Appl. Math. J. Chinese Univ.
2023, 38(1): 78-88

Stabilization of fractional bilinear systems with multiple inputs

Thouraya Kharrat   Fehmi Mabrouk   Fawzi Omri

Abstract. In this paper, we study in a constructive way the stabilization problem of fractional bilinear systems with multiple inputs. Using the quadratic Lyapunov functions and some additional hypotheses on the unit sphere, we construct stabilizing feedback laws for the considered fractional bilinear system. A numerical example is given to illustrate the efficiency of the obtained result.

§1 Introduction

The history of fractional systems is more than three centuries old, yet it only receives much attention and interest in the past 20 years, the reader may refer to [6,15] for the theory and applications of fractional calculus. The earliest more or less systematic studies seem to have been made in the 19$^{\text{th}}$ century by Liouville, Riemann, Leibniz,... [14,16].

The Stability analysis of nonlinear systems attracts the attention of many researchers [1,17]. Recently, the stability analysis of fractional systems is more developed. As in classical calculus, stability analysis is a central task in the study of fractional differential systems and fractional control [12,21]. For the stability of nonlinear classical ordinary differential equations, the researches are in general based on Lyapunov theory, see for instance [4]. Following Lyapunov's seminal 1892 thesis, these two methods are expected to also work for fractional differential equations.

Lyapunov's first method: the method of linearization of the nonlinear equation along its solutions. The asymptotic stability of the linearized system implies the local asymptotic stability of the initial system.

Lyapunov's second method: the method of Lyapunov candidate functions, i.e. there exists a positive definite scalar function such that the derivative of this function decreases along the orbits of the system.

Received: 2019-11-30.   Revised: 2020-05-01.
MR Subject Classification: 93D15.
Keywords: stabilization, fractional system, bilinear system, multiple inputs, Lyapunov functions.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-023-3976-5.

A similar tool is developed for the stability study of fractional systems. There has been many researches on Lypunov's second method for fractional differential equations [9] or [10]. The relation between the Lyapunov function and the fractional differential equation is not elementary nor simple [7,8] . In [2], the author proposes some other Lyapunov functionals, where the relation between them and the fractional differential system is more elementary, but these functionals are neither simple, and they are valid for fractional systems with specific characteristics. In nonlinear systems, only Lyapunov's direct method (also called the second method of Lyapunov) provides a way to analyze the stability of a system without explicitly solving the differential equation. This method generalizes the idea which shows that the system is asymptotically stable if there exists some Lyapunov function for the system. The Lyapunov direct method is a sufficient tool to show the stability of a nonlinear system, which means the system may be stable even one cannot find a Lyapunov function to conclude the system stability property.

The problems of stability and stabilization of fractional-order systems have also great attractions due to the inherent memory advantage of fractional derivatives. In [11], the authors provided a method for the asymptotic stabilization of fractional-order linear systems with saturation nonlinearity. In [18], Shahri and al. proposed a new stability condition for estimating the domain of attraction via ellipsoid approach based on saturation functions. In [19], Esmat and al. study the stability and the stabilization for a class of uncertain fractional order (FO) systems subject to input saturation. The authors investigate the problem of the robust stability of saturation control. In [20], the authors used the Lyapunov approach for the study of uncertain FO system stability analysis. To the best of our knowledge, the researches on the stability and stabilization of the fractional-order systems using the Lyapunov approach are not abundant enough.

In this paper, we will study in a constructive way the stabilization of the following fractional bilinear system with multiple inputs:

$$^{C}D_{t_0}^{\alpha}x(t) = Ax(t) + \sum_{i=1}^{p} u_i B_i x(t), \quad t \in \mathbb{R} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $\forall i \in \{1, ..., p\}$, $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times n}$. We will show that the above fractional bilinear system can be made globally asymptotically stable by means of homogeneous feedback functions of degree zero.

The paper is organized as follows. In section 2, some basic notations and preliminaries are given. The stabilization problem and the construction of feedback functions, which make the fractional bilinear systems globally asymptotically stable, are presented in section 3. In section 4, two examples are presented to illustrate the results. Conclusion is given in section 5.

§2 Notations and preliminaries

We start by introducing some notations that will be useful throughout the paper.

Notation:

$\mathbb{R}^n$: the real n-dimensional vector space.
$\mathbb{R}^{n\times n}$: the set of all $n \times n$ real matrices.
$\langle .,. \rangle$: the usual inner product on $\mathbb{R}^n$.
$\|x\|$: the norm of the vector x that belongs to $\mathbb{R}^n$, i.e. $\sqrt{\langle x,x\rangle} = \|x\|$.
$\mathbf{S}^{n-1}$: the unit sphere, i.e. $\mathbf{S}^{n-1} = \{x \in \mathbb{R}^n,\ \|x\|^2 = 1\}$.
Let $A \in \mathbb{R}^{n\times n}$:
the matrix A is positive semi-definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$;
A is negative semi definite if $-A$ is positive semi definite;
A^T denotes the transpose of the matrix A;
A is symmetric if $A^T = A$;
A_s: the symmetric part of A where A is any square matrix, i.e. $A_s = \frac{A+A^T}{2}$.
$\lambda(A)$ denotes the set of all the eigenvalues of A;
$\lambda_{max}(A) = max\{Re(\lambda) : \lambda \in \lambda(A)\}$, $\lambda_{min}(A) = \min\{Re(\lambda) : \lambda \in \lambda(A)\}$.

In the following, we recall some classical definitions and results which will play important roles in our study.

Definition 1 (Caputo fractional derivative [5]). *Let $k \in \mathbb{N}^*$ and $k-1 \leq \alpha < k$, the Caputo fractional derivative of a function x of order $\alpha > 0$ is defined as*

$$^{C}D^{\alpha}_{t_0}x(t) = \frac{1}{\Gamma(k-\alpha)}\int_{t_0}^{t}(t-s)^{k-\alpha-1}x^{(k)}(s)ds. \tag{2}$$

Let the system described by

$$^{C}D^{\alpha}_{t_0}x(t) = f(t,x), \tag{3}$$

where the map $f : \mathbb{R}\times U \to \mathbb{R}^n$ is continuous locally Lipschitz, $f(t,0) = 0$, $\forall\ t \geq 0$ and U is an open set of $\mathbb{R}^n$. Denote $x(t,t_0)$ the solution of (3) starting at x_0 at time t_0.

Definition 2. *The equilibrium point $x = 0$ of the system (3) is said to be:*

i) *stable if*

$$\forall\varepsilon > 0, \forall t_0 \geq 0, \exists\delta = \delta(t_0,\varepsilon) > 0, such\ \ that \parallel x_0 \parallel < \delta \Longrightarrow \parallel x(t,t_0) \parallel < \varepsilon,\ \forall t \geq t_0$$

ii) *attractive if there exists a neighborhood $\mathcal{V}$ of 0 such that for any initial condition x_0 belonging to $\mathcal{V}$, the corresponding solution $x(t,t_0)$ is defined for all $t \geq t_0$ and $\lim_{t\to+\infty} x(t,t_0) = 0$. If $\mathcal{V} = \mathbb{R}^n$, $x = 0$ is globally attractive.*

iii) *asymptotically stable if it is stable and attractive.*

iv) *globally asymptotically stable (GAS) if it is stable and globally attractive.*

Definition 3. *Let us consider the following control system :*

$$\begin{cases} ^{C}D^{\alpha}_{t_0}x = X(x,u) \\ x \in U, u \in \mathcal{U} \end{cases} \tag{4}$$

where U is an open set of $\mathbb{R}^n$,$\mathcal{U} \subset \mathbb{R}^m$, x is called the state of (4), u is called the control and $X : U \times \mathcal{U} \longrightarrow \mathbb{R}^n$ is a smooth function satisfying $X(0,0) = 0$.
We say that the system (4) is stabilizable (respectively globally stabilizable), if there exists a feedback function $u = u(x)$ such that the vector field $X(x, u(x))$ is at least continuous and the closed-loop system:

$$^{C}D^{\alpha}_{t_0}x = X(x, u(x))$$

admits the origin as an asymptotically stable equilibrium point (respectively globally asymptotically stable).

Definition 4. *[4] A continuous function $\gamma : [0, t) \longrightarrow [0, +\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$.*

Lemma 1. *[22] Let $V : D \to \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $B_d = \{x \in \mathbb{R}^n : \ \|x\| < d\} \subset D$ for some $d > 0$. Then there exist class $\mathcal{K}$ functions λ_1 and λ_2 defined on $[0, d)$, such that*

$$\lambda_1(\|x\|) \leq V(x) \leq \lambda_2(\|x\|), \tag{5}$$

for all $x \in B_d$. If $D = \mathbb{R}^n$, the functions λ_1 and λ_2 are defined on $[0, \infty)$.

Theorem 1 (Fractional-order extension of Lyapunov direct method [10])**.**
Let $x = 0$ be the equilibrium point of the fractional-order system (3). Assume that there exists a fractional Lyapunov function $V(t, x(t)) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and class $\mathcal{K}$ functions λ_i, $i = 1, 2, 3$ satisfying:

(i) $\lambda_1(\|x\|) \leq V(t, x(t)) \leq \lambda_2(\|x\|)$,

(ii) $^{C}D^{\alpha}_{t_0}V(t, x(t)) \leq -\lambda_3(\|x\|)$.

Then the fractional-order system (3) is asymptotically stable.
Moreover, if $U = \mathbb{R}^n$, then the fractional-order system (3) is globally asymptotically stable.

Lemma 2. *[13] Let $x(t) \in \mathbb{R}$ be a real continuous and differentiable function. Then, for any time $t \geq t_0$,*

$$\frac{1}{2}{}^{C}D^{\alpha}_{t_0}x^2(t) \leq x(t)^{C}D^{\alpha}_{t_0}x(t), \ \textit{for all}\ 0 < \alpha < 1. \tag{6}$$

Remark 1. *[3] In the case when $x(t) \in \mathbb{R}^n$, lemma (2) is still valid. That is, $\alpha \in (0, 1)$ and $t \geq t_0$,*

$$\frac{1}{2}{}^{C}D^{\alpha}_{t_0}x^T(t)x(t) \leq x^T(t)^{C}D^{\alpha}_{t_0}x(t).$$

In addition, let $x(t) \in \mathbb{R}$ be a real continuous and differentiable function. Then, for any $p = 2n$, $n \in \mathbb{N}$, we have

$$^{C}D^{\alpha}_{t_0}x^p \leq p\, x^{p-1}\, {}^{C}D^{\alpha}_{t_0}x(t),$$

where $0 < \alpha < 1$.

§3 Stabilization via feedback laws

In this section, we give an explicit design of the stabilizing feedbacks and we present sufficient conditions for the stabilization of bilinear systems.

3.1 Stabilization of driftless bilinear systems

We consider the system (1) in the case where $A = 0$. This system is called without drift and can be written as:

$$^{C}D_{t_0}^{\alpha}x(t) = \sum_{i=1}^{p} u_i B_i x(t), \tag{7}$$

where $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $i \in \{1, 2, ..., p\}$, and $B_i \in \mathbb{R}^{n\times n}$. Denote

$$\mathbf{S}_i = \{x \in \mathbb{R}^n : \langle B_i x, x\rangle = 0\},\ i \in \{1, 2, ..., p\}.$$

In the sequel, we introduce the following condition:

$$(\mathbf{H}) : \bigcap_{i=1}^{p} \mathbf{S}_i = \{0\} \tag{8}$$

Lemma 3. *If the condition* (***H***) *holds, then the function* $f(x) = \sum_{i=1}^{p}\langle B_i x(t), x(t)\rangle^2$ *verifies: there exists two positive reals* $\mathfrak{m}$ *and* $\mathfrak{M}$ *such that*

$$0 < \mathfrak{m}\|x\|^4 \leq f(x) \leq \mathfrak{M}\|x\|^4,\ \forall x \in \mathbb{R}^n\backslash\{0\}.$$

Proof We suppose that the condition (**H**) holds. Let $x \in \mathbb{R}^n\backslash\{0\}$, denote $y = \frac{x}{\|x\|}$. It is easy to verify that the function $f(y)$ is positive and continuous on a the compact set $\mathbf{S}^{n-1}$, then $f(y)$ admits a maximum and a minimum on $\mathbf{S}^{n-1}$. Denote $\mathfrak{m} = \min\limits_{y\in\mathbf{S}^{n-1}} f(y)$ and $\mathfrak{M} = \max\limits_{y\in\mathbf{S}^{n-1}} f(y)$.
By hypothesis (**H**) and for all $y \in \mathbf{S}^{n-1}$, $y \neq 0$, we deduce that $0 < \mathfrak{m} \leq \mathfrak{M}$.
That implies $0 < \mathfrak{m} \leq f(y) \leq \mathfrak{M},\ \forall y \in \mathbf{S}^{n-1}$.
It follows that

$$0 < \mathfrak{m}\|x\|^4 \leq f(x) \leq \mathfrak{M}\|x\|^4,\ \forall x \in \mathbb{R}^n\backslash\{0\}.$$

□

Theorem 2. *If the condition* (***H***) *is satisfied, then there exist feedback laws*

$$u_i(x) = -\langle B_i x, x\rangle,\ i \in \{1, 2, ..., p\} \tag{9}$$

that makes the closed-loop system (7) GAS.

Proof Let us consider the quadratic function $V : \mathbb{R}^n \to \mathbb{R}$ defined by

$$V(x) = \frac{1}{2}\|x\|^2 \tag{10}$$

The function V is positive definite.
Using the lemma 2, 3 and Remark 1, the derivative of V along the trajectories of the system (7) induces:

$$\begin{aligned} ^{C}D_{t_0}^{\alpha}V(x) &\leq x^T(t)\,^{C}D_{t_0}^{\alpha}x(t) \\ &\leq x^T(t)\sum_{i=1}^{p} u_i(x) B_i x(t) \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^{p} u_i(x)\langle B_i x(t), x(t)\rangle \\ &\leq -\sum_{i=1}^{p}\langle B_i x(t), x(t)\rangle^2 \\ &\leq -\mathfrak{m}\|x\|^4. \end{aligned}$$

So, according to theorem 1, the closed loop system (7) by the feedback laws (9) is GAS. □

Theorem 3. *If the condition (**H**) is satisfied, then there exist bounded feedback laws*

$$\begin{cases} u_i(x) = -\dfrac{\langle B_i x, x\rangle}{\|x\|^2}, \ x \neq 0 \\ u_i(0) = 0 \end{cases}, \ i \in \{1, 2, ..., p\} \tag{11}$$

such that the closed loop system (7) becomes GAS.

Proof Let us consider the quadratic function

$$V(x) = \frac{1}{2}\|x\|^2,$$

V is positive definite. If the condition (**H**) holds, then, the derivative of V along the solutions of the closed loop system (7) by the feedback (11) becomes :

$$\begin{aligned} {}^{C}D_{t_0}^{\alpha}V(x) &\leq x^T(t)\,{}^{C}D_{t_0}^{\alpha}x(t) \\ &\leq x^T(t)\sum_{i=1}^{p} u_i B_i x(t) \\ &\leq -\sum_{i=1}^{p} u_i(x)\langle B_i x(t), x(t)\rangle \\ &\leq -\frac{1}{\|x\|^2}\sum_{i=1}^{p}\langle B_i x(t), x(t)\rangle^2 \\ &\leq -\mathfrak{m}\|x\|^2. \end{aligned}$$

The proof of the theorem is completed by using the theorem 1. □

3.2 Stabilization of bilinear systems with drift

We consider the system with drift (1):

$${}^{C}D_{t_0}^{\alpha}x(t) = Ax + \sum_{i=1}^{p} u_i B_i x(t),$$

Theorem 4. *If the condition (**H**) is satisfied, then there exist bounded feedback laws*

$$\begin{cases} u_i(x) = -c\dfrac{\langle B_i x, x\rangle}{\|x\|^2}, \ x \neq 0 \\ u_i(0) = 0 \end{cases}, \ i \in \{1, 2, ..., p\} \tag{12}$$

where c is a positive constant which will be chosen later, such that the closed-loop system (1) becomes GAS.

Proof Let us consider the quadratic function

$$V(x) = \frac{1}{2}\|x\|^2,$$

V is positive definite. If the condition (**H**) holds, then, the derivative of V along the solutions of the closed loop system (1) by the feedback (12) becomes :

$$\begin{aligned} {}^C D_{t_0}^{\alpha} V(x) &\leq x^T(t)^C D_{t_0}^{\alpha} x(t) \\ &\leq \langle Ax, x\rangle + \sum_{i=1}^{p} u_i(x)\langle B_i x, x\rangle \\ &\leq \langle Ax, x\rangle - \frac{c}{\|x\|^2}\sum_{i=1}^{p}\langle B_i x(t), x(t)\rangle^2 \end{aligned}$$

We have,

$$\langle Ax, x\rangle = \langle A_s x, x\rangle$$

where $A_s = \frac{A+A^T}{2}$. So, let

$$\lambda_{max} = \max_{x\neq 0}\frac{\langle A_s x, x\rangle}{\|x\|^2} = \max_{y\in\mathbf{S}^{n-1}}\langle A_s y, y\rangle$$

Therefore, we have

$${}^C D_{t_0}^{\alpha} V(x) \leq (\lambda_{max} - c\mathfrak{m})\|x\|^2.$$

If we choose $c > \frac{\lambda_{max}}{\mathfrak{m}}$, then the closed loop system (1) by the feedback laws (12) is GAS. □

Remark 2. *For the mono-input system,*

$${}^C D_{t_0}^{\alpha} x = Ax + uBx, \tag{13}$$

*our method is still effective. Actually, replacing the condition (**H**) by the condition*

$$\{x \in \mathbb{R}^n : \langle Bx, x\rangle = 0\} \subseteq \{x \in \mathbb{R}^n : \langle Ax, x\rangle < 0\},$$

we can construct a feedback $u_c(x) = -c\frac{\langle Bx,x\rangle}{\|x\|^2}$, *such that* ${}^C D_{t_0}^{\alpha} V(t, x(t)) \leq -\lambda_3(\|x\|)$. *Thus, the feedback* u_c *stabilizes the system (13).*

Proposition 1. *If there exist scalars* $k_1, k_2, ..., k_p \in \mathbb{R}$ *such that*

$$k_1 B_{1_s} + k_2 B_{2_s} + ... + k_p B_{p_s} > 0 \text{ or } k_1 B_{1_s} + k_2 B_{2_s} + ... + k_p B_{p_s} < 0$$

*then the condition (**H**) holds, i.e.* $\bigcap_{i=1}^{p} \boldsymbol{S}_i = \{0\}$.

Proof Without loss of generality, we suppose that there are p numbers $k_1, k_2, ..., k_p \in \mathbb{R}$ such that

$$k_1 B_{1_s} + k_2 B_{2_s} + ... + k_p B_{p_s} > 0$$

For $x \in \mathbf{S}_1 \cap ... \cap \mathbf{S}_p$, we have

$$\langle B_i x, x\rangle = \langle B_{i_s} x, x\rangle = 0, i \in \{1, 2, ..., p\}$$

Then

$$\langle (k_1 B_{1_s} + k_2 B_{2_s} + ... + k_p B_{p_s})x, x\rangle = 0$$

Thus, $x = 0$, which implies that $\bigcap_{i=1}^{p} \mathbf{S}_i = \{0\}$, i.e. the condition (**H**) is checked. □

§4 Illustrating examples

Example 1. *Let us consider the system*

$$^{C}D^{\alpha}_{t_0}x(t) = Ax + u_1(x)_1x + u_2(x)B_2x. \tag{14}$$

where

$$A = \begin{pmatrix} 0.25 & 0.25 & 0.5 \\ 0.25 & 0 & -0.5 \\ -0.25 & 0.25 & 0.75 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0.25 & -0.75 & 0 \\ 0,75 & -0.5 & 0.25 \\ 0 & -0.25 & 0.25 \end{pmatrix}$$

,

$$B_2 = \begin{pmatrix} 0.5 & 0 & -0.25 \\ 0 & -1.5 & 0.25 \\ 0.75 & -0.25 & -1 \end{pmatrix}$$

and $\alpha \in (0, 1)$. *We can easily verify that*

$B_{1_s} = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & 0.25 \end{pmatrix}$ *and* $B_{2_s} = \begin{pmatrix} 0.5 & 0 & 0.25 \\ 0 & -1.5 & 0 \\ 0.25 & 0 & -1 \end{pmatrix}$.

Let $k_1 = -2.5$ *and* $k_2 = 1$. *We have* $k_1B_{1_s} + k_2B_{2_s} = \begin{pmatrix} -0.125 & 0 & 0.25 \\ 0 & -0.25 & 0 \\ 0.25 & 0 & -1.62 \end{pmatrix}$.

We can verify that $k_1B_{1_s} + k_2B_{2_s} < 0$.

According to Theorem 4, the system (14) can be stabilized.

Example 2. *Consider the system*

$$^{C}D^{\alpha}_{t_0}x(t) = Ax + u_1B_1x + u_2B_2x + u_3B_3x, \text{ with } \alpha \in (0, 1), \tag{15}$$

where $A = \begin{pmatrix} 8.5 & 4.5 & 15.5 \\ -3.5 & 30.5 & 37 \\ 27.7 & -5 & 13 \end{pmatrix}$ $B_1 = \begin{pmatrix} 8.5 & 4,5 & 8.5 \\ 7.5 & 1.5 & 10 \\ 18.5 & -24 & -5.5 \end{pmatrix}$

$B_2 = \begin{pmatrix} -6.5 & -5 & 8 \\ -6 & 14,5 & -9,5 \\ 15 & -8.5 & 4.5 \end{pmatrix}$ *and* $B_3 = \begin{pmatrix} 10.5 & -10 & 7.5 \\ -12 & 12.5 & -27 \\ 5.5 & 11 & 34 \end{pmatrix}$.

Let $k_1 = 1$, $k_1 = -1$ *and* $k_3 = 2$. *We can get*

$k_1B_{1_s} + k_2B_{2_s} + k_3B_{3_s} = \begin{pmatrix} 36 & -10.5 & 15 \\ -10.5 & 12 & -14 \\ 15 & -14 & 58 \end{pmatrix}$, $k_1B_{1_s} + k_2B_{2_s} + k_3B_{3_s} > 0$.

According to Proposition 1, we have $\bigcap_{i=1}^{3} \boldsymbol{S}_i = \{0\}$. *So the closed-loop system*

$$^{C}D^{\alpha}_{t_0}x(t) = Ax + u_1(x)B_1x + u_2(x)B_2x + u_3(x)B_3x, \tag{16}$$

where

$$\begin{cases} u_1(x) = -\dfrac{1}{2}\dfrac{17x_1^2+24x_1x_2+54x_1x_3+3x_2^2-28x_2x_3-11x_3^2}{x_1^2+x_2^2+x_3^2}, \\ u_2(x) = \dfrac{1}{2}\dfrac{13x_1^2+22x_1x_2-46x_1x_3-29x_2^2+36x_2x_3-9x_3^2}{x_1^2+x_2^2+x_3^2}, \\ u_3(x) = -\dfrac{1}{2}\dfrac{20x_1^2-44x_1x_2+26x_1x_3+25x_2^2-32x_2x_3+68x_3^2}{x_1^2+x_2^2+x_3^2} \end{cases}$$

is asymptotically stable.

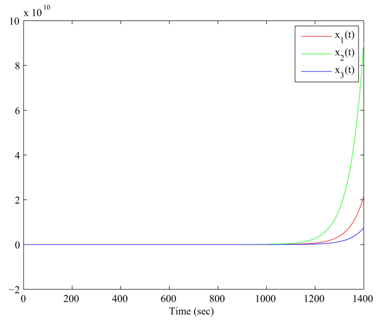

Figure 1. Evolution of the state $x_1(t)$, $x_2(t)$, and $x_3(t)$ of Example 2, without feedback and initial conditions $x_1(0) = -1,\ x_2(0) = 3$ *and* $x_3(0) = 2$.

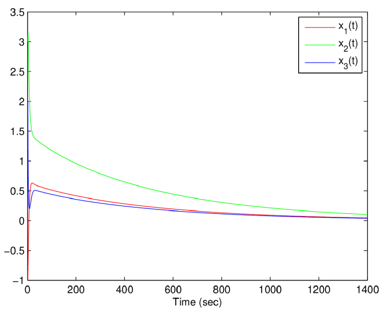

Figure 2. Evolution of the state $x_1(t)$, $x_2(t)$, and $x_3(t)$ of Example 2, with α=0.98, c = 0.67 and initial conditions $x_1(0) = -1,\ x_2(0) = 3$ *and* $x_3(0) = 2$.

The numerical solution to the system (16) is shown in the Figure 2 for some suitable value of fractional order $\alpha = 0.98$*. It indicates that the zero solution is asymptotically stable.*

§5 Conclusion

In this article, we studied the stabilization problem of fractional bilinear systems with multiple inputs by homogeneous feedbacks of degree zero. We construct under a given hypothesis stabilizing feedback laws for bilinear systems without drift and with drift. These results are obtained using Lyapunov fractional functions. Our future goal is to stabilize homogeneous fractional systems.

References

[1] Z Artstein. *Stabilization with relaxed controls*, Nonlinear Analysis: Theory, Methods and Applications, 1983, 7(11): 1163-1173.

[2] T A Burton. *Fractional differential equations and Lyapunov functionals*, Nonlinear Anal Theory Methods Appl, 2011, 74(16): 5648-5662.

[3] D S Ding, D L Qi, Q Wang. *Non-linear Mittag-Leffler stabilisation of commensurate fractional-order nonlinear systems*, IET Control Theory Appl, 2015, 9(5): 681-690.

[4] K Hassan. *Nonlinear Systems*, third ed, Prentice Hall, 2002.

[5] R Khalil, M Al Horani, A Yousef, M Sababheh. *A new definition of fractional derivative* , Journal of Computational and Applied Mathematics, 2014, 264: 65-70.

[6] A A Kilbas, H M Srivastava, J J Trujillo. *Theory and Application of Fractional Differential Equations*, Elsevier, 2006.

[7] V Lakshmikantham, S Leela, M Sambandham. *Lyapunov theory for fractional differential equations*, Commun Appl Anal, 2008, 12(4): 365-376.

[8] V Lakshmikantham, S Leela, S J Vasundara Devi. *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009.

[9] Y Li, Y Chen, I Podlubny. *Mittag-Leffler stability of fractional order nonlinear dynamic systems*, Automatica, 2009, 45(8): 1965-1969.

[10] Y Li, Y Chen, I Podlubny. *Stability of fractional-order nonlinear dynamic system: Lyapunov direct method and generalized Mittag-Leffler stability*, Comput Math Appl, 2010, 59(5): 1810-1821.

[11] Y H Lim, K K Oh, H S Ahn. *Stability and stabilization of fractional-order linear systems subject to input saturation*, IEEE Transactions on Automatic Control, 2013, 58(4): 1062-1067.

[12] J A T Machado. *Analysis and design of fractional-order digital control systems*, J Syst Anal Modelling Simul, 1997, 27(2): 107-122.

[13] Norelys Aguila-Camacho, A Manuel Duarte-Mermoud, A Javier Gallegos. *Lyapunov functions for fractional order systems*, Commun Nonlinear Sci Numer Simulat, 2014, 19: 2951-2957.

[14] K B Oldham, J Spanier. *The Fractional Calculus*, Academic Press, 1974.

[15] I Podlubny. *Fractional Differential Equations*, Academic Press, 1999.

[16] S G Samko, A A Kilbas, O I Marichev. *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Amsterdam, 1993.

[17] E D Sontag. *A "universal" construction of Artstein s Theorem on nonlinear stabilization*, Systems and Control Letter, 1989, 13(2): 117-123.

[18] E S A Shahri, A Alfi, J A Tenreiro Machado. *An extension of estimation of domain of attraction for fractional order linear system subject to saturation control*, Applied Mathematics Letters, 2015, 47: 26-34.

[19] E S A Shahri, A Alfi, J A Tenreiro Machado. *Robust stability and stabilization of uncertain fractional order systems subject to input saturation*, Journal of Vibration and Control, 2018, 24(16) : 3676-3683.

[20] E S A Shahri, A Alfi, J A Tenreiro Machado. *Lyapunov method for the stability analysis of uncertain FO systems under input saturation*, Applied Mathematical Modelling, 2020, 81: 663-672.

[21] B M Vinagre, A C Monje. *Introducci n al control fraccionario*, RIAI, 2006, 3(3): 5-23.

[22] Q Wang, J L Zhang, D S Ding, D L Qi. *Adaptive Mittag-Leffler stabilization of a class of fractional order uncertain nonlinear systems*, Asian J Control, 2016, 18(6): 2343-2351.

Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Tunisia.

Email: thouraya.kharrat@fss.rnu.tn, fehmi.mabrouki@gmail.com, fawzi_omri@yahoo.fr