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Local pointwise convergence of the 3D finite element

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Abstract. For an elliptic problem with variable coefficients in three dimensions, this article discusses local pointwise convergence of the three-dimensional (3D) finite element. First, the Green's function and the derivative Green's function are introduced. Secondly, some relationship of norms such as L^2-norms, $W^{1,\infty}$-norms, and negative-norms in locally smooth subsets of the domain Ω is derived. Finally, local pointwise convergence properties of the finite element approximation are obtained.

§1 Introduction

There have been many studies concerned with the superconvergence of finite element methods in three dimensions (see [1–10, 13–16, 18–26, 28]). Most of them focus on the global superconvergent properties. However, to obtain the global superconvergent properties, it is necessary to satisfy two fundamental conditions: C-uniform partition (or piecewise C-uniform partition) and highly smooth solution such as $u \in W^{m+2,p}$ $(2 \leq p \leq \infty)$. So-called C-uniform partition means that for each element e in a quasi-uniform partition $\mathcal{T}^h$, and its two adjacent vertices M and P, if $\overrightarrow{MM'}$ is an edge in $\mathcal{T}^h$, there exists another edge $\overrightarrow{PP'}$ such that $|\overrightarrow{MM'} + \overrightarrow{PP'}| \leq Ch^2$, which shows that $MPP'M'$ is almost a parallelogram. In fact, it is difficult to possess these two conditions in the whole domain Ω. Nevertheless, the above two conditions are easily satisfied in the interior subset of Ω. Thus, we may study the superconvergent properties in interior subsets of Ω (so-called local superconvergent properties). Actually, up to now, there have been some local superconvergence results (see [10, 25, 27]). However, to derive local superconvergent properties, we should first obtain the local estimates for the finite element approximation, which is the focus of this article. Most of the results for local estimates will be used in the study of the local superconvergent properties (see [10, 25, 27]).

In this article, we will introduce the definitions of Green's function and derivative Green's function, and discuss their properties. These properties play important roles in arguments of main conclusions, which similarly can be seen in [11, 12]. We shall use the letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

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Consider the following elliptic problem with variable coefficients:

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i u) + a_0 u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathcal{R}^3$ is a bounded domain with Lipschitz boundary. We assume that $a_{ij} = a_{ji}$ and that the matrix (a_{ij}) is uniformly positive definite and $a_0 \geq 0$.

The weak formulation of the above problem reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \text{ for all } v \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where

$$a(u\,,\,v) \equiv \int_\Omega (\sum_{i,j=1}^{3} a_{ij}\partial_i u \partial_j v + a_0 uv)\,dxdydz, \quad (f\,,\,v) \equiv \int_\Omega fv\,dxdydz.$$

We also assume that the given functions $a_{ij} \in W^{1,\infty}(\Omega)$, $a_0 \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. In addition, we write $\partial_1 u = \frac{\partial u}{\partial x}$, $\partial_2 u = \frac{\partial u}{\partial y}$, and $\partial_3 u = \frac{\partial u}{\partial z}$, which are generalized partial derivatives. For any direction $\ell \in \mathcal{R}^3$ and $|\ell| = 1$, we denote by $\partial_\ell v(Z)$ the onesided directional derivatives defined by

$$\partial_\ell v(Z) = \lim_{|\Delta Z|\to 0} \frac{v(Z+\Delta Z) - v(Z)}{|\Delta Z|}, \;\; \Delta Z = |\Delta Z|\ell. \tag{1.2}$$

For the above problem, we assume the following a priori estimate holds.

Lemma 1.1. *For the true solution u of (1.1), there exists a $q_0(1 < q_0 \leq \infty)$ such that for every $1 < q < q_0$,*

$$\|u\|_{2,\,q,\,\Omega} \leq C(q)\|\mathcal{L}u\|_{0,\,q,\,\Omega}. \tag{1.3}$$

Specially, if $\partial\Omega$ is smooth enough and the integer $k \geq 0$, then we have

$$\|u\|_{k+2,\,q,\,\Omega} \leq C(q)\|\mathcal{L}u\|_{k,\,q,\,\Omega}. \tag{1.4}$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$. We denote by h_e the size of an element $e \in \mathcal{T}^h$, and write $h = \max_{e\in\mathcal{T}^h} h_e$. In this article, we assume for every element e that $1 \leq \frac{h}{h_e} \leq C_0$ (C_0 is a constant independent of the element e). Denote by $S^h(\Omega)$ a continuous m-degree (or tensor-product m-degree) finite elements space regarding this kind of partitions and let $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$. For every $Z \in \bar{\Omega}$, we define the discrete δ function $\delta_Z^h \in S_0^h(\Omega)$, the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h \in S_0^h(\Omega)$, the regularized Green's function $G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$, the regularized derivative Green's function $\partial_{Z,\ell}G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$, and the L^2-projection $P_h u \in S_0^h(\Omega)$ such that (see [27])

$$(v,\,\delta_Z^h) = v(Z) \;\; \forall\, v \in S_0^h(\Omega), \tag{1.5}$$

$$(v,\,\partial_{Z,\ell}\delta_Z^h) = \partial_\ell v(Z) \;\; \forall\, v \in S_0^h(\Omega), \tag{1.6}$$

$$a(G_Z^*,\,v) = (\delta_Z^h,\,v) \;\; \forall\, v \in H_0^1(\Omega), \tag{1.7}$$

$$a(\partial_{Z,\ell}G_Z^*,\,v) = (\partial_{Z,\ell}\delta_Z^h,\,v) \;\; \forall\, v \in H_0^1(\Omega), \tag{1.8}$$

$$(u - P_h u,\,v) = 0 \;\; \forall\, v \in S_0^h(\Omega). \tag{1.9}$$

As for the operator P_h, we have the following results (see [16] and [19]):

$$\|P_h w\|_{0,\,q,\,\Omega} \leq C\|w\|_{0,\,q,\,\Omega}, \;\; 1 \leq q \leq \infty, \tag{1.10}$$

$$\|P_h w\|_{1,\,q,\,\Omega} \leq C\|w\|_{1,\,q,\,\Omega}, \;\; 3 < q \leq \infty. \tag{1.11}$$

In addition, similar to (1.2), we get for $\partial_{Z,\ell}\delta_Z^h$ that

$$\partial_{Z,\ell}\delta_Z^h = \lim_{|\Delta Z|\to 0} \frac{\delta_{Z+\Delta Z}^h - \delta_Z^h}{|\Delta Z|}, \;\; \Delta Z = |\Delta Z|\ell.$$

So do for $\partial_{Z,\ell}G_Z^*$ in (1.8), $\partial_{Z,\ell}G_Z$ in (2.2), and $\partial_{Z,\ell}G_Z^h$ in (2.4).

The rest of this article is organized as follows. In Section 2, we introduce the Green's function and the derivative Green's function as well as their properties. Local pointwise estimates for the finite element approximation are derived in Section 3.

§2 Green's Function and Derivative Green's Function

We introduce the Green's function G_Z such that $a(G_Z, v) = v(Z)$ for all $v \in C_0^\infty(\Omega)$. Moreover, we can prove the following Lemma 2.1.

Lemma 2.1. *There exists a unique $G_Z \in W_0^{1,p}(\Omega)$ $(1 \le p < \frac{3}{2})$ such that*

$$a(G_Z, v) = v(Z) \ \ \forall\, v \in W_0^{1,p'}(\Omega),\ \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.1}$$

In addition, we give a weight function $\tau = |X - Z|^{-3}$, and write $W_\beta(\Omega) = \{v : v|_{\partial\Omega} = 0, \|v\|_{1,\tau^\beta} < \infty\}$. We call $\partial_{Z,\ell}G_Z$ the derivative Green's function, which satisfies the following Lemma 2.2.

Lemma 2.2. *There exists a unique $\partial_{Z,\ell}G_Z \in W_{-\alpha}(\Omega)$ such that*

$$a(\partial_{Z,\ell}G_Z, v) = \partial_\ell v(Z) \ \ \forall\, v \in W_\alpha(\Omega) \cap C_0^\infty(\Omega), \tag{2.2}$$

where $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$, and $1 < \alpha < \frac{4}{3}$ when $q_0 \ge 6$.

Remark 1. The above two lemmas have been proved in [17].

For every $Z \in \bar{\Omega}$, we define the discrete Green's function $G_Z^h \in S_0^h(\Omega)$ and the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h \in S_0^h(\Omega)$ such that (see [27])

$$a(G_Z^h,\, v) = v(Z) \ \ \forall\, v \in S_0^h(\Omega), \tag{2.3}$$

$$a(\partial_{Z,\ell}G_Z^h,\, v) = \partial_\ell v(Z) \ \ \forall\, v \in S_0^h(\Omega). \tag{2.4}$$

As for G_Z and G_Z^h, we have for $q_0 > \frac{3}{2}$ and $\frac{1}{3} < \epsilon < \infty$

$$\|G_Z\|_{1,\tau^{-\epsilon}} + \left\|G_Z^h\right\|_{1,\tau^{-\epsilon}} \le C(\epsilon). \tag{2.5}$$

As for $\partial_{Z,\ell}G_Z$ and $\partial_{Z,\ell}G_Z^h$, we get

$$\|\partial_{Z,\ell}G_Z\|_{1,\tau^{-\alpha}} + \left\|\partial_{Z,\ell}G_Z^h\right\|_{1,\tau^{-\alpha}} \le C(\alpha), \tag{2.6}$$

where $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$, and $1 < \alpha < \frac{4}{3}$ when $q_0 \ge 6$.

Similar to the two-dimensional setting (see [27]), we can obtain Lemma 2.3.

Lemma 2.3. *Suppose $q_0 > 2$, $D \subset \Omega$, and Z is not in $\bar{D}$. Then we have*

$$\|G_Z\|_{2,D} + \|\partial_{Z,\ell}G_Z\|_{2,D} \le C(\rho), \tag{2.7}$$

where $\rho = \mathrm{dist}(Z, \bar{D})$.

Proof. Set $\rho = \mathrm{dist}(Z, \bar{D})$. We choose $D_1 \subset \Omega$ such that $D \subset\subset D_1$, Z being not in $\bar{D}_1$, and $d \equiv \mathrm{dist}(\partial D_1, \partial D) > \frac{1}{2}\rho$. From (2.1) and (2.2),

$$a(G_Z, v) = 0 \text{ and } a(\partial_{Z,\ell}G_Z, v) = 0 \ \ \forall v \in C_0^\infty(D_1).$$

Thus

$$\mathcal{L}G_Z \equiv 0 \text{ and } \mathcal{L}\partial_{Z,\ell}G_Z \equiv 0, \text{ in } \Omega \setminus \{Z\}. \tag{2.8}$$

Choosing $\mu \in C^\infty(\Omega)$ such that $\mathrm{supp}(\mu) \subset\subset D_1$ and $\mu|_D = 1$, we have $\mu G_Z \in H^2(D_1) \cap H_0^1(D_1)$. Thus, from (2.8),

$$\mathcal{L}(\mu G_Z) = -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i(\mu G_Z)) + a_0\mu G_Z = -\sum_{i,j=1}^{3} \left(\partial_j(a_{ij}G_Z\partial_i\mu) - \partial_j(a_{ij}\mu)\partial_i G_Z\right).$$

We have

$$\|\mathcal{L}(\mu G_Z)\|_{0,D_1} \le C(\rho)\,\|G_Z\|_{1,D_1}\,. \tag{2.9}$$

Since Z is not in $\bar{D}_1$, $\mathrm{dist}(Z,\bar{D}_1) > 0$. Thus from (2.5) and (2.9), we get for a fixed $\varepsilon_0 \in (\frac{1}{3},\,\infty)$

$$\|\mathcal{L}(\mu G_Z)\|_{0,D_1} \le C(\rho)\,\|G_Z\|_{1,D_1} \le C(\rho)\,\|G_Z\|_{1,\tau^{-\varepsilon_0}} \le C(\rho). \tag{2.10}$$

Combining (1.3) and (2.10) yields

$$\|G_Z\|_{2,D} = \|\mu G_Z\|_{2,D} \le \|\mu G_Z\|_{2,D_1} \le C\,\|\mathcal{L}(\mu G_Z)\|_{0,D_1} \le C(\rho).$$

Similar to the above arguments, we can obtain $\|\partial_{Z,\ell} G_Z\|_{2,D} \le C(\rho)$. Thus, the proof of Lemma 2.3 is completed.

§3 Local Pointwise Convergence for the Finite Element Approximation

In this section, we first give some lemmas, and then derive local estimates for the finite element approximation.

Lemma 3.1. *Suppose* $\mu \in C^\infty(\Omega)$, $D_0 \subset \mathrm{supp}(\mu) \subset\subset D \subset \Omega$, $\mu|_{D_0} = 1$, *and* $d \equiv \mathrm{dist}(\partial D_0, \partial D)$. *Let* Π *be the standard Lagrange interpolation operator. Then we have for every* $v \in S_0^h(\Omega)$

$$\|\hat{v} - \Pi\hat{v}\|_{s,D} \le C(d)h^{1-s}\,\|v\|_{0,D\setminus D_0}\,, \tag{3.1}$$

$$\|\hat{v} - \Pi\hat{v}\|_{s,D} \le C(d)h^{m+1-s}\,\|v\|^h_{m,D\setminus D_0}\,, \tag{3.2}$$

where $\hat{v} = \mu v$, $0 \le s \le m$, *and* $\|v\|^h_{m,D\setminus D_0} = \left(\sum_{e\cap(D\setminus D_0)\ne\phi} \|v\|^2_{m,e}\right)^{\frac{1}{2}}$.

Proof. Set $N = \{e : e \cap (D\setminus D_0) \ne \phi,\ e \in \mathcal{T}^h\}$. For all $e \in N$, when $Q \in e$,

$$\begin{aligned}\hat{v}(Q) - \Pi\hat{v}(Q) &= \textstyle\sum_{k=m+1}^r \frac{1}{k!} D^k\hat{v}(Q)\cdot\sum_{i=1}^n(-1)^{k+1}(Q-Q_i)^k\phi_i(Q) + R_r(\hat{v})\\ &= R_0(\hat{v}) + R_r(\hat{v}),\end{aligned} \tag{3.3}$$

where $\{Q_i\}_{i=1}^n$ is the set of interpolation nodes on e, $\{\phi_i\}_{i=1}^n$ is the set of shape functions of the interpolation, and $D^k\hat{v}(Q)$ is the k-order Fréchet derivative. Moreover, $R_r(\hat{v})$ satisfies

$$|R_r(\hat{v})|_{s,e} \le Ch^{r+1-s}|\nabla^{r+1}\hat{v}|_{0,e} \le C(d)h^{r+1-s}\|v\|_{r,e} \le C(d)h^{m+1-s}\|v\|_{m,e}, \tag{3.4}$$

where $s = 0, 1, \cdots, m$. Obviously, when $v \in S_0^h(\Omega)$, $R_0(v) = 0$. Thus we have

$$\begin{aligned}\nabla^s R_0(\hat{v}) &= \nabla^s(R_0(\hat{v}) - \mu R_0(v))\\ &= \sum_{k=m+1}^r \frac{1}{k!}\nabla^s[(D^k\hat{v} - \mu D^k v)(Q)\cdot\sum_{i=1}^n(-1)^{k+1}(Q-Q_i)^k\phi_i(Q)].\end{aligned}$$

Further,

$$\begin{aligned}|\nabla^s R_0(\hat{v})| &\le C\sum_{k=m+1}^r\sum_{t=0}^s h^{k-s+t}\left|\nabla^t(D^k\hat{v} - \mu D^k v)\right|\\ &\le C(d)\sum_{k=m+1}^r\sum_{t=0}^s h^{k-s+t}\sum_{i=0}^{k-1+t}\left|\nabla^i v\right|.\end{aligned}$$

Choosing the L^2-norm with respect to the above inequality and applying the inverse estimate, we get

$$\begin{aligned}|\nabla^s R_0(\hat{v})|_{0,e} &\le C(d)\textstyle\sum_{k=m+1}^r\sum_{t=0}^s h^{k-s+t}\,\|v\|_{k-1+t,e}\\ &\le C(d)h^{m+1-s}\,\|v\|_{m,e}\,.\end{aligned} \tag{3.5}$$

From (3.3)–(3.5),

$$\|\hat{v}-\Pi\hat{v}\|_{s,e} \le C(d)h^{m+1-s}\|v\|_{m,e}. \tag{3.6}$$

Note that $\|\hat{v}-\Pi\hat{v}\|_{s,D_0}=0$. Thus we have $\|\hat{v}-\Pi\hat{v}\|_{s,D}=\|\hat{v}-\Pi\hat{v}\|_{s,D\setminus D_0}$. Summing over all elements of N in (3.6) yields the result (3.2). Applying the inverse estimate in (3.6), we have

$$\|\hat{v}-\Pi\hat{v}\|_{s,e} \le C(d)h^{1-s}\|v\|_{0,e}. \tag{3.7}$$

Summing over all elements of N in (3.7) yields the result (3.1). Thus the proof of Lemma 3.1 is completed.

Lemma 3.2. *Suppose $D \subset\subset D' \subset \Omega$, $d \equiv \mathrm{dist}(\partial D, \partial D')$, $0<\varepsilon \ll 1$, and $\chi \in S_0^h(\Omega)$ satisfies $a(\chi,v)=0$ for all $v \in S_0^h(D')$. Then we have when $q_0>\frac{3}{2}$,*

$$\|\chi\|_{0,\infty,D} \le C(d)\|\chi\|_{0,D'}, \tag{3.8}$$

and

$$\|\chi\|_{1,\infty,D} \le C(d)h|\ln h|^r\|\chi\|_{0,D'} + C(d)\|\chi\|_{-1,D'}, \tag{3.9}$$

where $r=([\frac{2q_0}{q_0-3}]+1)\frac{6+(3\varepsilon-1)q_0}{6q_0}$ when $3<q_0<6$, and $r=\varepsilon$ when $q_0\ge 6$.

Proof. Choosing D_1 such that $D\subset\subset D_1 \subset\subset D'$, $\mathrm{dist}(\partial D_1,\partial D')=\mathrm{dist}(\partial D_1,\partial D)=\frac{1}{2}d$, $\mu\in C^\infty(\Omega)$ satisfying $\mathrm{supp}(\mu)\subset\subset D'$ and $\mu|_{D_1}=1$, and setting $\hat{\chi}=\mu\chi$, we have for $Z\in D_1$

$$\chi(Z)=\hat{\chi}(Z)=\Pi\hat{\chi}(Z)=\Pi\chi(Z). \tag{3.10}$$

For every $Z\in D$, from (2.3), (2.4), and (3.10),

$$\chi(Z)=\Pi\hat{\chi}(Z)=a(G_Z^h,\Pi\hat{\chi}) \text{ and } \partial_\ell\chi(Z)=\partial_\ell\Pi\hat{\chi}(Z)=a(\partial_{Z,\ell}G_Z^h,\Pi\hat{\chi}). \tag{3.11}$$

Thus, from (3.2), (3.11), and the triangle inequality,

$$\begin{aligned}
|\chi(Z)| &= |a(G_Z^h,\Pi\hat{\chi})| = |a(G_Z^h,\Pi\hat{\chi}-\hat{\chi})| + |a(G_Z^h,\hat{\chi})| \\
&\le C\|\Pi\hat{\chi}-\hat{\chi}\|_{1,D'\setminus D_1}\|G_Z^h\|_{1,D'\setminus D_1} + |a(G_Z^h,\hat{\chi})| \\
&\le C(d)h\|\chi\|_{1,D'\setminus D_1}\|G_Z^h\|_{1,D'\setminus D_1} + |a(G_Z^h,\hat{\chi})|.
\end{aligned}$$

In addition

$$\begin{aligned}
a(G_Z^h,\hat{\chi}) &= \int_\Omega (\sum_{i,j=1}^3 a_{ij}\partial_i G_Z^h \partial_j\hat{\chi} + a_0 G_Z^h\hat{\chi})\,dxdydz \\
&= \int_\Omega (\sum_{i,j=1}^3 a_{ij}\partial_i(\mu G_Z^h)\partial_j\chi + a_0\mu G_Z^h\chi)\,dxdydz \\
&\quad + \int_\Omega \sum_{i,j=1}^3 (-G_Z^h a_{ij}\partial_i\mu\partial_j\chi + \chi a_{ij}\partial_i G_Z^h\partial_j\mu)\,dxdydz \\
&= \int_\Omega (\sum_{i,j=1}^3 a_{ij}\partial_i(\mu G_Z^h)\partial_j\chi + a_0\mu G_Z^h\chi)\,dxdydz \\
&\quad + \int_\Omega \sum_{i,j=1}^3 (-\partial_j(\chi G_Z^h a_{ij}\partial_i\mu) + \chi\partial_j(G_Z^h a_{ij}\partial_i\mu) + \chi a_{ij}\partial_i G_Z^h\partial_j\mu)\,dxdydz \\
&= a(\hat{G_Z^h},\chi) + J,
\end{aligned}$$

where $\hat{G_Z^h}=\mu G_Z^h$. By the conditions of Lemma 3.2 and the result (3.2), we have

$$|a(\hat{G_Z^h},\chi)| = |a(\hat{G_Z^h}-\Pi\hat{G_Z^h},\chi)| \le C(d)h\|\chi\|_{1,D'\setminus D_1}\|G_Z^h\|_{1,D'\setminus D_1}.$$

By the above arguments, we get

$$|\chi(Z)| \le C(d)h\|\chi\|_{1,D'\setminus D_1}\|G_Z^h\|_{1,D'\setminus D_1} + |J|. \tag{3.12}$$

Since $\chi \in S_0^h(\Omega)$, thus

$$|J| = \left| \int_\Omega \chi \sum_{i,j=1}^{3} (\partial_j (G_Z^h a_{ij}\partial_i \mu) + a_{ij}\partial_i G_Z^h \partial_j \mu)\, dxdydz \right| \le C(d)\|\chi\|_{0,D'\setminus D_1}\|G_Z^h\|_{1,D'\setminus D_1}. \tag{3.13}$$

Since $\mathrm{dist}(Z, D' \setminus D_1) > 0$, from (2.5), we have

$$\|G_Z^h\|_{1,D'\setminus D_1} \le C\|G_Z^h\|_{1,\tau^{-\epsilon}} \le C. \tag{3.14}$$

By (3.12)–(3.14) and the inverse estimate, we immediately obtain the result (3.8). When $q_0 > 3$, from (2.6), $\|\partial_{Z,\ell} G_Z^h\|_{1,\tau^{-\alpha}} \le C(\alpha)$. Thus, similar to the above arguments, we obtain

$$|\partial_\ell \chi(Z)| = |a(\partial_{Z,\ell} G_Z^h, \Pi\hat{\chi})| \le C(d)\|\chi\|_{0,D'}. \tag{3.15}$$

In fact, we have $a(\chi, v) = 0\ \ \forall v \in S_0^h(D_1) \subset S_0^h(D')$. Choosing $D_{\frac{1}{2}}$ such that $D \subset\subset D_{\frac{1}{2}} \subset\subset D_1$, and $\mathrm{dist}(\partial D_{\frac{1}{2}}, \partial D_1) = \mathrm{dist}(\partial D_{\frac{1}{2}}, \partial D) = \frac{1}{4}d$, similar to (3.12) and (3.13), we have

$$|\partial_\ell \chi(Z)| = |a(\partial_{Z,\ell} G_Z^h, \Pi\hat{\chi})| \le C(d)h\|\chi\|_{1,D_1\setminus D_{\frac{1}{2}}}\|\partial_{Z,\ell} G_Z^h\|_{1,D_1\setminus D_{\frac{1}{2}}} + |J'|, \tag{3.16}$$

where $\hat{\chi} = \mu\chi$, $\mu \in C^\infty(\Omega)$ satisfying $\mathrm{supp}(\mu) \subset\subset D_1$ and $\mu|_{D_{\frac{1}{2}}} = 1$, and

$$J' = \int_\Omega \chi \sum_{i,j=1}^{3} (\partial_j (\partial_{Z,\ell} G_Z^h a_{ij}\partial_i \mu) + a_{ij}\partial_i \partial_{Z,\ell} G_Z^h \partial_j \mu)\, dxdydz.$$

Further,

$$|J'| \le C(d)\,\|\chi\|_{0,D_1\setminus D_{\frac{1}{2}}} \left\|\partial_{Z,\ell} G_Z - \partial_{Z,\ell} G_Z^h\right\|_{1,D_1\setminus D_{\frac{1}{2}}} + C(d)\,\|\chi\|_{-1,D_1\setminus D_{\frac{1}{2}}} \,\|\partial_{Z,\ell} G_Z\|_{2,D_1\setminus D_{\frac{1}{2}}}. \tag{3.17}$$

In [18], we have obtained

$$\left\|\partial_{Z,\ell} G_Z - \partial_{Z,\ell} G_Z^h\right\|_{1,\tau^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} |\ln h|^{\frac{4-3\alpha}{6}},$$

where $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$, and $1 < \alpha < \frac{4}{3}$ when $q_0 \ge 6$. Moreover, since $\mathrm{dist}(Z, D_1 \setminus D_{\frac{1}{2}}) > 0$, we have

$$\left\|\partial_{Z,\ell} G_Z - \partial_{Z,\ell} G_Z^h\right\|_{1,D_1\setminus D_{\frac{1}{2}}} \le C\left\|\partial_{Z,\ell} G_Z - \partial_{Z,\ell} G_Z^h\right\|_{1,\tau^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} |\ln h|^{\frac{4-3\alpha}{6}}. \tag{3.18}$$

We get by (2.6)

$$\left\|\partial_{Z,\ell} G_Z^h\right\|_{1,D_1\setminus D_{\frac{1}{2}}} \le C\left\|\partial_{Z,\ell} G_Z^h\right\|_{1,\tau^{-\alpha}} \le C(\alpha). \tag{3.19}$$

From (2.7) and (3.16)–(3.19),

$$|\partial_\ell \chi(Z)| \le C(d)h^{\frac{3(\alpha-1)}{2}} |\ln h|^{\frac{4-3\alpha}{6}} \|\chi\|_{1,D_1\setminus D_{\frac{1}{2}}} + C(d)\|\chi\|_{-1,D_1\setminus D_{\frac{1}{2}}}.$$

Further,

$$\|\chi\|_{1,\infty,D} \le C(d)h^{k_1} |\ln h|^{k_2} \|\chi\|_{1,D_1} + C(d)\|\chi\|_{-1,D_1}, \tag{3.20}$$

where $k_1 = \frac{3(\alpha-1)}{2}$ and $k_2 = \frac{4-3\alpha}{6}$.

Choosing $\{D_i\}_{i=2}^s$ such that $D \subset\subset D_1 \subset\subset D_2 \subset\subset \cdots \subset\subset D_s = D'$, and $\mathrm{dist}(\partial D_i, \partial D_{i+1}) = \frac{d}{2(s-1)}$, $i = 1, \cdots, s-1$, we have

$$\|\chi\|_{1,\infty,D_i} \le C(d)h^{k_1} |\ln h|^{k_2} \|\chi\|_{1,D_{i+1}} + C(d)\|\chi\|_{-1,D_{i+1}},\ \ i = 1, 2, \cdots, s-1. \tag{3.21}$$

Combining (3.20) and (3.21), and noting $\|\chi\|_{1,D_i} \le C\,\|\chi\|_{1,\infty,D_i}$, we have

$$\|\chi\|_{1,\infty,D} \le C(d)h^{sk_1} |\ln h|^{sk_2} \|\chi\|_{1,D'} + C(d)\|\chi\|_{-1,D'}. \tag{3.22}$$

When $3 < q_0 < 6$, we have $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$. Taking $\alpha = \frac{5}{3} - \frac{2}{q_0} - \varepsilon$, $0 < \varepsilon \ll 1$, and $s = [\frac{2q_0}{q_0-3}]+1$ such that $sk_1 > 2$, we get by (3.22) and the inverse estimate

$$\|\chi\|_{1,\infty,D} \le C(d)h\,|\ln h|^r\,\|\chi\|_{0,D'} + C(d)\|\chi\|_{-1,D'}, \tag{3.23}$$

where $r = ([\frac{2q_0}{q_0-3}] + 1)\frac{6+(3\varepsilon-1)q_0}{6q_0}$.

When $q_0 \ge 6$, we have $1 < \alpha < \frac{4}{3}$. Taking $\alpha = \frac{4}{3} - \frac{2\varepsilon}{5}$, $0 < \varepsilon \ll 1$, and $s = 5$ such that $sk_1 > 2$, we get by (3.22) and the inverse estimate

$$\|\chi\|_{1,\infty,D} \le C(d)h\,|\ln h|^t\,\|\chi\|_{0,D'} + C(d)\|\chi\|_{-1,D'}, \tag{3.24}$$

where $t = \varepsilon$. Combining (3.23) and (3.24) yields the result (3.9). The proof of Lemma 3.2 is completed.

Remark 2. In (3.20) and (3.21), for a suitable α, there exits an $\epsilon > 0$ such that $k_1 - \epsilon > 0$ and $|\ln h|^{k_2} \le h^{-\epsilon}$ when h is suitable small. Thus we can choose s such that $s(k_1 - \epsilon) > 2$. Further, we obtain when $q_0 > 3$,

$$\|\chi\|_{1,\infty,D} \le C(d)h\|\chi\|_{0,D'} + C(d)\|\chi\|_{-1,D'},$$

which is the better result than (3.9).

Lemma 3.3. *Suppose $D \subset\subset D' \subset \Omega$ and the integer $k \ge 0$. Then we have*

$$\|v\|_{0,D} \le Ch^{-k}\,\|v\|_{-k,D'} \ \ \forall\, v \in S_0^h(\Omega). \tag{3.25}$$

Proof. Set $D^* = \cup_e\{e : e \cap D \ne \phi,\ e \in \mathcal{T}^h\}$. For an element $e \subset D^*$, we define a negative-norm as follows:

$$\|v\|_{-k,e} = \sup_{\varphi \in C_0^\infty(e)} \frac{|(v,\varphi)_e|}{\|\varphi\|_{k,e}}. \tag{3.26}$$

Further, we define an affine transformation by

$$F : \tilde{X} \in \tilde{e} \longrightarrow X = \mathbf{B}\tilde{X} + \mathbf{b} \in e,$$

where $\tilde{e}$ is the standard reference element and $\mathbf{B} = (b_{ij})$ is a 3×3 matrix. We write $\tilde{\varphi}(\tilde{X}) = \varphi(F(\tilde{X}))$ and $\tilde{v}(\tilde{X}) = v(F(\tilde{X}))$. In addition, we have (see [27])

$$|w|_{k,p,e} \le C\|\mathbf{B}^{-1}\|^k|\mathrm{det}\mathbf{B}|^{\frac{1}{p}}|\tilde{w}|_{k,p,\tilde{e}} \ \ \forall\, \tilde{w} \in W^{k,p}(\tilde{e}).$$

Thus we get

$$|\varphi|_{k,e} \le Ch_e^{\frac{3}{2}-k}|\tilde{\varphi}|_{k,\tilde{e}}. \tag{3.27}$$

From (3.27),

$$\|\varphi\|_{k,e}^2 = \sum_{i=0}^{k}|\varphi|_{i,e}^2 \le Ch_e^{3-2k}\sum_{i=0}^{k}|\tilde{\varphi}|_{i,\tilde{e}}^2 = Ch_e^{3-2k}\,\|\tilde{\varphi}\|_{k,\tilde{e}}^2\,.$$

Namely,

$$\|\varphi\|_{k,e} \le Ch_e^{\frac{3-2k}{2}}\,\|\tilde{\varphi}\|_{k,\tilde{e}}\,. \tag{3.28}$$

By (3.28), the definition of the negative-norm (3.26), and the equivalence of norms in the finite-dimensional space, we have

$$\begin{aligned}\|v\|_{0,e} &\le Ch_e^{\frac{3}{2}}\,\|\tilde{v}\|_{0,\tilde{e}} \le Ch_e^{\frac{3}{2}}\,\|\tilde{v}\|_{-k,\tilde{e}} \le Ch_e^{\frac{3}{2}} \sup_{\tilde{\varphi}\in C_0^\infty(\tilde{e})}\frac{|(\tilde{v},\tilde{\varphi})_{\tilde{e}}|}{\|\tilde{\varphi}\|_{k,\tilde{e}}}\\ &\le Ch_e^{\frac{3}{2}-3+\frac{3-2k}{2}}\sup_{\varphi\in C_0^\infty(e)}\frac{|(v,\varphi)_e|}{\|\varphi\|_{k,e}}.\end{aligned}$$

Namely,

$$\|v\|_{0,e} \le Ch_e^{-k}\,\|v\|_{-k,e}\,. \tag{3.29}$$

Thus, from (3.29) and $1 \leq \frac{h}{h_e} \leq C_0$,

$$\|v\|_{0,D^*}^2 = \sum_e \|v\|_{0,e}^2 \leq Ch^{-2k} \sum_e \|v\|_{-k,e}^2 . \tag{3.30}$$

For every $\varepsilon > 0$, choosing $\varepsilon_e > 0$ such that $\sum_e \varepsilon_e = \varepsilon$, thus we have

$$\|v\|_{-k,e}^2 - \varepsilon_e \leq |(v, \varphi_e)_e|^2 , \ \varphi_e \in C_0^\infty(e) \ \text{ and } \|\varphi_e\|_{k,e} = 1. \tag{3.31}$$

We write $\omega = \sum_e (v, \varphi_e)_e \varphi_e \in C_0^\infty(D')$, and then

$$(v, \omega)_{D'} = \sum_e |(v, \varphi_e)_e|^2 . \tag{3.32}$$

Combining (3.30)–(3.32) yields

$$\|v\|_{0,D^*}^2 \leq Ch^{-2k}((v,\omega)_{D'} + \varepsilon) \leq Ch^{-2k}(\|v\|_{-k,D'} \|\omega\|_{k,D'} + \varepsilon). \tag{3.33}$$

In addition,

$$\begin{aligned}
\|\omega\|_{k,D'}^2 &= \int_{D'} \sum_{0\leq s\leq k} |\sum_e (v,\varphi_e)_e \nabla^s \varphi_e|^2 \, dX \\
&= \sum_{0\leq s\leq k} \int_{D'} |\sum_e (v,\varphi_e)_e \nabla^s \varphi_e|^2 \, dX \\
&= \sum_{0\leq s\leq k} \sum_e |(v,\varphi_e)_e|^2 \int_e |\nabla^s \varphi_e|^2 \, dX \\
&= \sum_e |(v,\varphi_e)_e|^2 = (v,\omega)_{D'} \leq \|v\|_{-k,D'} \|\omega\|_{k,D'} .
\end{aligned}$$

Thus,

$$\|\omega\|_{k,D'} \leq \|v\|_{-k,D'} . \tag{3.34}$$

When $\varepsilon \to 0$, we have by (3.33) and (3.34)

$$\|v\|_{0,D^*} \leq Ch^{-k} \|v\|_{-k,D'} .$$

Obviously, $D \subset D^*$, thus $\|v\|_{0,D} \leq \|v\|_{0,D^*} \leq Ch^{-k} \|v\|_{-k,D'}$. The proof of Lemma 3.3 is completed.

Lemma 3.4. *Suppose $D \subset\subset D' \subset \Omega$, $d \equiv \text{dist}(\partial D, \partial D')$, and $\partial D'$ is smooth enough. Let the integer $k \geq 0$, $q_0 > 2$, $a_{ij} \in W^{k+2,\infty}(\Omega)$, and $\chi \in S_0^h(\Omega)$ satisfies $a(\chi, v) = 0$ for all $v \in S_0^h(D')$. Then we have*

$$\|\chi\|_{-k,D} \leq C(d)h \|\chi\|_{1,D'} + C(d) \|\chi\|_{-k-1,D'} . \tag{3.35}$$

Proof. Choosing D_1 such that $D \subset\subset D_1 \subset\subset D'$, $\text{dist}(\partial D_1, \partial D') = \text{dist}(\partial D_1, \partial D) = \frac{1}{2}d$, and $\mu \in C^\infty(\Omega)$ satisfying $\text{supp}(\mu) \subset\subset D'$ and $\mu|_{D_1} = 1$, and setting $\hat{\chi} = \mu\chi$, we have by (1.4)

$$\|\chi\|_{-k,D} \leq \|\hat{\chi}\|_{-k,D'} = \sup_{\varphi \in C_0^\infty(D')} \frac{|(\varphi, \hat{\chi})_{D'}|}{\|\varphi\|_{k,D'}} \leq C \sup_{w\in\mathcal{H}} \frac{|a(w,\hat{\chi})_{D'}|}{\|w\|_{k+2,D'}}, \tag{3.36}$$

where $\mathcal{L}w = \varphi$ and $w \in \mathcal{H} \equiv H^{k+2}(D') \cap H_0^1(D')$. Similar to the arguments of Lemma 3.2, we get by the conditions of Lemma 3.4

$$a(w, \hat{\chi})_{D'} = a(\hat{w}, \chi)_{D'} + I_{D'} = a(\hat{w} - \Pi\hat{w}, \chi)_{D'} + I_{D'}, \tag{3.37}$$

where $\hat{w} = \mu w$ and

$$I_{D'} = \int_{D'} \sum_{i,j=1}^3 (-\partial_j(\chi w a_{ij} \partial_i \mu) + \chi \partial_j (w a_{ij} \partial_i \mu) + \chi a_{ij} \partial_i w \partial_j \mu) \, dxdydz.$$

Since $w \in \mathcal{H}$, thus we have

$$|I_{D'}| = \left| \int_{D'} \sum_{i,j=1}^{3} (\chi \partial_j (w a_{ij} \partial_i \mu) + \chi a_{ij} \partial_i w \partial_j \mu) \, dxdydz \right| \le C(d) \|\chi\|_{-k-1,D'} \|w\|_{k+2,D'} . \quad (3.38)$$

From (3.37) and (3.38),

$$\begin{aligned} |a(w,\hat{\chi})_{D'}| &\le C \|\chi\|_{1,D'} \|\hat{w} - \Pi \hat{w}\|_{1,D'} + C(d) \|\chi\|_{-k-1,D'} \|w\|_{k+2,D'} \\ &\le C(d) h \|\chi\|_{1,D'} \|w\|_{k+2,D'} + C(d) \|\chi\|_{-k-1,D'} \|w\|_{k+2,D'} . \end{aligned} \quad (3.39)$$

Combining (3.36) and (3.39) yields the result (3.35). The proof of Lemma 3.4 is completed.

Lemma 3.5. *Suppose $D' \subset \Omega$ and $\partial D'$ is smooth enough. Let the integer $k \ge 0$, $q_0 > 3$, $a_{ij} \in W^{k+2,\infty}(\Omega)$, and $\chi \in S_0^h(\Omega)$ satisfies $a(\chi, v) = 0$ for all $v \in S_0^h(D')$. For every D^* and D^{**} satisfying $D^* \subset\subset D^{**} \subset\subset D'$, we have*

$$\|\chi\|_{1,\infty,D^*} + \|\chi\|_{-k,D^*} \le C(d) \|\chi\|_{-k-1,D^{**}} , \quad (3.40)$$

where $d \equiv \operatorname{dist}(\partial D^*, \partial D^{**})$.

Proof. When $k = 0$, choosing $\tilde{D}$ such that $D^* \subset\subset \tilde{D} \subset\subset D^{**}$ and $\operatorname{dist}(\partial \tilde{D}, \partial D^{**}) = \operatorname{dist}(\partial \tilde{D}, \partial D^*) = \frac{1}{2} d$, we have by Remark 2 and Lemma 3.3

$$\|\chi\|_{1,\infty,D^*} \le C(d) h \|\chi\|_{0,\tilde{D}} + C(d) \|\chi\|_{-1,\tilde{D}} \le C(d) \|\chi\|_{-1,D^{**}} . \quad (3.41)$$

From (3.41),

$$\|\chi\|_{0,D^*} \le \|\chi\|_{1,\infty,D^*} \le C(d) \|\chi\|_{-1,D^{**}} . \quad (3.42)$$

Thus, from (3.41) and (3.42), when $k = 0$, the result (3.40) holds. Now when $k = t$, we suppose the result (3.40) holds. Namely,

$$\|\chi\|_{1,\infty,D^*} + \|\chi\|_{-t,D^*} \le C(d) \|\chi\|_{-t-1,D^{**}} . \quad (3.43)$$

We consider the case of $k = t+1$. Choosing $\{D_i\}_{i=0}^{t+2}$ such that $D^* \subset\subset \tilde{D} \subset\subset D_0 \subset\subset D_1 \subset\subset D_2 \subset\subset \cdots \subset\subset D_{t+2} \subset\subset D^{**}$, and $\operatorname{dist}(\partial \tilde{D}, \partial D_0) = \operatorname{dist}(\partial D_i, \partial D_{i+1}) = \frac{d}{2(t+4)}, i = 0, \cdots, t+1$, we have by (3.35) and (3.43)

$$\begin{aligned} \|\chi\|_{-t-1,\tilde{D}} &\le C(d) h \|\chi\|_{1,D_0} + C(d) \|\chi\|_{-t-2,D_0} \\ &\le C(d) h \|\chi\|_{1,\infty,D_0} + C(d) \|\chi\|_{-t-2,D_0} \\ &\le C(d) h \|\chi\|_{-t-1,D_1} + C(d) \|\chi\|_{-t-2,D_1} . \end{aligned} \quad (3.44)$$

Similarly,

$$\|\chi\|_{-t-1,D_i} \le C(d) h \|\chi\|_{-t-1,D_{i+1}} + C(d) \|\chi\|_{-t-2,D_{i+1}} , \; i = 1, 2, \cdots, t+1. \quad (3.45)$$

From (3.25), (3.44), and (3.45),

$$\begin{aligned} \|\chi\|_{-t-1,\tilde{D}} &\le C(d) h^{t+2} \|\chi\|_{-t-1,D_{t+2}} + C(d) \|\chi\|_{-t-2,D_{t+2}} \\ &\le C(d) h^{t+2} \|\chi\|_{0,D_{t+2}} + C(d) \|\chi\|_{-t-2,D_{t+2}} \\ &\le C(d) \|\chi\|_{-t-2,D^{**}} . \end{aligned} \quad (3.46)$$

In addition, from (3.43) and (3.46),

$$\|\chi\|_{1,\infty,D^*} \le C(d) \|\chi\|_{-t-1,\tilde{D}} \le C(d) \|\chi\|_{-t-2,D^{**}} . \quad (3.47)$$

Thus, from (3.46) and (3.47),

$$\|\chi\|_{1,\infty,D^*} + \|\chi\|_{-t-1,D^*} \le C(d) \|\chi\|_{-t-2,D^{**}} ,$$

which shows when $k = t+1$, the result (3.40) holds. The proof of Lemma 3.5 is completed.

Lemma 3.6. *Suppose $D \subset\subset D' \subset \Omega$ and $\partial D'$ is smooth enough. Let the integer $k \ge 0$, $q_0 > 3$,*

$a_{ij} \in W^{k+2,\infty}(\Omega)$, and $\chi \in S_0^h(\Omega)$ satisfies $a(\chi, v) = 0$ for all $v \in S_0^h(D')$. Then we have

$$\|\chi\|_{0,D} \le C(d)\,\|\chi\|_{-k-1,D'}\,, \tag{3.48}$$

and

$$\|\chi\|_{1,\infty,D} \le C(d)\,\|\chi\|_{-k-1,D'}\,, \tag{3.49}$$

where $d \equiv \mathrm{dist}(\partial D, \partial D')$.

Proof. Choosing $\{D_i\}_{i=1}^{k+1}$ such that $D \subset\subset D_1 \subset\subset D_2 \subset\subset \cdots \subset\subset D_k \subset\subset D_{k+1} = D'$, and $\mathrm{dist}(\partial D, \partial D_1) = \mathrm{dist}(\partial D_i, \partial D_{i+1}) = \frac{d}{k+1}, i = 1, \cdots, k$, we have by (3.40)

$$\|\chi\|_{1,\infty,D} \le C(d)\,\|\chi\|_{-1,D_1} \le C(d)\,\|\chi\|_{-2,D_2} \le C(d)\,\|\chi\|_{-3,D_3} \le \cdots \le C(d)\,\|\chi\|_{-k-1,D'}\,,$$

which is the result (3.49). Obviously,

$$\|\chi\|_{0,D} \le \|\chi\|_{1,\infty,D}\,.$$

Combined with (3.49), we immediately obtain the result (3.48). The proof of Lemma 3.6 is completed.

Theorem 3.1. *For every $Z \in \bar{\Omega}$, let $\mathcal{U}_r = \{X : |X - Z| < r,\ X \in \Omega\}$, and u and u_h be the solution of (1.1) and the m-degree (or tensor-product m-degree) finite element approximation. When $q_0 > 3$ and $u \in W^{m+1,\infty}(\mathcal{U}_r) \cap H_0^1(\Omega)$, we have*

$$|(u - u_h)(Z)| \le C(r)h^{m+1}\,|\ln h|^{\frac{2}{3}}\,\|u\|_{m+1,\infty,\mathcal{U}_r} + C(r)\,\|u - u_h\|_{-1,\mathcal{U}_r}\,, \tag{3.50}$$

and

$$|\nabla(u - u_h)(Z)| \le C(r)h^m\,\|u\|_{m+1,\infty,\mathcal{U}_r} + C(r)\,\|u - u_h\|_{-1,\mathcal{U}_r}\,. \tag{3.51}$$

Proof. Let $\mathcal{U}_{r_1} = \{X : |X - Z| < \frac{r}{4},\ X \in \Omega\}$, $\mathcal{U}_{r_2} = \{X : |X - Z| < \frac{3r}{4},\ X \in \Omega\}$. Thus, $\mathcal{U}_{r_1} \subset\subset \mathcal{U}_{r_2} \subset\subset \mathcal{U}_r$. Choosing $\mu \in C^\infty(\Omega)$ satisfying $\mathrm{supp}(\mu) \subset\subset \mathcal{U}_r$ and $\mu|_{\mathcal{U}_{r_2}} = 1$, and setting $\hat{u} = \mu u$ and $\tilde{u} = u - \hat{u}$, we easily obtain $\tilde{u}|_{\mathcal{U}_{r_2}} = 0$. In (1.10), for every $v \in S_0^h(\Omega)$, we replace w with $w - v$. Thus,

$$\|P_h w - v\|_{0,\infty,\Omega} \le C\,\|w - v\|_{0,\infty,\Omega}\,.$$

Further,

$$\|w - P_h w\|_{0,\infty,\Omega} \le \|w - v\|_{0,\infty,\Omega} + \|P_h w - v\|_{0,\infty,\Omega} \le C\,\|w - v\|_{0,\infty,\Omega}\,.$$

So we have

$$\|w - P_h w\|_{0,\infty,\Omega} \le C \inf_{v \in S_0^h(\Omega)} \|w - v\|_{0,\infty,\Omega}\,. \tag{3.52}$$

In addition, from (1.5), (1.7), and (1.9), we get for $v \in S_0^h(\Omega)$

$$\begin{aligned} |(P_h w - w_h)(Z)| &= |a(G_Z^*, w - w_h)| = |a(G_Z^* - G_Z^h, w - v)| \\ &\le C\left\|G_Z^* - G_Z^h\right\|_{1,1,\Omega}\|w - v\|_{1,\infty,\Omega}\,. \end{aligned}$$

Thus,

$$|(P_h w - w_h)(Z)| \le C\left\|G_Z^* - G_Z^h\right\|_{1,1,\Omega} \inf_{v \in S_0^h(\Omega)} \|w - v\|_{1,\infty,\Omega}\,. \tag{3.53}$$

As for G_Z^* and G_Z^h defined by (1.7) and (2.3), respectively, we have (see [16])

$$\left|G_Z^* - G_Z^h\right|_{1,1,\Omega} \le Ch|\ln h|^{\frac{2}{3}}. \tag{3.54}$$

By (3,52)–(3.54), the triangle inequality, and the Poincaré inequality, we have

$$\begin{aligned} \|w - w_h\|_{0,\infty,\Omega} &\le \|w - P_h w\|_{0,\infty,\Omega} + \|P_h w - w_h\|_{0,\infty,\Omega} \\ &\le C \inf_{v \in S_0^h(\Omega)} \|w - v\|_{0,\infty,\Omega} + Ch|\ln h|^{\frac{2}{3}} \inf_{v \in S_0^h(\Omega)} \|w - v\|_{1,\infty,\Omega} \\ &\le C\,\|w - \Pi w\|_{0,\infty,\Omega} + Ch|\ln h|^{\frac{2}{3}}\,\|w - \Pi w\|_{1,\infty,\Omega}\,, \end{aligned}$$

where Π is an interpolation operator. Further, by the interpolation error estimate, we get

$$\|w - w_h\|_{0,\infty,\Omega} \le Ch^{m+1}|\ln h|^{\frac{2}{3}} \|w\|_{m+1,\infty,\Omega}. \tag{3.55}$$

As for $\partial_{Z,\ell}G^*_Z$ and $\partial_{Z,\ell}G^h_Z$ defined by (1.8) and (2.4), respectively, we have (see [19])

$$\left|\partial_{Z,\ell}G^*_Z - \partial_{Z,\ell}G^h_Z\right|_{1,1,\Omega} \le C. \tag{3.56}$$

Similarly, by (1.6), (1.8), (1.9), (1.11), and (3.56), we get

$$\|w - w_h\|_{1,\infty,\Omega} \le Ch^m \|w\|_{m+1,\infty,\Omega}. \tag{3.57}$$

Obviously, $\hat{u} \in W^{m+1,\infty}(\Omega)$. From (3.55) and (3.57),

$$\begin{aligned}\|\hat{u} - \hat{u}_h\|_{0,\infty,\Omega} &\le Ch^{m+1}|\ln h|^{\frac{2}{3}} \|\hat{u}\|_{m+1,\infty,\Omega} \le Ch^{m+1}|\ln h|^{\frac{2}{3}} \|\hat{u}\|_{m+1,\infty,\mathcal{U}_r} \\ &\le C(r)h^{m+1}|\ln h|^{\frac{2}{3}} \|u\|_{m+1,\infty,\mathcal{U}_r},\end{aligned} \tag{3.58}$$

$$\|\hat{u} - \hat{u}_h\|_{1,\infty,\Omega} \le Ch^m \|\hat{u}\|_{m+1,\infty,\Omega} \le Ch^m \|\hat{u}\|_{m+1,\infty,\mathcal{U}_r} \le C(r)h^m \|u\|_{m+1,\infty,\mathcal{U}_r}. \tag{3.59}$$

For every $v \in S^h_0(\mathcal{U}_{r_2})$, since $\tilde{u}|_{\mathcal{U}_{r_2}} = 0$, we have $a(\tilde{u}, v) = 0$. Further, $a(\tilde{u}_h, v) = a(\tilde{u}, v) = 0$. Obviously, $\tilde{u}|_{\mathcal{U}_{r_1}} = 0$. Let $\mathcal{U}_{r^*} = \{X : |X - Z| < \frac{r}{2}, X \in \Omega\}$. Obviously, $\mathcal{U}_{r_1} \subset\subset \mathcal{U}_{r^*} \subset\subset \mathcal{U}_{r_2}$. Thus,

$$a(\tilde{u}_h, v) = 0 \;\; \forall v \in S^h_0(\mathcal{U}_{r^*}) \subset S^h_0(\mathcal{U}_{r_2}).$$

From Remark 2 and (3.25),

$$\begin{aligned}\|\tilde{u} - \tilde{u}_h\|_{1,\infty,\mathcal{U}_{r_1}} &= \|\tilde{u}_h\|_{1,\infty,\mathcal{U}_{r_1}} \le C(r)h \|\tilde{u}_h\|_{0,\mathcal{U}_{r^*}} + C(r) \|\tilde{u}_h\|_{-1,\mathcal{U}_{r^*}} \\ &\le C(r) \|\tilde{u}_h\|_{-1,\mathcal{U}_{r_2}} = C(r) \|\tilde{u} - \tilde{u}_h\|_{-1,\mathcal{U}_{r_2}} \\ &\le C(r) \|u - u_h\|_{-1,\mathcal{U}_r} + C(r) \|\hat{u} - \hat{u}_h\|_{-1,\mathcal{U}_r}.\end{aligned} \tag{3.60}$$

Moreover,

$$\begin{aligned}\|\hat{u} - \hat{u}_h\|_{-1,\mathcal{U}_r} &\le \|\hat{u} - \hat{u}_h\|_{-1,\Omega} \le \|\hat{u} - \hat{u}_h\|_{0,\Omega} \\ &\le Ch^{m+1} \|\hat{u}\|_{m+1,\Omega} \le C(r)h^{m+1} \|u\|_{m+1,\mathcal{U}_r}.\end{aligned} \tag{3.61}$$

Combining (3.60) and (3.61) yields

$$\|\tilde{u} - \tilde{u}_h\|_{1,\infty,\mathcal{U}_{r_1}} \le C(r)h^{m+1} \|u\|_{m+1,\mathcal{U}_r} + C(r) \|u - u_h\|_{-1,\mathcal{U}_r}. \tag{3.62}$$

Since $u = \hat{u} + \tilde{u}$, by (3.59) and (3.62), we immediately obtain the result (3.51). In addition,

$$\|\tilde{u} - \tilde{u}_h\|_{0,\infty,\mathcal{U}_{r_1}} \le \|\tilde{u} - \tilde{u}_h\|_{1,\infty,\mathcal{U}_{r_1}}. \tag{3.63}$$

Combining (3.58), (3.62), and (3.63) immediately yields the result (3.50). The proof of Theorem 3.1 is completed.

Similar to the arguments of Theorem 3.1, using (3.49), we easily obtain the following results.

Theorem 3.2. *For every $Z \in \bar{\Omega}$, let $\mathcal{U}_r = \{X : |X - Z| < r, X \in \Omega\}$, and u and u_h be the solution of (1.1) and the m-degree (or tensor-product m-degree) finite element approximation. When the integer $k \ge 0$, $q_0 > 3$, $a_{ij} \in W^{k+2,\infty}(\Omega)$, $u \in W^{m+1,\infty}(\mathcal{U}_r) \cap H^1_0(\Omega)$, and $\partial\mathcal{U}_r$ is smooth enough, we have*

$$|(u - u_h)(Z)| \le C(r)h^{m+1} |\ln h|^{\frac{2}{3}} \|u\|_{m+1,\infty,\mathcal{U}_r} + C(r) \|u - u_h\|_{-k-1,\mathcal{U}_r}, \tag{3.64}$$

and

$$|\nabla(u - u_h)(Z)| \le C(r)h^m \|u\|_{m+1,\infty,\mathcal{U}_r} + C(r) \|u - u_h\|_{-k-1,\mathcal{U}_r}. \tag{3.65}$$

Remark 3. As for the negative-norms in Theorems 3.1 and 3.2, we now give their bounds. For each $\varphi \in H^{k+1}(\Omega)$, we have

$$|(u - u_h, \varphi)| = |a(u - u_h, \tilde{\varphi} - \Pi\tilde{\varphi})| \le C \|u - u_h\|_1 \|\tilde{\varphi} - \Pi\tilde{\varphi}\|_1,$$

where $\tilde{\varphi} \in H^{k+3}(\Omega) \cap H^1_0(\Omega)$, $\mathcal{L}\tilde{\varphi} = \varphi$ in Ω, $\tilde{\varphi} = 0$ on $\partial\Omega$, and Π is the m-degree (or tensor-product m-degree) interpolation operator. When $m \ge 2$ and $0 \le k \le m - 2$, we have by the interpolation error estimate, the optimal approximation estimate, and the a priori estimate

$$|(u-u_h,\varphi)| \le Ch^{m+k+2}\|u\|_{m+1}\|\tilde{\varphi}\|_{k+3} \le Ch^{m+k+2}\|u\|_{m+1}\|\varphi\|_{k+1}.$$

Thus we obtain

$$\|u-u_h\|_{-k-1,\Omega} \le Ch^{m+k+2}\|u\|_{m+1}.$$

Hence

$$\|u-u_h\|_{-k-1,\mathcal{U}_r} \le \|u-u_h\|_{-k-1,\Omega} \le Ch^{m+k+2}\|u\|_{m+1}.$$

When $m=1$, we have

$$\|u-u_h\|_{-1,\mathcal{U}_r} \le \|u-u_h\|_{-1,\Omega} \le C\|u-u_h\|_{0,\Omega} \le Ch^2\|u\|_2.$$

The above results show that the negative norms do not spoil the order of superconvergence.

Declarations

Conflict of interest The authors declare no conflict of interest.

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