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On the partial stability of nonlinear impulsive Caputo fractional systems

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Abstract. In this work, stability with respect to part of the variables of nonlinear impulsive Caputo fractional differential equations is investigated. Some effective sufficient conditions of stability, uniform stability, asymptotic uniform stability and Mittag Leffler stability. The approach presented is based on the specially introduced piecewise continuous Lyapunov functions. Furthermore, some numerical examples are given to show the effectiveness of our obtained theoretical results.

§1 Introduction

The theory of differential equations with impulse effects of integer order has found extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. However, impulsive differential equations of fractional order have not been extensively studied and many aspects of these equations are yet to be explored. The study of these equations was initially carried out very slowly. This was due to the great problems caused by the specific properties of the impulsive equations such as beating, bifurcation, merging and dying of the solutions. Despite these difficulties, a boom in the development of the theory of impulsive differential equations of fractional order is observed recently.

In addition, in recent decades, the study of fractional order systems has gained importance [6,9,24]. The fractional calculus has been considered as the generalization of the classical integer-order calculus which motivates several authors to take a serious interest in this calculus. In [21,22], some comparison theorems are established, and the author investigated the problem of asymptotic stability and Mittag-Leffler stability of impulsive fractional differential equations. As for [11], several sufficient conditions are given to guarantee proprieties of the global asymptotic stability and global Mittag-Leffler stability for impulsive fractional-order differential equations on networks. In [23], the author investigated the global Mittag-Leffler stability and synchronization for the proposed delayed fractional-order neural networks by means of appropriate impulsive perturbations. Furthermore, stability analysis of Hilfer fractional differential

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systems are shown in [15]. On the other hand, in [27], the authors described the asymptotical stability of a nonlinear fractional differential system with Caputo derivative.

In the literature, the traditional Lyapunov theory attempts to assess the stability, asymptotic stability and instability of a system with respect to all of the variables. In recent years considerable attention has been paid to the generalization of these stability concepts in several directions. In particular, the concept of partial stability or stability with respect to a part of the variables has been studied by several authors for ordinary differential equations and nonlinear fractional differential equations, as e.g. V V Roumiantzeff [17], C Corduneanu [4, 5], N Rouche and K Peiffer [16], O Akinyele [1] and [3,7,18-20,25]. Recently, A B Makhlouf [3] investigated the problems of partial stability or, briefly, the *PSt-problem* related to nonlinear Caputo fractional differential equations without impulse effects. Sufficient conditions of stability, uniform stability, Mittag Leffler stability and asymptotic uniform stability are obtained within the method of Lyapunov-like functions. However, to the best of our knowledge, no paper in the literature has tackled the *PSt-problem* analysis for fractional order systems with impulse effects. By this fact, the main contribution of this paper is to study *PSt-problem* of nonautonomous systems with impulse effects in the sense of Caputo fractional derivative. This paper is organized as follows: In Section 2, some definitions and notations are given, and the concept of stability with respect to part of the variables is presented. Sufficient conditions for stability, uniform stability, asymptotic uniform stability and Mittag-Leffler stability with respect to part of the variables of nonlinear impulsive fractional systems are the focus of Section 3. In Section 4, some examples are worked out to illustrate the main results.

§2 Preliminary Notes

In this section a brief description of the main classes of fractional equations that will be used in the paper.

Definition 2.1. [6] The Riemann-Liouville fractional integral of an arbitrary integrable function l on the interval $[a, b]$ of order $\alpha \in (0, 1)$ is defined by

$$I_a^{\alpha} l(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} l(s) ds, \ t \in [a, b]$$

where $\Gamma(z) = \displaystyle\int_0^{+\infty} t^{z-1} e^{-t} dt$ is the Gamma function which converges in the right-half of the complex plane $Re(z) > 0$.

Definition 2.2. [6] Given an interval $[a, b]$ of $\mathbb{R}$, the Caputo fractional derivative of a function l of order $\alpha > 0$ is defined by

$${}^{c}D_{a,t}^{\alpha} l(t) = I_a^{m-\alpha} l^{(m)}(t)$$

where m stands for the smallest integer not less than α. When $\alpha \in (0, 1)$, the Caputo fractional derivative of order α, for a function $l \in C^1\big([a, b], \mathbb{R}^n\big)$, $b > a$, is defined as

$${}^{c}D_{a,t}^{\alpha} l(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{l'(s)}{(t-s)^{\alpha}} ds, \ t \in [a, b].$$

In the theory of integer-order differential systems, the exponential function is frequently used. In this work, we shall use the Mittag-Leffler function which plays an important role in the theory of non-integer order differential equations.

Definition 2.3. [9] The Mittag-Leffler function with two parameters $\alpha, \beta > 0$ is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}$$

where $z \in \mathbb{C}$. For $\beta = 1$, we have $E_\alpha(z) = E_{\alpha,1}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + 1)}$ and $E_{1,1}(z) = e^z$

2.1 Definitions of stability with respect to part of the variables

Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space with norm $\|.\|$ and Ω be an open set in $\mathbb{R}^m, m \leq n$ containing the origin $0_{\mathbb{R}^m}$. Consider the nonlinear systems of impulsive fractional differential equations for $\beta \in (0, 1)$

$$\begin{cases} {}^cD^\beta_{t_0,t}x(t) = f(t, (y(t), z(t))),\ t \neq \tau_k,\ k = \pm 1, \pm 2, ... \\ \\ \triangle x(\tau_k) = I_k(x(\tau_k^-)),\ t = \tau_k,\ k = \pm 1, \pm 2, ... \end{cases} \tag{1}$$

where $x^T = (y^T, z^T) \in \mathbb{R}^m \times \mathbb{R}^p,\ f : \mathbb{R} \times \Omega \times \mathbb{R}^p \to \mathbb{R}^n,\ \tau_k < \tau_{k+1},\ \ k = \pm 1, \pm 2, ...$ such that $\lim\limits_{k \to \pm\infty} \tau_k = \pm\infty$ and $I_k : \Omega \times \mathbb{R}^p \to \mathbb{R}^n$. Throughout this paper, suppose the variables constituting the phase vector x of system (1) are divided into two groups $x^T = (y^T, z^T) = (y_1, .., y_m, z_1, ..., z_p),\ y \in \mathbb{R}^m,\ z \in \mathbb{R}^p,\ m > 0, p \geq 0, n = m + p$, namely the $y-$variables with respect to which the stability of $x = 0$ is to be investigated and the remaining $z-$variables.

More precisely, this partitioning depends on the nature of the problem under study. In general, we assume that the choice of variables has already been made at the moment when a partial stability problem has to be analyzed. The $z-$variables are correspondingly called the uncontrollable variables.

Throughout this paper, the following notation is adopted

$$\|y\| = \Big(\sum_{i=1}^m y_i^2\Big)^{\frac{1}{2}},\ \|z\| = \Big(\sum_{i=1}^p z_i^2\Big)^{\frac{1}{2}} \text{ and } \|x\| = \Big(\|y\|^2 + \|z\|^2\Big)^{\frac{1}{2}}.$$

Let $t_0 \in \mathbb{R}$, $x_0 \in \Omega \times \mathbb{R}^p$. Denote by $x(t) \triangleq x(t, t_0, x_0)$ the solution of system (1) satisfying the initial condition $x(t_0^+, t_0, x_0) = x_0$. In this paper, the functions f and $I_k,\ k = \pm 1, \pm 2, ...$, are smooth enough on $\mathbb{R} \times \Omega \times \mathbb{R}^p$ and $\Omega \times \mathbb{R}^p$, respectively, to guarantee existence, uniqueness and continuability of the solution $x(t)$ of the system (1) on the interval $[t_0, +\infty)$ for all suitable initial data $x_0 \in \Omega \times \mathbb{R}^p$ and $t_0 \in \mathbb{R}$. We also assume that the functions $E + I_k : \Omega \times \mathbb{R}^p \to \Omega \times \mathbb{R}^p, k = \pm 1, \pm 2, ...$, where E is the identity in $\Omega \times \mathbb{R}^p$.

Note that, the solutions $x(t)$ of system (1) are, in general, piecewise continuous functions with points of discontinuity of the first kind at which they are left continuous, that is, at the moments $\tau_k, k = \pm 1, \pm 2, ...$, the following relations are satisfied

$$x(\tau_k^-) = x(\tau_k) \text{ and } x(\tau_k^+) = x(\tau_k) + I_k(x(\tau_k)) = x(\tau_k) + \triangle x(\tau_k)$$

where $x(\tau_k^+) := \lim\limits_{h \to 0^+} x(\tau_k + h)$ and $x(\tau_k^-) := \lim\limits_{h \to 0^+} x(\tau_k - h)$ represent the right- and left-sided limits of $x(t)$ at $t = \tau_k$, respectively. Note that in theory of differential and fractional impulsive systems, the elements of the sequence $(\tau_k)_{k \in \mathbb{Z}}$ are the moments of impulsive perturbations due to which the state $x(t)$ changes from the position $x(\tau_k)$ into the position $x(\tau_k^+)$ and the functions I_k characterize the magnitude of the impulse effect at the moments τ_k.

In the following, we recall the next Lyapunov stability definitions for the system (1) that will be used in this paper.

Definition 2.4. (Partial Boundedness) The solutions of system (1) are uniformly bounded with respect to y ($y-$uniformly bounded, $y-$UB) if there exists a positive constant c, independent of t_0, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of t_0, such that for all $x_0 \in \mathbb{R}^n$, $\|x_0\| < a$, we have

$$\|y(t, t_0, x_0\| \leq \beta,\ t \geq t_0.$$

Definition 2.5. [25](Partial Stability)
The equilibrium point $x = 0$ of impulsive fractional-order system (1) is said to be:

(a) Stable with respect to y (or, briefly, $y-$Stable ($y-$St)), if for any numbers $\varepsilon > 0$, $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that from $\|x_0\| < \delta$ it follows that $\|y(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$.

(b) Uniformly stable with respect to y ($y-$Uniformly stable, $y-$ USt), if is $y-$St and δ does not depend on t_0.

Definition 2.6. [3,24](Partial Attractivity) The equilibrium point $x = 0$ of impulsive fractional-order system (1) is said to be:

(a) Attractive with respect to y or briefly $y-$Attractive ($y-$A), if for all $t_0 \in \mathbb{R}$, there exists $\lambda = \lambda(t_0) > 0$ such that for all $x_0 \in \Omega \times \mathbb{R}^p$, $\|x_0\| < \lambda$, we have

$$\lim_{t \to +\infty} \|y(t, t_0, x_0)\| = 0. \tag{2}$$

The domain $\|x_0\| < \lambda$, being contained in the domain of $y-$attraction of the point $x = 0$ for the initial time t_0.

(b) Equi-attractive with respect to y or briefly $y-$Eq-Attractive ($y-$Eq-A), if for all $(t_0, \epsilon) \in \mathbb{R} \times \mathbb{R}_+^*$, there exist $\lambda = \lambda(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that, for all $x_0 \in \Omega$, $\|x_0\| < \lambda$, we have

$$\|y(t, t_0, x_0\| < \epsilon,\ \forall\, t \geq T + t_0.$$

(c) Uniformly attractive with respect to y ($y-$UA), if it is $y-$Eq-A with λ and T are independent of $t_0 \in \mathbb{R}$.

(d) Globally equi-attractive with respect to y, if for all $t_0 \in \mathbb{R}$, $\nu, \epsilon \in \mathbb{R}_+^*$, there exist $\gamma = \gamma(t_0, \nu, \epsilon) > 0$ such that, for all $x_0 \in \mathbb{R}^n$, $\|x_0\| < \nu$, we have

$$\|y(t, t_0, x_0\| < \epsilon,\ \forall\, t \geq t_0 + \gamma.$$

(e) Globally uniformly attractive with respect to y ($y-$GUA) if the number γ in (d) is independent of $t_0 \in \mathbb{R}$.

Definition 2.7. [20, 24, 25](Partial Asymptotic Stability)
The equilibrium point $x = 0$ of impulsive fractional-order system (1) is said to be:

(a) Asymptotically stable with respect to y ($y-$ASt), if it is $y-$St and $y-$A

(b) Uniformly asymptotically stable with respect to y ($y-$ UASt), if it is $y-$ USt and $y-$UA

(c) Globally equi-asymptotically stable with respect to y , if it is $y-$St and globally equi-attractive;

(d) Globally uniformly asymptotically stable with respect to y ($y-$GUASt), if it is $y-$USt and $y-$GUA and the solutions of system (1) are $y-$UB.

Definition 2.8. [3](Partial Mittag-Leffler Stability)
The equilibrium point $x = 0$ of impulsive fractional-order system (1) is said to be Mittag-Leffler stable with respect to y ($y-$M-LSt) , if given $\delta > 0$, we have $\|x_0\| < \delta$ implies

$$\|y(t, t_0, x_0)\| \leq \Big[m(x_0)E_\alpha(-\lambda(t-t_0)^\alpha\Big]^b, \text{ for all } t \geq t_0 \tag{3}$$

where $\alpha \in (0, 1)$, $\lambda > 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$ and m is locally Lipschitz continuous with respect to $x \in \Omega \times \mathbb{R}^p$, $\|x\| < \delta$.

Remark 2.1. *Mittag-Leffler stability of a solution of system (1), implies its asymptotic stability (see [12, 13]).*

Remark 2.2. *Geometrically, the definitions introduced mean the following. In the case of $y-$St, inside any $\epsilon-$cylinder $\|y\| < \epsilon$ is a $\delta-$sphere $\|x_0\| = \delta$ such that any solution $x(t, t_0, x_0)$ of system (1), once originated from the $\delta-$sphere at $t = t_0$, will remain inside the $\epsilon-$cylinder for all $t \geq t_o$. In the case of $y-$ASt, the solution $x(t)$ will, in addition, asymptotically approach the $\epsilon-$cylinder axis.*

The fact that the solutions of (1) are piecewise continuous functions requires introducing some analogous of the classical Lyapunov functions which have discontinuities of the first kind [2, 10]. By means of such functions it becomes possible to solve basic problems related to the application of Lyapunov second method to impulsive fractional systems.
Let $\tau_0 = t_0 \in \mathbb{R}$ and introduce the sets

$$G_k = \{(t, x) \in \mathbb{R} \times \Omega \times \mathbb{R}^p : \tau_{k-1} < t < \tau_k\}, k = \pm 1, \pm 2, ... \text{ and } G = \bigcup_{k=\pm 1,\pm 2,....} G_k.$$

Definition 2.9. [24] A function $V : \mathbb{R} \times \Omega \times \mathbb{R}^p \to \mathbb{R}_+$ belongs to the class $\mathcal{V}_0$, if

1. $V(t, x)$ is continuous in G and locally Lipschitz continuous with respect to its second argument on each of the sets $G_k, k = \pm 1, \pm 2, ...$

2. For each $k = \pm 1, \pm 2, ...$ and $x \in \Omega \times \mathbb{R}^p$, there exist the finite limits
$$V(\tau_k^-, x) = \lim_{\substack{t\to\tau_k \\ t<\tau_k}} V(t, x), \quad V(\tau_k^+, x) = \lim_{\substack{t\to\tau_k \\ t>\tau_k}} V(t, x)$$
and the following equalities are valid
$$V(\tau_k^-, x) = V(\tau_k, x).$$

Definition 2.10. [24] Let $t \in [\tau_k, \tau_{k+1}), k = \pm 1, \pm 2, ...$ and $x \in \Omega \times \mathbb{R}^p$. The upper right-hand derivative of V in the Caputo sense of order β, $0 < \beta < 1$, with respect to system (1) is defined by

$$^cD_+^\beta V(t, x) = \lim_{\chi\to 0^+} \sup \frac{1}{\chi^\beta}\Big[V(t, x) - V(t-\chi, x-\chi^\beta f(t, x))\Big]$$

Remark 2.3. *[6, 26] By simple calculation , if $x = x(t)$ is a solution of system (1), then for $t \neq \tau_k,\ k = 1, 2, ...$*

$$^cD_+^\beta V(t, x(t)) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{D_{(1)}^+ V(\tau, x(\tau))}{(t-\tau)^\beta} d\tau$$

where

$$D_{(1)}^+ V(t, x(t)) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta} \Big[V(t+\delta, x(t+\delta)) - V(t, x) \Big].$$

Definition 2.11. [8] A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is to belong to class $\mathcal{K}_\infty$ if in addition $\lim\limits_{s\to+\infty} \alpha(s) = +\infty$.

The following lemmas will also be required in the investigations of the paper.

Lemma 2.1. *[24] Assume that the function $V \in \mathcal{V}_0$ is such that:*

i) $V(t^+, x + I_k(x)) \leq V(t, x),\ x \in \Omega \times \mathbb{R}^p,\ t = \tau_k,\ \tau_k > t_0$

ii) $^cD_+^\beta V(t, x) \leq qV(t, x),\ (t, x) \in G,\ t \in [t_0, +\infty)$.

Then,

$$V(t, x(t, t_0, x_0)) \leq V(t_0^+, x_0) E_\beta\big(q(t-t_0)^\beta\big),\ t \in [t_0, +\infty).$$

Lemma 2.2. *[24] Assume that the function $V \in \mathcal{V}_0$ is such that:*

i) $V(t^+, x + I_k(x)) \leq V(t, x),\ x \in \Omega \times \mathbb{R}^p,\ t = \tau_k,\ \tau_k > t_0$

ii) $^cD_+^\beta V(t, x) \leq 0,\ (t, x) \in G,\ t \in [t_0, +\infty)$.

Then,

$$V(t, x(t, t_0, x_0)) \leq V(t_0^+, x_0),\ t \in [t_0, +\infty).$$

The goal of this paper, is to investigate the stability of the zero solution $x(t) \equiv 0$ of system (1). That is why the following conditions will be assumed :

(C1) The origin $x = 0$ is an equilibrium point of fractional-order system (1); that is, $f(t, 0) = 0,\ t \geq t_0$.

(C2) $I_k(0) = 0,\ k = \pm 1, \pm 2, ...$

In addition, in the proofs of our main results, we shall use piecewise continuous Lyapunov functions $V : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}_+,\ V \in \mathcal{V}_0$ for which the following condition is true :

(C3) $V(t, 0) = 0,\ t \geq t_0$.

§3 Main results

In this section, some sufficient conditions are given to guarantee $y-$stability, $y-$asymptotic stability and $y-$Mittag-Leffler stability of the nonlinear systems of impulsive fractional differential equations (1).

3.1 Partial stability

Theorem 3.1. *Assume that conditions (**C1**) and (**C2**) are met. Let $V \in \mathcal{V}_0$ be such that (**C3**) holds and*

i) $\alpha_1(\|y\|) \leq V(t,x),\ \alpha_1 \in \mathcal{K},\ (t,x) \in [t_0,+\infty[\times\Omega\times\mathbb{R}^p$,

ii) ${}^cD^{\beta}_{t_0,t}V(t,x) \leq 0,\ (t,x) \in G_k,\ k = \pm1, \pm2, ...,$

iii) $V(t^+, x + I_k(x)) \leq V(t,x),\ x \in \Omega\times\mathbb{R}^p,\ t = \tau_k,\ k = \pm1, \pm2, ...$

where $\beta \in (0,1)$. Then, the zero solution of system (1) is $y-$St.

Proof. Let $\epsilon > 0$. From the properties of the function V, it follows that there exists a constant $\delta = \delta(t_0, \epsilon) > 0$ such that if $\|x\| \leq \delta$, then

$$\sup_{\|x\|\leq\delta} V(t_0^+, x) < \alpha_1(\epsilon) \tag{4}$$

Let $x_0 \in \Omega\times\mathbb{R}^p$ such that $\|x_0\| \leq \delta$; and let $x(t) = x(t,t_0,x_0) = (y(t,t_0,x_0), z(t,t_0,x_0))$ be the solution of (1). We shall prove that $\|y(t,t_0,x_0)\| \leq \epsilon$ for $t \geq t_0$. Suppose that this is not true. Then there exists $t^* > t_0$, $t^* \in [\tau_k, \tau_{k+1}[$, for some fixed integer k such that

$$\|y(t^*)\| > \epsilon \text{ and } \|y(t,t_0,x_0\| \leq \epsilon,\ t \in [t_0, \tau_k].$$

By using the condition *iii*) and the properties of functions $E + I_k, k = \pm1, \pm2, ...$, it's possible to find $\breve{t} \in]\tau_k, t^*]$, such that

$$\|y(\breve{t})\| > \epsilon \text{ and } y(\breve{t}, t_0, x_0) \in \Omega.$$

Then, for $t \in [t_0, \breve{t}]$ it follows from Lemma 2.2 that

$$V(t, x(t,t_0,x_0)) \leq V(t_0^+, x_0)$$

and

$$\alpha_1(\epsilon) < \alpha_1(\|y(\breve{t},t_0,x_0)\|) \leq V(\breve{t}, x(\breve{t},t_0,x_0)) \leq V(t_0^+, x_0) < \alpha_1(\epsilon).$$

The contradiction obtained shows that

$$\|y(t,t_0,x_0)\| \leq \epsilon$$

for $\|x_0\| \leq \delta$ and $t \geq t_0$. This implies that the equilibrium point $x = 0$ of system (1) is $y-$St. □

Theorem 3.2. *Let the conditions of Theorem 3.1 hold, and let a function $\alpha_2 \in \mathcal{K}$ exists such that*

$$V(t,x) \leq \alpha_2(\|w\|),\ (t,x) \in [t_0,+\infty[\times\Omega\times\mathbb{R}^p \tag{5}$$

where $w = (x_1, x_2, ..., x_k)^T \in \mathbb{R}^k$, $m \leq k \leq n$. Then, the zero solution of system (1) is $y-$USt.

Proof. Let $\epsilon > 0$ be chosen. Choose $\delta = \delta(\epsilon) > 0$ so that $\alpha_2(\delta) < \alpha_1(\epsilon)$. Let $x_0 \in \Omega\times\mathbb{R}^p$ such that $\|x_0\| < \delta$ and $x(t) = x(t,t_0,x_0)$ be the solution of problem (1). It follows from Corollary 1.4 that

$$V(t, x(t)) \leq V(t_0^+, x_0),\ t \geq t_0.$$

From the above inequalities and (5), we get to the inequalities

$$\alpha_1(\|y(t,t_0,x_0)\|) \leq V(t_0^+, x_0) \leq \alpha_2(\|w_0\|) \leq \alpha_2(\|x_0\|) \leq \alpha_2(\delta) < \alpha_1(\epsilon)$$

where $x_0 = (x_{10}, x_{20}, ..., x_{k0}, x_{k+10}, ..., x_{n0})^T \in \Omega\times\mathbb{R}^p$ and $w_0 = (x_{10}, x_{20}, ..., x_{k0})^T \in \mathbb{R}^k$. From which it follows that

$$\|y(t,t_0,x_0)\| \leq \epsilon \text{ for } t \geq t_0.$$

This proves the uniform $y-$stability of the zero solution of system (1). □

Theorem 3.3. *Assume that conditions (**C1**) and (**C2**) are met. In addition, suppose there exists a function $V \in \mathcal{V}_0$ such that (**C3**) holds and*

i) $\alpha_1(\|y\|) \le V(t,x) \le \alpha_2(\|w\|)$, $t \in [t_0, +\infty)$, $x \in \Omega \times \mathbb{R}^p$,

ii) ${}^cD_+^\beta V(t,x) \le -\alpha_3(\|w\|)$, $\alpha_3 \in \mathcal{K}$, $(t,x) \in G_k$, $k = \pm 1, \pm 2, ...$

iii) $V(t^+, x + I_k(x)) \le V(t,x)$, $x \in \Omega \times \mathbb{R}^p$, $t = \tau_k$, $k = \pm 1, \pm 2, ...$

where $\beta \in (0,1)$ and $w = (x_1, x_2, ..., x_k)^T \in \mathbb{R}^k$, $m \le k \le n$. Then, the zero solution of system (1) is $y-$UASt.

Proof. Firstly, it is easy to see that all conditions of Theorem 3.2 are satisfied. Then, the zero solution $x = 0$ is $y-$USt.
Now, consider $h > 0$ and $x_0 \in \Omega$ such that $\|x_0\| \le h$. Let $\epsilon > 0$ be chosen. Choose $\eta = \eta(\epsilon)$ so that $\alpha_2(\eta) < \alpha_1(\epsilon)$, and let $T > \left[\frac{\alpha_2(h)\Gamma(\beta+1)}{\alpha_3(\eta)}\right]^{\frac{1}{\beta}}$. If we assume that for each $t \in [t_0, t_0 + T]$ the inequality $\|w(t, t_0, x_0\| \ge \eta$ is valid, then from *ii*) and *iii*), we get

$$\begin{aligned} V(t, x(t, t_0, x_0)) &\le V(t_0^+, x_0) - \frac{1}{\Gamma(\beta)} \int_{t_0}^t \alpha_3(\|w(s, t_0, x_0)\|)(t-s)^{\beta-1} ds, \\ &\le \alpha_2(\|w_0\|) - \frac{\alpha_3(\eta)}{\Gamma(\beta)} \int_{t_0}^t (t-s)^{\beta-1} ds, \\ &\le \alpha_2(\|x_0\|) - \frac{\alpha_3(\eta)}{\Gamma(\beta)} \int_{t_0}^t (t-s)^{\beta-1} ds, \\ &\le \alpha_2(h) - \frac{\alpha_3(\eta)}{\Gamma(\beta+1)} (t-t_0)^\beta. \end{aligned}$$

If $t = t_0 + T$, then

$$V(t_0 + T, x(t_0 + T, t_0, x_0)) \le \alpha_2(h) - \frac{\alpha_3(\eta)}{\Gamma(\beta+1)} T^\beta < 0$$

which contradicts (i) of Theorem 3.3. Then, there exists $t^* \in [t_0, t_0 + T]$, such that

$$\|w(t^*, t_0, x_0\| < \eta.$$

It follows that for $t \ge t^*$, in particular for any $t \ge t_0 + T$ the following inequalities hold

$$\begin{aligned} \alpha_1(\|y(t, t_0, x_0)\| &\le V(t, x(t, t_0, x_0) \\ &\le V(t^*, x(t^*, t_0, x_0)), \\ &\le \alpha_2(\|w(t^*, t_0, x_0)\|) \le \alpha_2(\eta) < \alpha_1(\epsilon). \end{aligned}$$

Therefore, $\|y(t, t_0, x_0)\| < \epsilon$ for $t \ge t_0 + T$. It follows that the zero solution of system (1) is $y-$UA. Since it is $y-$USt, then the solution $x = 0$ is $y-$UASt. □

3.2 Partial Mittag-Leffler Stability

Let $t_0 = 0$. In this part, we extend the problem of Mittag-Leffler stability introduced by Podlubny and his co-authors. Precisely, we shall investigate the problem of Mittag-Leffler stability with respect to y of the equilibrium point $x = 0$ of system (1). Using the fractional Lyapunov method, some sufficient conditions for Mittag-Leffler stability with respect to a part of the variables are given.

Theorem 3.4. *Assume that :*

1. *Conditions (C1) and (C2) hold for* $t_0 = 0$.

2. *There exists a function* $V \in \mathcal{V}_0$ *such that **(C3)** holds, and*

   i) $c_1\|y\|^a \le V(t,x) \le c_2\|w\|^{ab}$, $t \in \mathbb{R}_+$, $x \in \Omega \times \mathbb{R}^p$,

   ii) ${}^cD_+^{\beta}V(t,x) \le -c_3\|w\|^{ab}$, $(t,x) \in G_k$, $t \in \mathbb{R}_+$,

   iii) $V(t^+, x + I_k(x)) \le V(t,x)$, $x \in \Omega \times \mathbb{R}^p$, $t = \tau_k > 0$

where $\beta \in (0,1)$, $w = (x_1, x_2, ..., x_k)^T \in \mathbb{R}^k$, $m \le k \le n$, c_1, c_2, c_3, a *and* b *are positive constants. Then, the zero solution of system (1) is* $y-$*M-LSt. If the assumptions hold globally on* $\mathbb{R}^n$, *then the zero solution of system (1) is* $y-$*GM-LSt.*

Proof. let $x(t) = x(t, 0, x_0)$ be the solution of (1) with initial condition $(0, x_0)$ where $x_0 \in \Omega \times \mathbb{R}^p$ such that $\|x_0\| < \delta$. By using conditions (i) and (ii) it follows that

$$ {}^cD_+^{\beta}V(t,x(t)) \le -\frac{c_3}{c_2}V(t,x(t)),\ t \ne \tau_k,\ t > 0. $$

Therefore, for $t \in [0, +\infty)$ there exists a nonnegative function $W(t)$ satisfying

$$ {}^cD_+^{\beta}V(t,x(t)) + W(t) = -c_3c_2^{-1}V(t,x(t)),\ t \ne \tau_k. \tag{6} $$

Taking the Laplace transform of (6) for $t \ne \tau_k$, , $t > 0$ gives

$$ s^{\beta}\mathcal{V}(s) - s^{\beta-1}V(0) + W(s) = -c_3c_2^{-1}\mathcal{V}(s), $$

where $V(0) = V(0, x(0))$ and $\mathcal{V}(s) = \mathcal{L}[V(t,x(t))](s)$. From the last equality we obtain

$$ \mathcal{V}(s) = \frac{V(0)s^{\beta-1} - W(s)}{s^{\beta} + \frac{c_3}{c_2}}. $$

The unique solution of (6) is (see [14])

$$ V(t,x(t)) = V(0,x(0))E_{\beta}\big(-\frac{c_3}{c_2}t^{\beta}\big) - W(t) * \Big[t^{\beta-1}E_{\beta,\beta}\big(-\frac{c_3}{c_2}t^{\beta}\big)\Big],\ t \ne \tau_k, t > 0 $$

where $*$ denotes the convolution operator. Since both $t^{\beta-1}$ and $E_{\beta,\beta}\big(-\frac{c_3}{c_2}t^{\beta}\big)$ are nonnegative for $t \in (\tau_{k-1}, \tau_k]$, $k = \pm1, \pm2, ...,$ $t > 0$, it follows that for any closed interval contained in $(\tau_{k-1}, \tau_k]$

$$ V(t,x(t)) \le V(0,x(0))E_{\beta}\big(-\frac{c_3}{c_2}t^{\beta}\big). $$

Set $R = V(0,x(0))E_{\beta}\big(-\frac{c_3}{c_2}t^{\beta}\big)$. From condition $iii)$ it follows that, if $V(\tau_k, x(\tau_k)) \le R$, then

$$ V(\tau_k^+, x(\tau_k^+)) = V(\tau_k^+, x(\tau_k) + I_k(x(\tau_k))) \le V(\tau_k, x(\tau_k)) \le R $$

which implies the solution $x(t)$ cannot exceed R by jump. Therefore

$$ V(t,x(t)) \le V(0,x(0))E_{\beta}\big(-\frac{c_3}{c_2}t^{\beta}\big),\ \forall\, t \ge 0. $$

From the last inequality and $i)$ we have

$$ \|y(t,t_0,x_0)\| \le \Big[\frac{V(0,x(0))}{c_1}E_{\beta}\big(-\frac{c_3}{c_2}t^{\beta}\big)\Big]^{\frac{1}{a}},\ t \ge 0. $$

Let $m(x) = \frac{V(0,x)}{c_1} \ge 0$. Then, we have

$$ \|y(t,t_0,x_0)\| \le \Big[m(x(0))E_{\beta}\big(-\frac{c_3}{c_2}t^{\beta}\big)\Big]^{\frac{1}{a}},\ t \ge 0. $$

From the properties of the Lyapunov function $V(t,x)$ it follows that m is Lipschitz with respect to x and $m(0)=0$, which imply the Mittag-Leffler stability with respect to y of the zero solution of system (1). This completes the proof of Theorem 3.5. □

3.3 Global partial Stability

Consider the system of impulsive fractional differential equations (1) where the open set $\Omega=\mathbb{R}^{n-p}$. Suppose again that the functions $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ and $I_k:\mathbb{R}^n\to\mathbb{R}^n$, $k=\pm1,\pm2,...$ are smooth enough on $[t_0,+\infty)\times\mathbb{R}^n$ and $\mathbb{R}^n$, respectively, to guarantee the existence, uniqueness and continuability of the solution of (1) on the interval $[t_0,+\infty)$ for all $x_0\in\mathbb{R}^n$ and $t\ge t_0$.
In this subsection the global equi-asymptotic stability and global asymptotic stability with respect to y of (1) will be considered.

Theorem 3.5. *Suppose that conditions **(C1)** and **(C2)** hold for system (1). Let $V\in\mathcal{V}_0$ be a Lyapunov function such that **(C3)** holds, and*

i) $\alpha_1(\|y\|)\le V(t,x)$, $\alpha_1\in\mathcal{K}_\infty,(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$,

ii) $V(t^+,x+I_k(x))\le V(t,x)$, $x\in\mathbb{R}^n$, $t=\tau_k$,

iii) ${}^cD^\beta_+V(t,x)\le -cV(t,x)$, $\alpha_3\in\mathcal{K}$, $(t,x)\in G_k$, $k=1,2,...$

where $c>0$ and $\beta\in(0,1)$. Then, the zero solution of system (1) is $y-$GEq-ASt.

Proof. Let $\epsilon>0$. From the properties of the function $V(t,x)$, it follows that there exists a constant $\delta=\delta(t_0,\epsilon)>0$ such that if $x\in\mathbb{R}^n$ such that $\|x\|<\delta$, then $\sup\limits_{\|x\|<\delta}V(t_0^+,x)<\alpha_1(\epsilon)$.
Let $x_0\in\mathbb{R}^n$ such that $\|x_0\|<\delta$. By Lemma 2.2, the following hold

$$V(t,x(t,t_0,x_0))\le V(t_0^+,x_0),\ t\in[t_0,+\infty).$$

Consequently

$$\alpha_1(\|y(t,t_0,x_0)\|)\le V(t,x(t,t_0,x_0))\le V(t_0^+,x_0)<\alpha_1(\epsilon),$$

which imply that $\|y(t,t_0,x_0)\|<\epsilon$ for $t\ge t_0$. Then, the zero solution of system (1) is $y-$St.

Now we shall prove that it is globally equi-attractive with respect to y.
Let $\nu>0$ and $x_0\in\mathbb{R}^n$ such that $\|x_0\|<\nu$. By Lemma 2.1, it follows that for $t\ge t_0$, the following inequality is valid

$$V(t,x(t,t_0,x_0))\le V(t_0^+,x_0)E_\beta\big(-c(t-t_0)^\beta\big). \tag{7}$$

Let consider

$$\aleph(t_0,\nu)=\sup\Big\{V(t_0^+,x),\ \|x\|<\nu\Big\}$$

and

$$\gamma=\gamma(t_0,\nu,\epsilon)>\Big[\frac{-1}{c}E_\beta^{-1}\big(\frac{\alpha_1(\epsilon)}{\aleph(t_0,\nu)}\big)\Big]^{\frac{1}{\beta}}$$

For $t\ge t_0+\gamma$, from (7) it follows that

$$V(t,x(t,t_0,x_0))<\alpha_1(\epsilon).$$

From the last inequality and condition *i)* of Theorem 3.5 we have

$$\|y(t,t_0,x_0)\|<\epsilon$$

which means that the zero solution of system (1) is $y-$GEq-A. □

In the next Theorem, some sufficient conditions are given to ensure the global asymptotic stability with respect to y of the zero solution of system (1).

Theorem 3.6. *Assume that conditions (**C1**) and (**C2**) hold for system (1). Let $V \in \mathcal{V}_0$ be a Lyapunov function such that (**C3**) holds, and*

i) $\alpha_1(\|y\|) \le V(t,x) \le \alpha_2(\|w\|)$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, (t,x) \in [t_0, +\infty) \times \mathbb{R}^n$,

ii) ${}^cD_+^\beta V(t,x) \le -\alpha_3(\|w\|)$, $\alpha_3 \in \mathcal{K}_\infty$, $(t,x) \in G$, $t \in [t_0, +\infty)$

iii) $V(t^+, x + I_k(x)) \le V(t,x)$, $x \in \mathbb{R}^n$, $t = \tau_k$, $k = \pm 1, \pm 2, ...,$

where $w = (x_1, x_2, ..., x_k)^T \in \mathbb{R}^k$, $m \le k \le n$, $\beta \in (0,1)$. *Then, the zero solution of system (1) is $y-$GASt.*

Proof. Firstly, by conditions *i*) and *ii*), the zero solution of system (1) is $y-$USt.
Now, we shall prove that the solutions of system (1) are uniformly bounded with respect to y. Let consider a nonnegative constant r and $x_0 \in \mathbb{R}^n$ such that $\|x_0\| < r$. By using the fact $\lim_{s \to +\infty} \alpha_1(s) = +\infty$, then it is possible to choose $\rho = \rho(r)$ so that $\alpha_1(\rho) > \alpha_2(r)$. From conditions *ii*) and *iii*), all hypothesis of Lemma 2.2 are satisfied. Then,

$$V(t, x(t, t_0, x_0)) \le V(t_0^+, x_0),\ t \in [t_0, +\infty).$$

From *i*) and *ii*) and the above inequality, we obtain the following

$$\begin{aligned} \alpha_1(\|y(t,t_0,x_0\|) \le V(t, x(t,t_0,x_0)) &\le V(t_0^+, x_0), \\ &\le \alpha_2(\|w_0\|, \\ &\le \alpha_2(\|x_0\|), \\ &\le \alpha_2(r) < \alpha_1(\rho). \end{aligned}$$

Therefore, $\|y(t,t_0,x_0\|) \le \rho$, for $t \ge t_0$, which implies that the solutions of system (1) are $y-$UB. To complete the proof of this Theorem, we shall prove that the zero solution $x(t) \equiv 0$ of system (1) is $y-$UGA. Let $\nu > 0$ be arbitrarily chosen and $\epsilon > 0$ be given small. Let the number $\eta = \eta(\epsilon) > 0$ be chosen such that $\alpha_1(\epsilon) < \alpha_2(\eta)$ and let $\gamma = \gamma(\nu, \epsilon) > 0$ be satisfying the following

$$\gamma > \Big[\frac{\alpha_2(\nu)\Gamma(\beta+1)}{\alpha_3(\eta)}\Big]^{\frac{1}{\beta}}.$$

Similar to the proof of Theorem 3.3, we obtain $\|y(t,t_0,x_0)\| < \epsilon$ for $t \ge t_0 + \gamma$, whenever $\|x_0\| < \nu$. Then, the zero solution of (1) is $y-$UGA. □

Remark 3.1. *In Theorem 3.6, if $\alpha_1(s) = c_1 s^a$, $\alpha_2(s) = c_2 s^{ab}$ and $\alpha_3(s) = c_3 s^{ab}$ where c_1, c_2, c_3, a and b are positive constants, then the system (1) is $y-$globally Mittag-Laffler stable.*

§4 Examples

Example 1. *Consider the following fractional order system*

$$\begin{cases} {}^cD_{0,t}^\beta x_1(t) = -x_1(t) + \sin\big(x_1(t) + x_2(t) + x_3(t)\big)x_1(t),\ t \ge 0,\ t \ne \tau_k \\ {}^cD_{0,t}^\beta x_2(t) = -x_2(t) + \sin\big(x_1(t)) - x_2(t) + 2x_3(t)\big)x_2(t),\ t \ge 0,\ t \ne \tau_k \\ {}^cD_{0,t}^\beta x_3(t) = 2x_3(t),\ t \ge 0,\ t \ne \tau_k \\ x_1(\tau_k^+) = \frac{k}{\sqrt{2+k^2}} x_1(\tau_k),\ \tau_k > 0 \\ x_2(\tau_k^+) = \frac{k}{\sqrt{1+k^2}} x_2(\tau_k),\ \tau_k > 0 \\ x_3(\tau_k^+) = a_k,\ \tau_k > 0 \end{cases} \tag{8}$$

where $0 < \beta < 1$ *and* $x(t) = \big(x_1(t), x_2(t), x_3(t)\big) \in \mathbb{R}^3$. *Consider the Lyapunov-like function :*

$$V(t,x) = \frac{x_1^2 + x_2^2}{2}$$

for $t \geq 0$ *and* $t \neq \tau_k$, *we have*

$$\begin{aligned} {}^{c}\mathcal{D}_{+}^{\beta} V(t, x(t)) &= x_1(t)\,{}^{c}\mathcal{D}_{0,t}^{\beta} x_1(t) + x_2(t)\,{}^{c}\mathcal{D}_{0,t}^{\beta} x_2(t) \\ &= -x_1^2(t) + \sin\big(x_1(t) + x_2(t) + x_3(t)\big)x_1^2(t) - x_2^2(t) \\ &+ \sin\big(x_1(t)) - x_2(t) + 2x_3(t)\big)x_2^2(t) \\ &\leq 0. \end{aligned}$$

For $t = \tau_k$, $t > 0$,

$$V(\tau_k^+, x_1(\tau_k^+), x_2(\tau_k^+), x_3(\tau_k^+)) = \frac{k^2}{2+k^2}\frac{x_1^2}{2} + \frac{k^2}{1+k^2}\frac{x_2^2}{2} \leq V(\tau_k, x_1, x_2, x_3).$$

Then, the assumptions of Theorem 3.2 are satisfied. Hence, $x = 0$ *is* $(x_1, x_2)-$*USt.*

Remark 4.1. *In the previous example, a simple calculation leads to*

$$x_3(t, t_0, x_0) = x_{30} E_\beta\big(2(t - t_0)^\beta\big) + \sum_{i=1}^{k} a_i$$

where $x_{30} = x_3(t_0, t_0, x_0)$. *Then* $x = 0$ *of system (8) is unstable.*

Example 2. *Consider the following system*

$$\begin{cases} {}^{c}\mathcal{D}_{0,t}^{\beta} x_1(t) = -x_2(t)\cos^2(x_1(t)) - 4x_1(t) + x_2(t), \ t \geq 0, \ t \neq \tau_k \\ {}^{c}\mathcal{D}_{0,t}^{\beta} x_2(t) = x_2(t)\cos^2(x_1(t)) - 5x_2(t), \ t \geq 0, \ t \neq \tau_k \\ {}^{c}\mathcal{D}_{0,t}^{\beta} x_3(t) = x_3(t), \ t \geq 0, \ t \neq \tau_k \\ \Delta x_1(\tau_k) = c_k x_1(\tau_k), \ \tau_k > 0 \\ \Delta x_2(\tau_k) = d_k x_2(\tau_k), \ \tau_k > 0 \\ \Delta x_3(\tau_k) = a_k, \ \tau_k > 0 \end{cases} \tag{9}$$

where $\beta \in (0,1)$, $x_1, x_2, x_3 \in \mathbb{R}$, $c_k, d_k \in [-2, 0]$ *and* $a_k \neq 0$.
Define the function $V(t, x_1, x_2, x_3) = |x_1| + |x_2|$.*Then, for* $t \geq 0$ *and* $t \neq \tau_k$, *we have*

$$\begin{aligned} {}^{c}\mathcal{D}_{+}^{\beta} V(t, x(t)) &= sign x_1(t)\,{}^{c}\mathcal{D}_{0,t}^{\beta} x_1(t) + sign x_2(t)\,{}^{c}\mathcal{D}_{0,t}^{\beta} x_2(t) \\ &\leq -4|x_1(t)| - 4|x_2(t)| \\ &= -4V(t, x). \end{aligned}$$

For $t = \tau_k$, $t > 0$,

$$V(\tau_k^+, x_1 + c_k x_1, x_2 + d_k x_2, x_3 + a_k) = |1 + c_k||x_1| + |1 + d_k||x_2| \leq V(\tau_k, x_1, x_2, x_3).$$

Then, the assumptions of Theorem 3 are satisfied. Hence, $x = 0$ *is Mittag-Leffler stable with respect to* (x_1, x_2).

§5 Conclusion

In this paper, Lyapunov functions is used to study the stability with respect to part of the variables of the zero solution of a nonlinear impulsive fractional system. We introduce the derivative of the Lyapunov function based on the Caputo fractional Dini derivative of a function. By using the Lyapunov technique, some sufficient conditions for stability, uniform stability , Mittag Leffler stability and asymptotic uniform stability are obtained. Furthermore, the theoretical conclusions have been verified by some numerical examples.

Declarations

Conflict of interest The authors declare no conflict of interest.

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