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On new classes of strongly Janowski type functions of complex order

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Abstract. We investigate some new subclasses of analytic functions of Janowski type of complex order. We also study inclusion properties, distortion theorems, coefficient bounds and radius of convexity of the functions. Moreover, analytic properties of these classes under certain integral operator are also discussed. Our findings are more comprehensive than the existing results in the literature.

§1 Introduction and Preliminaries

Suppose $\mathcal{A}$ denotes the class of all analytic functions in the open unit disk $\mathrm{E} = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = 0$, and $f'(0) = 1$. Its series representation is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathrm{E}. \tag{1.1}$$

A function $g_1(z)$ analytic in E is called subordinate to a function $g_2(z)$, denoted as $g_1(z) \prec g_2(z)$, if there exists a Schwartz function $w(z)$ analytic in E with $w(0) = 0$ and $|w(z)| < 1$. Where $g_1(z) = g_2(w(z))$, for all $z \in \mathrm{E}$. Note that if $g_2(z)$ is univalent in E, then $g_1(z) \prec g_2(z)$ if and only if $g_1(0) = g_2(0)$, $g_1(\mathrm{E}) \subset g_2(\mathrm{E})$ implies $g_1(\mathrm{E}_r) \subset g_2(\mathrm{E}_r)$. Where $\mathrm{E}_r = \{z : |z| < r, \ 0 < r < 1\}$[4][5][11].

Let $\mathcal{P}$ denotes the well known class of Carathódory functions $p(z)$ such that $p(z)$ is analytic with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, for $z \in \mathrm{E}$. If $A, B \in \mathbb{R}$ with $-1 \leq B < A \leq 1$, then we say that a function $p(z)$ belongs to the class $\mathcal{P}[A, B]$ with $p(0) = 1$, if and only if $p(z) \prec \dfrac{1 + Az}{1 + Bz}$. Geometrically, $p \in \mathcal{P}[A, B]$ if and only if $p(0) = 1$ and $p(\mathrm{E})$ lies inside the open disk centered on the real axis and diameter with end points $\dfrac{1 - A}{1 - B}$ and $\dfrac{1 + A}{1 + B}$(for $B \neq -1$), or in the half plane,

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$\operatorname{Re} w > \frac{1-A}{2}$, if $B = -1$, respectively. Special choices of the parameters A and B lead to many familiar classes. For example $\mathcal{P}[1,-1] = \mathcal{P}$ [4][7]. Moreover, Noor [9] proved that $\mathcal{P}[A,B]$ is convex. We say that p belongs to the class $\widetilde{\mathcal{P}}^{\beta}[A,B]$ if and only if $p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta}$, for $z \in E$ and any arbitrary fixed numbers $-1 \leq B < A \leq 1$ and $0 < \beta \leq 1$. For $-1 \leq B < A \leq 1$, $\mathcal{C}[A,B]$ and $\mathcal{S}^*[A,B]$ are classes of Janowski convex and starlike functions respectively [7]. Moreover, $\widetilde{\mathcal{C}}^{\beta}[A,B]$ and $\mathcal{S}^{*^{\beta}}[A,B]$ with $-1 \leq B < A \leq 1$ and $0 < \beta \leq 1$ are defined by $\widetilde{\mathcal{C}}^{\beta}[A,B] = \left\{f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta}\right\}$ and $\mathcal{S}^{*^{\beta}}[A,B] = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta}\right\}$. Furthermore, classes of convex, starlike of complex order and Janowski convex and starlike functions of complex order b, ($b \neq 0$ is complex) with $-1 \leq B < A \leq 1$ are studied in [8][13][10]. Recently, in [6], some new classes of normalized analytic functions with bounded radius and bounded boundary rotation by using the subordination are introduced, discussed and explored. It is observed that most of the obtained results are best possible.

In [12], we have the following lemma:

Lemma 1. *Let* $f(z)$ *be defined as in* (1.1) *and* $F(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ $(z \in \mathrm{E})$, *are analytic functions such that* $f(z) \prec F(z)$. *If* $F(z)$ *is univalent in* E *and* $F(z)$ *is convex then* $|a_n| \leq |b_1|$, *for all* $n \geq 1$.

§2 Main Results

Before proceeding towards our main results, first we define the following:

Definition 1. *Let* $\widetilde{\mathcal{S}_b}^{\beta}[A,B]$ *denotes the family of functions* $f(z) \in \mathcal{A}$ *such that* $f(z)$ *belongs to* $\widetilde{\mathcal{S}_b}^{\beta}[A,B]$ *if and only if* $\frac{f(z)}{z} \neq 0$ *and*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \in \widetilde{\mathcal{P}}^{\beta}[A,B], (\textit{for complex number } b \neq 0).$$

Special Cases:

(i) For $\beta = 1$, $\widetilde{\mathcal{S}_b}^{\beta}[A,B]$ reduces to $\mathcal{S}_b^*[A,B]$.

(ii) For $A = 1$, $B = -1$, $\widetilde{\mathcal{S}_b}^{\beta}[A,B]$ becomes $\widetilde{\mathcal{S}_b}(\beta)$.

(iii) For $\beta = \frac{1}{2}$, $A = c$, $B = 0$, $\widetilde{\mathcal{S}_b}^{\beta}[A,B]$ reduces to $\mathcal{S}_b^{\frac{1}{2}}(c)$, for $b = 1$, it was introduced in [1].

(iv) For $b = 1$, $\widetilde{\mathcal{S}_b}^{\beta}[A,B]$ becomes $\mathcal{S}^{*^{\beta}}[A,B]$.

Definition 2. *Let* $\widetilde{\mathcal{C}_b}^{\beta}[A,B]$ *denotes the family of functions* $f(z) \in \mathcal{A}$ *such that* $f(z)$ *belongs to* $\widetilde{\mathcal{C}_b}^{\beta}[A,B]$ *if and only if* $\frac{f(z)}{z} \neq 0$ *and*

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \in \widetilde{\mathcal{P}}^{\beta}[A,B] (\text{where } b \neq 0 \text{ and is complex number })$$

Special Cases:

(i) For $\beta = 1$, $\widetilde{\mathcal{C}_b}^{\beta}[A,B]$ reduces to $\mathcal{C}_b[A,B]$.

(ii) For $A = 1$, $B = -1$, $\widetilde{\mathcal{C}_b}^{\beta}[A,B]$ becomes $\widetilde{\mathcal{C}}_b(\beta)$.

(iii) For $b = 1$, $\widetilde{\mathcal{C}_b}^{\beta}[A,B]$ becomes $\mathcal{C}^{*^{\beta}}[A,B]$.

Following result gives the coefficient bounds for the strongly Janowski functions in the class $\widetilde{\mathcal{P}}^{\beta}[A,B]$. By choosing special values of A, B and β, we obtain the coefficient bounds for the functions belong to the classes $p^{\sim}[\beta]$ and $P[A,B]$.

Lemma 2. *Suppose* $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ *is in* $\widetilde{\mathcal{P}}^{\beta}[A,B]$*, for* $-1 \le B < A \le 1$*, with* $0 < \beta \le 1$*. Then*

$$|c_n| \le \beta\,|A - B|\,, \text{ for all } n \ge 1. \tag{2.1}$$

Following lemma consists of distortion result for the functions belong to the class $\widetilde{\mathcal{P}}^{\beta}[A,B]$. Some existing results of well-known classes are special cases of this result.

Lemma 3. *Consider* $p(z)$ *belongs to* $\widetilde{\mathcal{P}}^{\beta}[A,B]$*, for* $-1 \le B < A \le 1$*, with* $0 < \beta \le 1$ *and* $z = re^{i\theta}$*. Then*

$$\left(\frac{1-Ar}{1-Br}\right)^{\beta} \le \operatorname{Re} p(z) \le |p(z)| \le \left(\frac{1+Ar}{1+Br}\right)^{\beta}. \tag{2.2}$$

Theorem 4. *Suppose* $f(z) \in \widetilde{\mathcal{C}_b}^{\beta}[A,B]$*. Then* $f(z) \in \mathcal{C}_b(\gamma)$*. Where* $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$*, for* $-1 \le B < A \le 1$ *and* $0 < \beta \le 1$*, with* $b \ne 0$*.*

Proof. Suppose $f(z) \in \widetilde{\mathcal{C}_b}^{\beta}[A,B]$. Then by definition

$$1 + \frac{1}{b}\frac{zf^{''}(z)}{f'(z)} \in \widetilde{\mathcal{P}}^{\beta}[A,B].$$

If $p(z) \in \widetilde{\mathcal{P}}^{\beta}[A,B]$ then $p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta}$. This follows that there exists an analytic function $w(z)$, $w(0) = 1$ with $|w(z)| < 1$ such that $p(z) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\beta}$.

Now

$$\operatorname{Re}(p(z)) = \operatorname{Re}\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\beta} \ge \left(\frac{1-A\,|w(z)|}{1-B\,|w(z)|}\right)^{\beta} > \left(\frac{1-A}{1-B}\right)^{\beta}. \tag{2.3}$$

This shows that $p(z) \in \mathcal{P}(\gamma)$, where $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$. It follows that $1 + \frac{1}{b}\frac{zf^{''}(z)}{f'(z)} \in \mathcal{P}(\gamma)$, where $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$. This implies that $f(z) \in \mathcal{C}_b(\gamma)$ with $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$. This completes the proof. □

Corollary 5. *For* $\beta = 1$, $f(z) \in \mathcal{C}_b[A,B]$*. Above Theorem 4 gives that* $f(z) \in \mathcal{C}_b(\gamma)$*, for* $\gamma = \frac{1-A}{1-B}$.

Corollary 6. *Put* $\beta = \frac{1}{2}$, $A = c$, $c \in (0,1]$, $B = 0$*. We get* $f(z) \in \mathcal{C}_b^{\frac{1}{2}}(c)$*, where* $\mathcal{C}_b^{\frac{1}{2}}(c) = \left\{ f \in A : \ 1 + \frac{1}{b}\frac{zf^{''}(z)}{f'(z)} \prec \sqrt{1+cz} \right\}$.

Using Theorem 4, we have $f(z) \in \mathcal{C}_b(\gamma)$, where $\gamma = \sqrt{1-c}$.

Corollary 7. *Put $b = 1$ in Theorem 4, we have $\widetilde{\mathcal{C}}^{\beta}[A, B] \subset \mathcal{C}(\gamma)$, for $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$.*

Theorem 8. *If $f(z) \in \widetilde{\mathcal{S}_b}^{\beta}[A, B]$ then $f(z) \in \mathcal{S}_b(\gamma)$. Where $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$, for $-1 \leq B < A \leq 1$ and $0 < \beta \leq 1$, $(b \neq 0)$.*

Proof. Suppose $f(z) \in \widetilde{\mathcal{S}_b}^{\beta}[A, B]$. Then by definition, we get

$$1 + \frac{1}{b}\left(\frac{zf^{'}(z)}{f(z)} - 1\right) \in \widetilde{\mathcal{P}}^{\beta}[A, B].$$

If $p(z) \in \widetilde{\mathcal{P}}^{\beta}[A, B]$. Then by (2.3), we get $p(z) \in \mathcal{P}(\gamma)$, where $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$. This implies that $1 + \frac{1}{b}\left(\frac{zf^{'}(z)}{f(z)} - 1\right) \in p(\gamma)$, where $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$.

Therefore, $f(z) \in \mathcal{S}_b(\gamma)$, for $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$. Hence the proof. □

Corollary 9. *For $\beta = 1$, Theorem 8 gives $\mathcal{S}_b^*[A, B] \subset \mathcal{S}_b^*(\gamma)$, for $\gamma = \frac{1-A}{1-B}$.*

Corollary 10. *If $\beta = \frac{1}{2}$, $A = c$, $c \in (0, 1]$, $B = 0$, then $f(z) \in \mathcal{S}_b^{\frac{1}{2}}(c)$, where $\mathcal{S}_b^{\frac{1}{2}}(c) = \left\{f \in A : 1 + \frac{1}{b}\left(\frac{zf^{'}(z)}{f(z)} - 1\right) \prec \sqrt{1+cz}\right\}$. Theorem 8 implies that $f(z) \in \mathcal{C}_b(\gamma)$, for $\gamma = \sqrt{1-c}$ [1].*

Corollary 11. *For $b = 1$, Theorem 8 follows that $\widetilde{\mathcal{S}}^{\beta}[A, B] \subset \mathcal{S}(\gamma)$, for $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$.*

Theorem 12. *Let $f(z) \in \widetilde{\mathcal{S}_b}^{\beta}[A, B]$ with $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, for $z \in \mathrm{E}, -1 \leq B < A \leq 1, 0 < \beta \leq 1$ and $b \neq o$ (complex). Then*

$$|a_n| \leq \frac{(b\beta\,|A - B|)_{n-1}}{(n-1)!}, \text{ for } n \geq 2. \tag{2.4}$$

Proof. Suppose $f(z) \in \widetilde{\mathcal{S}_b}^{\beta}[A, B]$. Then by definition, we have $1 + \frac{1}{b}\left(\frac{zf^{'}(z)}{f(z)} - 1\right) \in \widetilde{\mathcal{P}_b}^{\beta}[A, B]$.

Consider $p(z) = 1 + \frac{1}{b}\left(\frac{zf^{'}(z)}{f(z)} - 1\right) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$, where $p(z)$ is analytic in E with $p(0) = 1$. This follows that

$$\begin{aligned}
1 + \frac{1}{b}\left(\frac{zf^{'}(z)}{f(z)} - 1\right) &= 1 + \sum_{n=1}^{\infty} c_n z^n \\
zf^{'}(z) &= f(z)\left(1 + \sum_{n=1}^{\infty} bc_n z^n\right) \\
z + \sum_{n=2}^{\infty} na_n z^n &= \left(z + \sum_{n=2}^{\infty} a_n z^n\right)\left(1 + \sum_{n=1}^{\infty} bc_n z^n\right) \\
&= z + \sum_{n=2}^{\infty}\sum_{i=0}^{n-1} bc_i a_{n-i} z^n.
\end{aligned}$$

Comparing the coefficient of 'z^n' on both sides, we obtain

$$(n-1)\,a_n = \sum_{i=1}^{n-1} bc_i a_{n-i}, \quad c_0 = 1.$$

This implies that

$$|a_n| = \frac{1}{n-1}\sum_{i=1}^{n-1} |b|\,|c_i|\,|a_{n-i}|\,. \tag{2.5}$$

Since $p(z) \in \widetilde{\mathcal{P}_b}^{\beta}[A,B]$, then by Lemma 2, we get $|c_i| \le b\beta(A-B)$, for all $n \ge 1$. From (2.5), we have $|a_n| \le \frac{b\beta(A-B)}{n-1}\sum_{i=1}^{n-1}|a_i|$. Particularly,

$$|a_2| \le b\beta(A-B),$$

$$|a_3| \le \frac{b\beta(A-B)}{2}\left(b\beta(A-B)+1\right) = \frac{(b\beta(A-B))_2}{(2)!} \quad .$$

For $n = k$,

$$|a_k| \le \frac{b\beta(A-B)}{k-1}\prod_{j=1}^{k-2}\left(\frac{b\beta(A-B)}{j}+1\right) = \frac{(b\beta(A-B))_{k-1}}{(k-1)!}, \quad k \ge 3. \tag{2.6}$$

Where $(d)_k$ is Pochhammer notation represented as,

$$(d)_k = \begin{cases} 1, & k = 0, d \in C \setminus \{0\} \\ d(d+1)...(d+k-1), & k \in N, d \in C. \end{cases}$$

Consider

$$|a_{k+1}| \le \frac{b\beta(A-B)}{k}\sum_{i=1}^{k-1}|a_i| + \frac{b\beta(A-B)}{k}|a_k| \quad .$$

Using (2.6), we obtain the following,

$$|a_{k+1}| \le \frac{b\beta(A-B)}{k}\prod_{j=1}^{k-2}\left(\frac{b\beta(A-B)}{j}+1\right) + \frac{(b\beta(A-B))^2}{k\,(k-1)}\prod_{j=1}^{k-2}\left(\frac{b\beta(A-B)}{j}+1\right),$$

$$= \frac{b\beta(A-B)}{k}\prod_{j=1}^{k-1}\left(\frac{b\beta(A-B)}{j}+1\right) = \frac{(b\beta(A-B))_k}{(k)!.} \quad .$$

By induction, we get (2.4). This completes the proof. □

Corollary 13. *If we take* $A = 1, B = -1, b = 1,\ f \in \widetilde{\mathcal{S}}^{\beta}[1,-1] = \widetilde{\mathcal{S}}(\beta)$, *then we have*

$$|a_n| \le \frac{(2\beta)_{n-1}}{(n-1)!}, \text{ for } n \ge 2.$$

The bound is sharp for the function $f_\beta(z) = \frac{z}{(1-z)^{2\beta}}$ [2].

Note that by putting $\beta = 1$ in above Corollary 13, we obtain the coefficients bounds $|a_n| \le n$, $n \ge 2$ for the function $f \in \mathcal{S}^*$ with extremal function $f_1(z) = \frac{z}{(1-z)^2}$ [4].

Corollary 14. *If* $A = 1, B = -1$, *then* $f \in \widetilde{\mathcal{S}}_b(\beta)$. *Further, (2.4) reduces to* $|a_n| \le \frac{(2b\beta)_{n-1}}{(n-1)!}$, *for* $n \ge 2$.

Corollary 15. *If* $\beta = 1, b = 1$, *then* $f \in \mathcal{S}^*[A,B]$ *and Theorem 12 follows that*

$$|a_n| \le \frac{(A-B)_{n-1}}{(n-1)!}, \text{ for } n \ge 2.$$

Corollary 16. *Put $A = 1 - 2\alpha$, $B = -1$ in above Corollary 15, we achieve the coefficient bounds $|a_n| \leq \frac{(2(1-\alpha))_{n-1}}{(n-1)!}$, for $n \geq 2$ and $f \in \mathcal{S}^*(\alpha)$, for $0 \leq \alpha < 1$ with extremal function $f_1(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$ [4][3].*

Theorem 17. *If $f(z) \in \widetilde{\mathcal{C}_b}^{\beta}[A, B]$ with $f(z)$ defined in (1.1), for $-1 \leq B < A \leq 1$, $0 < \beta \leq 1$ and $b \neq 0$ (complex), then*

$$|a_n| \leq \frac{(b\beta\,|A - B|)_{n-1}}{(n)!}, \text{ for } n \geq 2.$$

Proof. Suppose $f(z) \in \widetilde{\mathcal{C}_b}^{\beta}[A, B]$. Then $zf^{'}(z) \in \widetilde{\mathcal{S}_b}^{\beta}[A, B]$. By Theorem 12, we get

$$|a_n| \leq \frac{(b\beta\,|A - B|)_{n-1}}{(n)!}, \text{ for } n \geq 2.$$

Hence the proof. □

Corollary 18. *For $b = 1$ and $f(z) \in \widetilde{\mathcal{C}}^{\beta}[A, B]$, Theorem 17 gives $|a_n| \leq \frac{(\beta|A-B|)_{n-1}}{(n)!}$, for $n \geq 2$.*

Corollary 19. *For $\beta = 1$ and $f(z) \in \mathcal{C}_b[A, B]$, Theorem 17 gives $|a_n| \leq \frac{(b|A-B|)_{n-1}}{(n)!}$, for $n \geq 2$.*

Corollary 20. *For $A = 1, B = -1$ and $f(z) \in \widetilde{\mathcal{C}}_b^{\beta}[1, -1] = \widetilde{\mathcal{C}}_b(\beta)$, Theorem 17 gives $|a_n| \leq \frac{(2b\beta)_{n-1}}{(n)!}$, for $n \geq 2$.*

Theorem 21. *Suppose $f, g \in \widetilde{\mathcal{C}_b}^{\beta}[A, B]$ and $H(z) = \int_0^z \left(f^{'}(t)\right)^{\alpha} \left(g^{'}(t)\right)^{\gamma} dt$ with $\alpha + \gamma = 1$. Then $H(z) \in \widetilde{\mathcal{C}_b}^{\beta}[A, B]$.*

Proof. Since

$$H(z) = \int_0^z \left(f^{'}(t)\right)^{\alpha} \left(g^{'}(t)\right)^{\gamma} dt, \text{with } \alpha + \gamma = 1,$$

$$\Longrightarrow H^{'}(z) = \left(f^{'}(z)\right)^{\alpha} \left(g^{'}(z)\right)^{\gamma}.$$

Differentiating logarithmatically, we have

$$\frac{H^{''}(z)}{H'(z)} = \alpha\frac{f^{''}(z)}{f'(z)} + \gamma\frac{g^{''}(z)}{g'(z)}$$

$$1 + \frac{1}{b}\frac{zH^{''}(z)}{H'(z)} = \alpha\left(1 + \frac{1}{b}\frac{zf^{''}(z)}{f'(z)}\right) + \gamma\left(1 + \frac{1}{b}\frac{zg^{''}(z)}{g'(z)}\right). \tag{2.7}$$

Since $f, g \in \widetilde{\mathcal{C}_b}^{\beta}[A, B]$, then there exists $p_1, p_2 \in \widetilde{\mathcal{P}}^{\beta}[A, B]$ such that $p_1 = 1 + \frac{1}{b}\frac{zf^{''}(z)}{f'(z)}$ and $p_2 = 1 + \frac{1}{b}\frac{zg^{''}(z)}{g'(z)}$. Then (2.7) becomes, $1 + \frac{1}{b}\frac{zH^{''}(z)}{H'(z)} = \alpha p_1 + \gamma\, p_2$. Since $p_1, p_2 \in \widetilde{\mathcal{P}}^{\beta}[A, B]$ and $\widetilde{\mathcal{P}}^{\beta}[A, B]$ is convex, then $1 + \frac{1}{b}\frac{zH^{''}(z)}{H'(z)} \in \widetilde{\mathcal{P}}^{\beta}[A, B]$. This follows that $H(z) \in \widetilde{\mathcal{C}_b}^{\beta}[A, B]$. This completes the proof. □

Corollary 22. *For $\beta = 1$ and $f, g \in \mathcal{C}_b[A, B]$, Theorem 21 gives $H(z) \in \mathcal{C}_b[A, B]$.*

Corollary 23. *For $A = 1,\ B = -1,\ b = 1$ and $f, g \in \widetilde{\mathcal{C}}(\beta)$, Theorem 21 gives $H(z) \in \widetilde{\mathcal{C}}(\beta)$.*

Corollary 24. *For $\beta = 1,\ A = 1 - 2\alpha,\ B = -1$ and $f, g \in \mathcal{C}_b(\alpha)$, Theorem 21 follows, $H(z) \in \mathcal{C}_b(\alpha)$.*

Theorem 25. *Suppose $f \in \widetilde{\mathcal{S}_b}^{\beta}[A, B]$ with $z = re^{i\theta}$ and $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$. Then we have the following inequalities:*

$$\frac{r}{(1+r)^{2b(1-\gamma)}} \leq |f(z)| \leq \frac{r}{(1-r)^{2b(1-\gamma)}} \quad . \tag{2.8}$$

$$\frac{1 - 2b(1-\gamma)r + (2b(1-\gamma) - 1)\, r^2}{(1-r)\,(1+r)^{2b(1-\gamma)+1}} \leq \left|f^{'}(z)\right| \leq \frac{1 + 2b(1-\gamma)r + (2b(1-\gamma) - 1)\, r^2}{(1+r)\,(1-r)^{2b(1-\gamma)+1}}. \tag{2.9}$$

Proof. Since $f \in \widetilde{\mathcal{S}_b}^{\beta}[A, B]$, then $1 + \frac{1}{b}\left(\frac{zf^{'}(z)}{f(z)} - 1\right) = p(z)$. Where $p(z) \in \widetilde{\mathcal{P}}^{\beta}[A, B] \subseteq p(\gamma)$, $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$. Also $p(z) \in \mathcal{P}(\gamma)$ implies that there exists $p_1 \in \mathcal{P}$ such that $p(z) = (1-\gamma)p_1 + \gamma$.

We know that each $p_1 \in \mathcal{P}$ can be written as

$$\left|p_1 - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2} \quad (\,[5]).$$

It follows that

$$\left|\frac{1}{1-\gamma}(p(z) - \gamma) - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}$$

$$\left|p(z) - \frac{1 + (1-2\gamma)r^2}{1-r^2}\right| \leq \frac{2r(1-\gamma)}{1-r^2} \quad . \tag{2.10}$$

$$\left|1 + \frac{1}{b}\left(\frac{zf^{'}(z)}{f(z)} - 1\right) - \frac{1 + (1-2\gamma)r^2}{1-r^2}\right| \leq \frac{2r(1-\gamma)}{1-r^2}$$

$$\left|\frac{zf^{'}(z)}{f(z)} - \frac{1 + (2b(1-\gamma) - 1)\, r^2}{1-r^2}\right| \leq \frac{2b(1-\gamma)r}{1-r^2}. \tag{2.11}$$

This inequality can be written in the following form

$$\frac{1 - 2b(1-\gamma)r + (2b(1-\gamma) - 1)\, r^2}{1-r^2} \leq \mathrm{Re}\{\frac{zf^{'}(z)}{f(z)}\} \leq \frac{1 + 2b(1-\gamma)r + (2b(1-\gamma) - 1)\, r^2}{1-r^2}. \tag{2.12}$$

On the other hand, we know that

$$\mathrm{Re}\{\frac{zf^{'}(z)}{f(z)}\} = r\frac{\partial}{\partial r}\log|f(z)|\,.$$

Thus we have,

$$\frac{1 - 2b(1-\gamma)r + (2b(1-\gamma) - 1)\, r^2}{r(1-r^2)} \leq \frac{\partial}{\partial r}\log|f(z)| \leq \frac{1 + 2b(1-\gamma)r + (2b(1-\gamma) - 1)\, r^2}{r(1-r^2)}. \tag{2.13}$$

Integrating both sides of (2.12), we get (2.8). The inequality (2.10) can be written in the form

$$\frac{1 - 2b(1-\gamma)r + (2b(1-\gamma) - 1)\, r^2}{1-r^2} \leq \left|\frac{zf^{'}(z)}{f(z)}\right| \leq \frac{1 + 2b(1-\gamma)r + (2b(1-\gamma) - 1)\, r^2}{1-r^2}.$$

In the above inequality, if we use (2.8), we obtain (2.9). Hence the proof. □

Theorem 26. *Suppose* $f \in \widetilde{\mathcal{C}_b}^{\beta}[A,B]$ *with* $z = re^{i\theta}$ *and* $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$. *Then*

$$\frac{1}{(1+r)^{2b(1-\gamma)}} \leq \left|f^{'}(z)\right| \leq \frac{1}{(1-r)^{2b(1-\gamma)}}. \tag{2.14}$$

$$\frac{1}{1-2b(1-\gamma)}\left(\frac{1}{(1+r)^{2b(1-\gamma)-1}}-1\right) \leq |f(z)| \leq \frac{1}{1-2b(1-\gamma)}\left(1-\frac{1}{(1-r)^{2b(1-\gamma)-1}}\right). \tag{2.15}$$

Proof. Suppose $f \in \widetilde{\mathcal{C}_b}^{\beta}[A,B]$, then $1+\frac{1}{b}\left(\frac{zf^{''}(z)}{f^{'}(z)}\right) = p(z)$. Where $p(z) \in \widetilde{\mathcal{P}}^{\beta}[A,B] \subseteq p(\gamma)$, $\gamma = \left(\frac{1-A}{1-B}\right)^{\beta}$. Using (2.10), we have

$$\left|p(z) - \frac{1+(1-2\gamma)r^2}{1-r^2}\right| \leq \frac{2r(1-\gamma)}{1-r^2}$$

$$\left|1+\frac{1}{b}\left(\frac{zf^{''}(z)}{f^{'}(z)}\right) - \frac{1+(1-2\gamma)r^2}{1-r^2}\right| \leq \frac{2r(1-\gamma)}{1-r^2}$$

$$\left|\frac{zf^{''}(z)}{f^{'}(z)} - \frac{2b(1-\gamma)r^2}{1-r^2}\right| \leq \frac{2b(1-\gamma)r}{1-r^2}.$$

This inequality can be written as

$$\frac{2b(\gamma-1)r}{1+r} \leq \operatorname{Re}\frac{zf^{''}(z)}{f^{'}(z)} \leq \frac{2b(1-\gamma)}{1-r}. \tag{2.16}$$

But we know that $\operatorname{Re}\{\frac{zf^{''}(z)}{f^{'}(z)}\} = r\frac{\partial}{\partial r}\log\left|f^{'}(z)\right|$. Using this result in (2.16) and integrating we get (2.14). Since $|z| = r$, integrating (2.14) over r. Therefore we obtain (2.15). This completes the proof. □

Theorem 27. *Suppose* $f(z) \in \widetilde{\mathcal{C}_b}^{\beta}[A,B]$. *Then* $f(z)$ *map* $|z| < \sigma$ *on to a convex domain. Where* $\sigma = (1-B)^{\beta}\left(b\left(\frac{1-A}{1-B}\right)^{\beta} + (1-b)\right)$, *for* $-1 \leq B < A \leq 1$, $b \neq 0$, $0 < \beta \leq 1$.

Proof. Suppose $f(z) \in \widetilde{\mathcal{C}_b}^{\beta}[A,B]$. Then for any $p(z) \in \widetilde{\mathcal{P}}^{\beta}[A,B]$, we have

$$1+\frac{1}{b}\frac{zf^{''}(z)}{f'(z)} = p(z),$$

$$1+\frac{zf^{''}(z)}{f'(z)} = bp(z)+1-b.$$

$$\operatorname{Re}\left(1+\frac{zf^{''}(z)}{f'(z)}\right) = \operatorname{Re}\left(bp(z)+1-b\right)$$

Since $p(z) \in \widetilde{\mathcal{P}}^{\beta}[A,B]$, then by using Lemma 3 we have

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq b \left(\frac{1-Ar}{1-Br}\right)^{\beta} + 1 - b$$
$$= \frac{b\,(1-Ar)^{\beta} + (1-b)\,(1-Br)^{\beta}}{(1-Br)^{\beta}}.$$

Suppose $h(r) = b\,(1-Ar)^{\beta} + (1-b)\,(1-Br)^{\beta}$. Then $h(0)=1$, and $h(1) = b\,(1-A)^{\beta} + (1-b)\,(1-B)^{\beta} = (1-B)^{\beta}\left(b\left(\frac{1-A}{1-B}\right)^{\beta} + (1-b)\right)$.

So,

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \text{ for } |z| < \sigma.$$

Where $\sigma = (1-B)^{\beta}\left(b\left(\frac{1-A}{1-B}\right)^{\beta} + (1-b)\right)$ is the least positive root of equation $b\,(1-A)^{\beta} + (1-b)\,(1-B)^{\beta} = 0$. Hence the proof. □

Corollary 28. *If $\beta = 1,\ f \in \widetilde{\mathcal{C}}_b[A,B]$, then Theorem 27 follows that*

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \text{ for } |z| < \frac{1}{B+b(A-B)}.$$

Corollary 29. *If $\beta = 1,\ A = 1,\ B = -1,\ f \in \widetilde{\mathcal{C}}_b[1,-1] = \mathcal{C}_b$, then Theorem 27 gives*

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \text{ for } |z| < \frac{1}{2b-1}.$$

Corollary 30. *By putting $\beta = 1,\ A = 1-2\alpha,\ B = -1,\ f \in \mathcal{C}_b[1-2\alpha,-1] = \mathcal{C}_b(\alpha)$, Theorem 27 follows that*

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \text{ for } |z| < \frac{1}{2b(1-\alpha)-1}.$$

Corollary 31. *If we put $A = 1,\ B = -1,\ f \in \widetilde{\mathcal{C}}_b^{\beta}[1,-1] = \widetilde{\mathcal{C}}_b(\beta)$. Then Theorem 27 gives*

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \text{ for } |z| < \sigma.$$

Where σ is the least positive root of equation $b\,(1-r)^{\beta} + (1-b)\,(1+r)^{\beta} = 0$.

§3 Conclusion

In this paper we defined a new classes of strongly Janowski type functions of complex order. We also derived growth and distortion theorems, coefficient bounds and radius of convexity of the functions. Moreover, we discussed analytic properties of these classes under certain integral operator.

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