4)-$tensor field by:

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

called the $(0, 4)-$curvature tensor field. If we consider the relationship among the $(0, 4)-$curvature tensor fields of ∇, ∇^{*} and ∇^{J}, we obtain the following.

Theorem 2. *Let (M, J, g, G) be an almost anti-Hermitian manifold. ∇^{*} and ∇^{J} denote respectively $g-$conjugation and $J-$conjugation of a linear connection ∇ on M. The relationship among the $(0, 4)-$curvature tensor fields R, R^{*} and R^{J} of ∇, ∇^{*} and ∇^{J} is as follow:*

$$R(X,Y,JZ,W) = -R^{*}(X,Y,W,JZ) = R^{J}(X,Y,Z,JW)$$

for all vector fields X, Y, Z, W on M.

Proof. Since the relation is linear in the arguments X, Y, W and Z, it suffices to prove it only on a basis. Therefore we assume $X, Y, W, Z \in \{\frac{\partial}{\partial x^{1}}, ..., \frac{\partial}{\partial x^{2n}}\}$ and take computational advantage of the following vanishing Lie brackets

$$[X,Y] = [Y,W] = [W,Z] = 0.$$

Then we get

$$\begin{aligned} XYG(Z,W) &= X(Yg(JZ,W)) \\ &= X(g(\nabla_Y JZ,W)) + X(g(JZ,\nabla^*_Y W)) \\ &= g(\nabla_X\nabla_Y JZ,W) + g(\nabla_Y JZ,\nabla^*_X W) \\ &\quad + g(\nabla_X JZ,\nabla^*_Y W) + g\left(JZ,\nabla^*_X\nabla^*_Y W\right) \end{aligned}$$

and by alternation

$$\begin{aligned} YXG(Z,W) &= g(\nabla_Y\nabla_X JZ,W) + g(\nabla_X JZ,\nabla^*_Y W) \\ &\quad + g(\nabla_Y JZ,\nabla^*_X W) + g\left(JZ,\nabla^*_Y\nabla^*_X W\right). \end{aligned}$$

Because of the above relations, we find

$$\begin{aligned} 0 &= [X,Y]G(Z,W) = XYG(Z,W) - YXG(Z,W) \\ 0 &= g(\nabla_X\nabla_Y JZ - \nabla_Y\nabla_X JZ,W) + g\left(JZ,\nabla^*_X\nabla^*_Y W - \nabla^*_Y\nabla^*_X W\right) \\ 0 &= R(X,Y,JZ,W) + R^*(X,Y,W,JZ) \end{aligned}$$

and similarly

$$\begin{aligned} 0 &= [X,Y]G(Z,W) = XYG(Z,W) - YXG(Z,W) \\ 0 &= G\left(J^{-1}\nabla_X J(J^{-1}\nabla_Y JZ) - J^{-1}\nabla_Y J(J^{-1}\nabla_X JZ),W\right) \\ &\quad + G\left(Z,\nabla^*_X\nabla^*_Y W - \nabla^*_Y\nabla^*_X W\right) \\ 0 &= G\left(\overset{J}{\nabla}_X\overset{J}{\nabla}_Y Z - \overset{J}{\nabla}_Y\overset{J}{\nabla}_X Z,W\right) \\ &\quad + G\left(Z,\nabla^*_X\nabla^*_Y W - \nabla^*_Y\nabla^*_X W\right) \\ 0 &= g\left(\overset{J}{\nabla}_X\overset{J}{\nabla}_Y Z - \overset{J}{\nabla}_Y\overset{J}{\nabla}_X Z,JW\right) \\ &\quad + g\left(\nabla^*_X\nabla^*_Y W - \nabla^*_Y\nabla^*_X W,JZ\right) \\ 0 &= R^J(X,Y,Z,JW) + R^*(X,Y,W,JZ). \end{aligned}$$

Hence, it follows that $R(X,Y,JZ,W) = -R^*(X,Y,W,JZ) = R^J(X,Y,Z,JW)$. □

§3 Codazzi Pairs

Let ∇ be an arbitrary linear connection on a pseudo-Riemannian manifold (M,g). Given the pair (∇,g), we construct respectively the $(0,3)-$tensor fields F and F^* by

$$F(X,Y,Z) := (\nabla_Z g)(X,Y)$$

and

$$F^*(X,Y,Z) := (\nabla^*_Z g)(X,Y),$$

where ∇^* is $g-$conjugation of ∇. The tensor field F (or F^*) is sometimes referred to as the cubic form associated to the pair (∇,g) (or (∇^*,g)). These tensors are related via

$$F(X,Y,Z) = g(X,(\nabla^* - \nabla)_Z Y)$$

so that

$$F^*(X,Y,Z) := (\nabla^*_Z g)(X,Y) = -F(X,Y,Z).$$

Therefore $F(X,Y,Z)=F^*(X,Y,Z)=0$ if and only if $\nabla^*=\nabla$, that is, ∇ is $g-$self-conjugate [2].

For an almost complex structure J, a pseudo-Riemannian metric g and a symmetric bilinear form ρ on a manifold M, we call (∇,J) and (∇,ρ), respectively, a Codazzi pair, if their covariant derivative (∇J) and $(\nabla\rho)$, respectively, is (totally) symmetric in X,Y,Z [12]:

$$(\nabla_Z J)\,X=(\nabla_X J)\,Z,(\nabla_Z\rho)\,(X,Y)=(\nabla_X\rho)\,(Z,Y)\,.$$

3.1 The Codazzi pair (∇,G)

Let ∇ be a linear connection ∇ on (M,J,g,G). Next we shall consider the Codazzi pair (∇,G). In here, the $(0,3)-$tensor field F is defined by

$$F(X,Y,Z):=(\nabla_Z G)\,(X,\ Y)\,.$$

Now we shall state the following proposition without proof, because its proof is easily obtained from some relations well-known concerning with the cubic form C and Codazzi condition. We omit standard calculations.

Proposition 1. *(See also [12]) Let ∇ be a linear connection on (M,J,g,G). Then the following statements are equivalent:*

i) (∇,G) is a Codazzi pair

ii) $(\nabla^\dagger,G)$ is a Codazzi pair,

iii) $F^\dagger(X,Y,Z)=\left(\nabla^\dagger_Z G\right)(X,Y)$ is totally symmetric,

iv) $T^\nabla=T^{\nabla^\dagger}$.

Proposition 2. *Let ∇ be a linear connection on (M,J,g,G). If (∇,G) is a Codazzi pair, then the following statements hold:*

i) $F(X,Y,Z)=(\nabla_Z G)(X,Y)$ is totally symmetric,

ii) $(\nabla^*_{JZ}G)(X,Y)=(\nabla^*_{JX}G)(Z,Y)$,

iii) $T^\nabla=T^{\nabla^*}$ if and only if (∇^*,J) is a Codazzi pair,

iv) $T^\nabla=T^{(\nabla^*)^J}$,

where ∇^* is the $g-$conjugation of ∇ and $(\nabla^*)^J$ is the $J-$conjugation of ∇^*.

Proof. *i)* Due to symmetry of G, $F(X,Y,Z)=(\nabla_Z G)(X,Y)=(\nabla_Z G)(Y,X)=F(Y,X,Z)$. Also for (∇,G) being a Codazzi pair, $F(X,Y,Z)=(\nabla_Z G)(X,Y)=F(X,Y,Z)=(\nabla_X G)(Z,Y)$ $=F(Z,Y,X)$, that is, F is totally symmetric in all of its indices.

ii) By virtue of the purity of g relative to J, we yield

$$(\nabla_Z G)\,(X,Y)=(\nabla_X G)\,(Z,\ Y)$$

$$\begin{aligned}&Zg\,(JX,Y)-g\,(J\nabla_Z X,Y)-g\,(JX,\nabla_Z Y)\\=\;&Xg\,(JZ,Y)-g\,(J\nabla_X Z,Y)-g\,(JZ,\nabla_X Y)\end{aligned}$$

$$g\,(\nabla^*_Z\,(JX)\,,Y)-g\,(J\nabla_Z X,Y)=g\,(\nabla^*_X\,(JZ)\,,\ Y)-g\,(J\nabla_X Z,Y)$$

$$\begin{aligned}&g\,(\nabla^*_Z\,(JX)\,,Y)-Zg\,(X,JY)+g\,(X,\nabla^*_Z\,(JY))\\=\;&g\,(\nabla^*_X\,(JZ)\,,Y)-Xg\,(Z,JY)+g\,(Z,\nabla^*_X\,(JY))\end{aligned}$$

$$\begin{aligned} & Zg\left(X, JY\right) - g\left(\nabla^*_Z\left(JX\right), Y\right) - g\left(X, \nabla^*_Z\left(JY\right)\right) \\ = \ & Xg\left(Z, JY\right) - g\left(\nabla^*_X\left(JZ\right),\ Y\right) - g\left(Z, \nabla^*_X\left(JY\right)\right). \end{aligned}$$

Putting $X = JX,\ Y = JY$ and $Z = JZ$ in the last relation, we find

$$\begin{aligned} & JZg\left(JX, J(JY)\right) - g\left(\nabla^*_{JZ}\left(J(JX)\right), JY\right) - g\left(JX, \nabla^*_{JZ}\left(J(JY)\right)\right) \\ = \ & JXg\left(JZ, J(JY)\right) - g\left(\nabla^*_{JX}\left(J(JZ)\right), JY\right) - g\left(JZ, \nabla^*_{JX}\left(J(JY)\right)\right) \end{aligned}$$

$$\begin{aligned} & JZg\left(JX, Y\right) - g\left(\nabla^*_{JZ}X, JY\right) - g\left(JX, \nabla^*_{JZ}Y\right) \\ = \ & JXg\left(JZ, Y\right) - g\left(\nabla^*_{JX}Z, JY\right) - g\left(JZ, \nabla^*_{JX}Y\right) \end{aligned}$$

$$\begin{aligned} & JZG\left(X, Y\right) - G\left(\nabla^*_{JZ}X, Y\right) - G\left(X, \nabla^*_{JZ}Y\right) \\ = \ & JXG\left(Z, Y\right) - G\left(\nabla^*_{JX}Z, Y\right) - G\left(Z, \nabla^*_{JX}Y\right) \end{aligned}$$

$$\left(\nabla^*_{JZ}G\right)\left(X, Y\right) = \left(\nabla^*_{JX}G\right)\left(Z, Y\right).$$

iii) Let T^{∇} and T^{∇^*} be respectively the torsion tensors of ∇ and its $g-$conjugation ∇^*. We calculate

$$\left(\nabla_Z G\right)\left(X, Y\right) = \left(\nabla_X G\right)\left(Z, Y\right)$$

$$\begin{aligned} & Zg\left(JX, Y\right) - g\left(J\nabla_Z X, Y\right) - g\left(JX, \nabla_Z Y\right) \\ = \ & Xg\left(JZ, Y\right) - g\left(J\nabla_X Z, Y\right) - g\left(JZ, \nabla_X Y\right) \end{aligned}$$

$$\begin{aligned} & g\left(\nabla^*_Z\left(JX\right), Y\right) - g\left(J\nabla_Z X, Y\right) \\ = \ & g\left(\nabla^*_X\left(JZ\right), Y\right) - g\left(J\nabla_X Z, Y\right) \end{aligned}$$

$$\begin{aligned} & G\left(J^{-1}\nabla^*_Z\left(JX\right), Y\right) - G\left(\nabla_Z X, Y\right) \\ = \ & G\left(J^{-1}\nabla^*_X\left(JZ\right), Y\right) - G\left(\nabla_X Z, Y\right) \end{aligned}$$

$$G\left(J^{-1}\left\{\nabla^*_Z\left(JX\right) - \nabla^*_X\left(JZ\right)\right\}, Y\right) = G\left(\nabla_Z X - \nabla_X Z, Y\right) \tag{3.1}$$

from which we get

$$J^{-1}\left\{\nabla^*_Z\left(JX\right) - \nabla^*_X\left(JZ\right)\right\} = \nabla_Z X - \nabla_X Z$$

$$J^{-1}\left\{\left(\nabla^*_Z J\right)X + J\nabla^*_Z X - \left(\nabla^*_X J\right)Z - J\nabla^*_X Z\right\} = \nabla_Z X - \nabla_X Z$$

$$\begin{aligned} & J^{-1}\left\{\left(\nabla^*_Z J\right)X - \left(\nabla^*_X J\right)Z\right\} + \left(\nabla^*_Z X - \nabla^*_X Z - [Z, X]\right) \\ = \ & \nabla_Z X - \nabla_X Z - [Z, X] \end{aligned}$$

$$J^{-1}\left\{\left(\nabla^*_Z J\right)X - \left(\nabla^*_X J\right)Z\right\} + T^{\nabla^*}\left(Z, X\right) = T^{\nabla}(Z, X).$$

This means that $T^{\nabla^*}\left(Z, X\right) = T^{\nabla}(Z, X)$ if and only if $\left(\nabla^*_Z J\right)X = \left(\nabla^*_X J\right)Z$.

iv) From (3.1), we can write

$$G\left(\left(\nabla^*\right)^J_Z X - \left(\nabla^*\right)^J_X Z, Y\right) = G\left(\nabla_Z X - \nabla_X Z, Y\right)$$

$$G(T^{(\nabla^*)^J}(Z, X), Y) = G(T^{\nabla}(Z, X), Y)$$

$$T^{(\nabla^*)^J}(Z, X) = T^{\nabla}(Z, X).$$

□

As a corollary to Proposition 1 and 2, we obtain the following conclusion.

Corollary 1. *Let (M,J,g,G) be an almost anti-Hermitian manifold. ∇^* and $\nabla^\dagger$ denote respectively $g-$conjugation and $G-$conjugation of a linear connection ∇ on M. If (∇,G) and (∇^*,J) are Codazzi pairs, then $T^\nabla = T^{\nabla^*} = T^{\nabla^\dagger}$.*

3.2 The Codazzi pair (∇, J)

Proposition 3. *Let ∇ be a linear connection on (M,J,g,G). $\nabla^\dagger$ denote $G-$conjugation of ∇ on M. Under the assumption that (∇,G) being a Codazzi pair, $(\nabla^\dagger,J)$ is a Codazzi pair if and only if (∇,g) is so.*

Proof. Using the definition of $G-$conjugation and $T^\nabla = T^{\nabla^\dagger}$, we find

$$\begin{aligned}
G\left((\nabla^\dagger_Z J)X-(\nabla^\dagger_X J)Z,Y\right) &= G(\nabla^\dagger_Z JX - J\nabla^\dagger_Z X,Y) - G(\nabla^\dagger_X JZ - J\nabla^\dagger_X Z,Y)\\
&= ZG(JX,Y)-G(JX,\nabla_Z Y)-G\left(J\nabla^\dagger_Z X,Y\right)-XG(JZ,Y)\\
&\quad +G(JZ,\nabla_X Y)+G(J\nabla^\dagger_X Z,Y)\\
&= ZG(JX,Y)-G(JX,\nabla_Z Y)-XG(JZ,Y)+G(JZ,\nabla_X Y)\\
&\quad +G\left(J(\nabla^\dagger_X Z-\nabla^\dagger_Z X-[Z,X])+J[Z,X],Y\right)\\
&= ZG(JX,Y)-G(JX,\nabla_Z Y)-XG(JZ,Y)+G(JZ,\nabla_X Y)\\
&\quad +G(J(\nabla_X Z-\nabla_Z X-[Z,X])+J[Z,X],Y)\\
&= -Zg(X,Y)+g(X,\nabla_Z Y)+Xg(Z,Y)\\
&\quad -g(Z,\nabla_X Y)+g(\nabla_Z X,Y)-g(\nabla_X Z,Y)\\
&= (\nabla_Z g)(X,Y)-(\nabla_X g)(Z,Y).
\end{aligned}$$

Consider the set $R(j)=\{a_0+a_1 j : j^2=-1; a_0,a_1\in R\}$, which is the algebra of complex numbers over the field of real numbers R. The canonical bases of this algebra has the form $\{1,j\}$. An $2k$-dimensional manifold M_{2k} with an integrable complex structure J is a real realization of the holomorphic manifold $M_k(R(j))$ over the algebra $R(j)$ with dimension k. Let t^* be a complex tensor field on $M_k(R(j))$. The real model of such a tensor field is a tensor field on M_{2k} of the same type and also is pure relative to the complex structure J (for pure tensor fields, see [11]). If J is a complex structure on M_{2k} and $\Phi_J t = 0$, then the complex tensor field t^* on $M_k(R(j))$ is said to be holomorphic [6, 11]. Therefore, we can say that the real model of a holomorphic tensor field t^* on $M_k(R(j))$ is the same type pure tensor field on M_{2k} such that $\Phi_J t = 0$. Now we consider the $\Phi-$operator (or Tachibana operator [15]) applied to the anti-Hermitian metric g:

$$(\Phi_J g)(X,Y,Z) = (L_{JX}g - L_X(g\circ J))(Y,Z). \tag{3.2}$$

Because of the fact that the twin metric G on an almost anti-Hermitian manifold (M,J,g) is an anti-Hermitian metric, we can apply the $\Phi-$operator to the twin metric G [11]:

$$\begin{aligned}
(\Phi_J G)(X,Y,Z) &= (L_{JX}G - L_X(G\circ J))(Y,Z) \\
&= (\Phi_J g)(X,JY,Z)+g(N_J(X,Y),Z),
\end{aligned} \tag{3.3}$$

where N_J is the Nijenhuis tensor field defined by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \tag{3.4}$$

for any vector fields X, Y on M.

Proposition 4. *Let ∇ be a torsion-free linear connection on (M, J, g, G). If (∇, J) is a Codazzi pair, then*

$$(\Phi_J G)(X,Y,Z) = (\Phi_J g)(X, JY, Z) = (\nabla_{JX} G)(Y,Z) - (\nabla_X g)(JY, JZ).$$

Proof. Using $\nabla_X Z - \nabla_Z X = [Z, X]$, from (3.2) we get

$$\begin{aligned}
(\Phi_J g)(X, JY, Z) &= (L_{JX} g - (L_X goJ)(JY, Z) \\
&= (L_{JX} g)(JY, Z) - (L_X goJ)(JY, Z) \\
&= JXg(JY, Z) - g(L_{JX} JY, Z) - g(JY, L_{JX} Z) - XgoJ(JY, Z) \\
&\quad + goJ(L_X JY, Z) + goJ(JY, L_X Z) \\
&= JXg(JY, Z) - g([JX, JY], Z) - g(JY, [JX, Z]) - XgoJ(JY, Z) \\
&\quad + goJ([X, JY], Z) + goJ(JY, [X, Z]) \\
&= JXg(JY, Z) - g(\nabla_{JX} JY - \nabla_{JY} JX, Z) - g(JY, \nabla_{JX} Z - \nabla_Z JX) \\
&\quad - XgoJ(JY, Z) + goJ(\nabla_X JY - \nabla_{JY} X, Z) + goJ(JY, \nabla_X Z - \nabla_Z X) \\
&= JXg(JY, Z) - g((\nabla_{JX} J)Y + J\nabla_{JX} Y - (\nabla_{JY} J)X - J\nabla_{JY} X, Z) \\
&\quad - g(JY, \nabla_{JX} Z - (\nabla_Z J)X - J\nabla_Z X) - Xg(JY, JZ) \\
&\quad + g((\nabla_X J)Y + J\nabla_X Y - \nabla_{JY} X, JZ) + g(JY, J\nabla_X Z - J\nabla_Z X) \\
&= JXg(JY, Z) - g((\nabla_{JX} J)Y, Z) - g(J\nabla_{JX} Y, Z) \\
&\quad + g((\nabla_{JY} J)X, Z) + g(J\nabla_{JY} X, Z) - g(JY, \nabla_{JX} Z) + g(JY, (\nabla_Z J)X) \\
&\quad + g(JY, J\nabla_Z X) - Xg(JY, JZ) + g((\nabla_X J)Y, JZ) + g(J\nabla_X Y, JZ) \\
&\quad - g(\nabla_{JY} X, JZ) + g(JY, J\nabla_X Z) - g(JY, J\nabla_Z X).
\end{aligned}$$

By virtue of the purity of g relative to J, $(\nabla_Z J)X = (\nabla_X J)Z$, the last relation reduces to

$$\begin{aligned}
&= JXg(JY, Z) - g((\nabla_{JX} J)Y, Z) - g(J\nabla_{JX} Y, Z) + g((\nabla_{JY} J)X, Z) \\
&\quad + g(J\nabla_{JY} X, Z) - g(JY, \nabla_{JX} Z) + g(JY, (\nabla_Z J)X) \\
&\quad + g(JY, J\nabla_Z X) - Xg(JY, JZ) + g((\nabla_X J)Y, JZ) \\
&\quad + g(J\nabla_X Y, JZ) - g(J\nabla_{JY} X, Z) + g(JY, J\nabla_X Z) - g(JY, J\nabla_Z X) \\
&= JXg(JY, Z) - g(J\nabla_{JX} Y, Z) - g(JY, \nabla_{JX} Z) + g(JY, (\nabla_Z J)X) \\
&\quad - Xg(JY, JZ) + g((\nabla_X J)Y, JZ) + g(J\nabla_X Y, JZ) + g(JY, J\nabla_X Z) \\
&= JXG(Y, Z) - G(\nabla_{JX} Y, Z) - G(Y, \nabla_{JX} Z) - Xg(JY, JZ) \\
&\quad + g(\nabla_X JY, JZ) + g(JY, \nabla_Z JX) \\
&= (\nabla_{JX} G)(Y, Z) - (\nabla_X g)(JY, JZ).
\end{aligned} \tag{3.5}$$

Relative to the torsion-free connection ∇, the Nijenhuis tensor has the following form:

$$N_J(X, Y) = -J\{(\nabla_{JY} J)JX - (\nabla_{JX} J)JY\} + J\{(\nabla_Y J)X - (\nabla_X J)Y\}.$$

From here, it is easy to show $N_J(X,Y)=0$ because (∇, J) is a Codazzi pair. Hence, taking account of (3.3) and (3.5) we have

$$(\Phi_J G)(X,Y,Z) = (\Phi_J g)(X,JY,Z) = (\nabla_{JX} G)(Y,Z) - (\nabla_X g)(JY,JZ).$$

As is well known, the anti-Kähler condition ($\nabla^g J = 0$) is equivalent to C-holomorphicity (analyticity) of the anti-Hermitian metric g, that is, $\Phi_J g = 0$. If the anti-Hermitian metric g is C-holomorphic, then the triple (M,J,g) is an anti-Kähler manifold [5]. □

Theorem 3. *Let ∇ be a torsion-free linear connection on (M,J,g,G). Under the assumption that (∇, J) being a Codazzi pair, (M,J,g,G) is an anti-Kähler manifold if and only if the following condition is fulfilled:*

$$(\nabla_{JX} G)(Y,Z) = (\nabla_X g)(JY,JZ).$$

Proof. The statement is a direct consequence of Proposition 4. □

§4 $J-$parallel Linear Connections

Given arbitrary linear connection ∇ on an almost complex manifold (M,J), if the following condition is satisfied:

$$\nabla_X JY = J\nabla_X Y$$

for any vector fields X,Y on M, then ∇ is called a $J-$parallel linear connection on M.

Proposition 5. *Let ∇ be a linear connection on (M,J,g,G). ∇^* and $\nabla^\dagger$ denote respectively $g-$conjugation and $G-$conjugation of ∇ on M. Then*

i) ∇ is $J-$parallel if and only if ∇^* is so.

ii) ∇ is $J-$parallel if and only if $\nabla^\dagger$ is so.

Proof. *i*) Using the definition of $g-$conjugation and the purity of g relative to J, we have

$$G(\nabla^*_X JY - J\nabla^*_X Y, Z) = g(\nabla^*_X JY, JZ) - g(J\nabla^*_X Y, JZ)$$
$$= -Xg(Y,Z) - g(JY, \nabla_X JZ) + Xg(Y,Z) - g(Y, \nabla_X Z)$$
$$= -g(JY, \nabla_X JZ) + g(JY, J\nabla_X Z) = -G(Y, \nabla_X JZ) + G(Y, J\nabla_X Z).$$

Hence, $\nabla^*_X JY = J\nabla^*_X Y$ if and only if $\nabla_X JZ = J\nabla_X Z$.

ii) Similarly, we get

$$G\left(\nabla^\dagger_X JY - J\nabla^\dagger_X Y, Z\right) = G\left(\nabla^\dagger_X JY, Z\right) - G(J\nabla^\dagger_X Y, Z)$$
$$= XG(JY,Z) - G(JY, \nabla_X Z) - XG(Y,JZ) + G(Y, \nabla_X JZ)$$
$$= G(Y, \nabla_X JZ) - G(JY, \nabla_X Z) = G(\nabla_X JZ - J\nabla_X Z, Y)$$

which gives the result. □

Proposition 6. *Let ∇ be a $J-$parallel linear connection on (M,J,g,G). ∇^* and $\nabla^\dagger$ denote respectively $g-$conjugation and $G-$conjugation of ∇ on M. The following statements hold:*

i) $\nabla^\dagger$ coincides with ∇^*,

ii) (∇, G) is a Codazzi pair if and only if (∇, g) is so.

Proof. *i*) Because ∇ is J-parallel, $\nabla J = 0$, $\nabla = \nabla^*$. It follows from Theorem 1 that $\nabla^\dagger = \nabla^*$.
ii) Using the purity of g relative to J, we get

$$(\nabla_Z G)(X,Y) = (\nabla_X G)(Z,Y)$$

$$Zg(JX,Y) - g(J\nabla_Z X, Y) - g(JX, \nabla_Z Y) = Xg(JZ,Y) - g(J\nabla_X Z, Y) - g(JZ, \nabla_X Y)$$

$$Zg(X,JY) - g(\nabla_Z X, JY) - g(X, J\nabla_Z Y) = Xg(Z,JY) - g(\nabla_X Z, JY) - g(Z, J\nabla_X Y)$$

$$Zg(X,JY) - g(\nabla_Z X, JY) - g(X, \nabla_Z JY) = Xg(Z,JY) - g(\nabla_X Z, JY) - g(Z, \nabla_X JY)$$

$$(\nabla_Z g)(X,JY) = (\nabla_X g)(Z,JY).$$

□

For the moment, we consider a torsion-free linear connection ∇ on a pseudo-Riemannian manifold (M,g). In the case, if (∇, g) is a Codazzi pair which characterizes what is known to information geometers as statistical structures, then the manifold M together with a statistical structure (∇, g) is called a statistical manifold. The notion of statistical manifold was originally introduced by Lauritzen [7]. Statistical manifolds are widely studied in affine differential geometry [7, 9] and plays a central role in information geometry.

Corollary 2. *Let ∇ be a $J-$parallel torsion-free linear connection on (M,J,g,G). ∇^* denote the $g-$conjugation of ∇ on M. (∇, G) is a statistical structure if and only if (∇^*, G) is so.*

Proof. The result immediately follows from Proposition 2, using the condition of ∇ being $J-$parallel. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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