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New hybrid inertial CQ projection algorithms with line-search process for the split feasibility problem

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Abstract. In this paper, we propose two hybrid inertial CQ projection algorithms with line-search process for the split feasibility problem. Based on the hybrid CQ projection algorithm, we firstly add the inertial term into the iteration to accelerate the convergence of the algorithm, and adopt flexible rules for selecting the stepsize and the shrinking projection region, which makes an optimal stepsize available at each iteration. The shrinking projection region is the intersection of three sets, which are the set C and two hyperplanes. Furthermore, we modify the Armijo-type line-search step in the presented algorithm to get a new algorithm.The algorithms are shown to be convergent under certain mild assumptions. Besides, numerical examples are given to show that the proposed algorithms have better performance than the general CQ algorithm.

§1 Introduction

Split Feasibility Problem (SFP) was firstly introduced by Censor and Elfving [4] in 1994, which is to find a point x^* satisfying

$$x^* \in C \; ; Ax^* \in Q, \tag{1}$$

where C and Q are nonempty convex sets in $\Re^N$ and $\Re^M$, respectively, and A is an M by N real matrix. Problem (1) together with many variants of it has received much attention from optimization community due to its broad applications to many disciplines, such as signal processing, image reconstruction, intensity-modulated radiation therapy, etc. [1,2,3,7]. Therefore, many effective methods have been proposed to solve the problem (1), see [5,6,8,16,17,22,23]. A very successful algorithm that solves the SFP seems to be the CQ algorithm of Byrne [2] which is defined as follows: Denote P_c by the orthogonal projection onto C, that is, $P_C(y) = \text{argmin}_{x\in C} ||x-y||$, for $y \in C$; then take an initial point arbitrarily and define the iterative step by

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$$x^{k+1} = P_C(I - \gamma A^T(I - P_Q)A)(x^k), \tag{2}$$

where $0 < \gamma < 2/\|A\|^2$. The CQ algorithm (2) can be regarded as a special case of the gradient projection method if we consider the convex minimization problem

$$\min_{x \in C} \frac{1}{2}\|(I - P_Q)\, Ax\|^2.$$

Observe that, the choice of stepsize in CQ algorithm depends on the norm of the operator, which is not a simple work. To avoid this computation, some modifications of the CQ algorithm and the self-adaptive method have been developed for solving the SFP. L pez et al. [12] introduced a new way of selecting the stepsizes for solving SFP (1) such that the information of operator norm is not necessary. Motivated and inspired by the work of [12,20,24,27], the authors of [14] introduced a self-adaptive CQ-type algorithm for solving the SFP in the setting of infinite dimensional real Hilbert spaces. Inspired by the projection and contraction method and the hybrid descent approximation method, Gibali et al. [11] investigated the problem of finding a common solution to a fixed point problem involving demi-contractive operator and a variational inequality with monotone and Lipschitz continuous mapping in real Hilbert spaces. Shehu et al. [19] introduced iterative algorithms and proved their strong convergence for solving proximal split feasibility problems and fixed point problems for k-strictly pseudocontractive mappings in Hilbert spaces. Dang et al. [9] proposed a hybrid CQ projection algorithm with Armijo-type line-search step, which is different from the general self-adaptive Armijo-type procedure [25,26].

On the other hand, in [15], Polyak firstly proposed the inertial term as an acceleration process to solve the smooth convex minimization problem. In recent years, scholars have proposed some inertial iterative algorithms for solving the SFP. Based on [12], Taddele et al. [21] proposed an iterative algorithm with inertial extrapolation to approximate the solution of multiple-set SFP. Shehu et al. [18] introduced new CQ methods with alternated inertial procedure and self-adaptive stepsize for solving SFP. Li et al. [13] proposed two inertial relaxed CQ algorithms for solving the SFP in real Hilbert spaces according to the previous experience of applying inertial technology to the algorithm. Godwin et al [10] introduced a new inertial extrapolation method for solving a certain class of generalized SFP without the prior knowledge of the operator norm or the coefficient of an underlying operator.

Inspired by the works mentioned above, we propose two hybrid inertial CQ projection algorithms with line-search process for the split feasibility problem. The main features of the proposed algorithms as follows:

1. Based on [9], we incorporate the inertial term into the iteration to construct Algorithm 3.1, which improves the efficiency of convergence. Furthermore, we adjust the Armijo-type line-search step in Algorithm 3.1 to get Algorithm 3.2.

2. Algorithms perform a computationally inexpensive Armijo-type linear search along the search direction to generate a hyperplane.

3. The next iteration is generated by the projection of the initial point on the intersection of the set C with the hyperplanes, which makes an optimal stepsize available at each iteration.

Different from the algorithms in [11,19], two hyperplanes are constructed in our algorithms to generate the shrinking projection region.

The paper is organized as follows. In section 2, Some useful definitions and results are collected for the convergence analysis of the proposed algorithms. In Section 3, we propose two hybrid inertial CQ projection algorithms for the split feasibility problem and show their convergence. In Section 4, numerical experiments are reported to conclude the effectiveness of our algorithms. Finally, some conclusions are given in Section 5.

§2 Preliminaries

We denote by I the identity operator and by $\mathrm{Fix}(T)$ the set of fixed points of an operator T, that is, $\mathrm{Fix}(T) := \{x|x = Tx\}$.

Recall that a mapping $T : \Re^n \to \Re$ is said to be monotone if

$$\langle T(x) - T(y), x - y\rangle \geq 0, \forall x, y \in \Re^n.$$

For a monotone mapping T, $\langle T(x) - T(y), x - y\rangle$ =0 iff $x = y$, then it is said to be strictly monotone.

A mapping $T : \Re^n \to \Re^n$ is called non-expansive if

$$\|T(x) - T(y)\| \leq \|x - y\| \ , \forall x, y \in \Re^n.$$

Lemma 2.1 [9,13] *Let Ω be a nonempty closed and convex subset in H. Then, for any $x, y \in H$ and $z \in \Omega$, the following hold:*

(1) $\langle x - P_\Omega(x), z - P_\Omega(x)\rangle \leq 0.$

(2) $\|P_\Omega(x) - P_\Omega(y)\|^2 \leq \langle P_\Omega(x) - P_\Omega(y), x - y\rangle.$

(3) $\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|, \forall x, y \in \Re^n,$ *or more precisely,*

$$\|P_\Omega(x) - P_\Omega(y)\|^2 \leq \|x - y\|^2 - \|P_\Omega(x) - x + y - P_\Omega(y)\|^2.$$

(4) $\|P_\Omega(x) - z\|^2 \leq \|x - z\|^2 - \|P_\Omega(x) - x\|^2.$

Remark 2.1 In fact, the projection property (1) also provides a sufficient and necessary condition for a vector $u \in K$ to be the projection of the vector x; that is,$u = P_K(x)$ if and only if

$$\langle u - z, x - u\rangle \geq 0, \forall z \in K.$$

Lemma 2.2 [18] *The following statements hold in $\Re^n$:*

(1) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2,$ *for all* $x, y \in \Re^n.$

(2) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$, *for all* $x, y \in \Re^n$.

Lemma 2.3 [3] *Let* $f(x) := \frac{1}{2}\|(I-P_Q)Ax\|^2$, $x \in C$. *Then*

(1) f *is convex and differentiable.*

(2) $\nabla f(x) = A^T(I-P_Q)Ax$, $x \in \Re^N$.

(3) f *is lower semicontinuous on* $\Re^N$.

(4) ∇f *Lipschitz continuous with Lipschitz constant* $\|A\|^2$.

Throughout the paper, the solution set of split feasibility problem is denoted by Γ, that is

$$\Gamma := \{x^* \in C | Ax^* \in Q\}. \tag{3}$$

§3 Algorithms and their convergence analysis

Let

$$F(x) := (A^T(I-P_Q)A)(x).$$

Then we know that F is Lipschitz-continuous with constant $\|A\|^2$. We first note that the solution set coincides with zeros of the following projected residual function:

$$e(x) := x - P_C(x - F(x)), e(x,\mu) := x - P_C(x - \mu F(x));$$

with this definition, we have $e(x,1) = e(x)$, and $x \in \Gamma$ if and only if $e(x,\mu) = 0$. For any $x \in \Re^N$ and $\alpha \ge 0$, define

$$x(\alpha) = P_C(x - \alpha F(x)), e(x,\alpha) = x - x(\alpha).$$

The following lemma is useful for the convergence analysis in the next section.

Lemma 3.1 [25] *Let* F *be a mapping from* $\Re^N$ *into* $\Re^N$. *For any* $x \in \Re^N$ *and* $\alpha \ge 0$, *we have*

$$\min\{1,\alpha\}\|e(x,1)\| \le \|e(x,\alpha)\| \le \max\{1,\alpha\}\|e(x,1)\|.$$

Now, we describe our first algorithm as follows:

Algorithm 3.1

Step 0. Choose arbitrary initial points $x^0, x^1 \in \Re^N$, and parameters $\eta_0 > 0$, $t_k \in (0,1)$, $\gamma \in (0,1)$, $\sigma \in (0,1)$, $\varepsilon \in (0,1)$ and $\theta > 1$, and set $k = 0$.

Step 1. Assuming x^{k-1}, x^k have been constructed, compute

$$\mathrm{w}^k = P_C[x^k + t_k(x^k - x^{k-1})], \tag{4}$$

$$z^k = P_C[w^k - \mu_k F(w^k)], \tag{5}$$

where μ_k is a positive number satisfying $\varepsilon < \mu_k \le \min\left\{\frac{2}{\rho(A^TA)} - \varepsilon, 1\right\}$. Obviously, $e(w^k, \mu_k) = w^k - z^k$. If $e(w^k, \mu_k) = 0$, then stop;

Step 2. Compute

$$y^k = w^k - \eta_k e(w^k, \mu_k), \tag{6}$$

where $\eta_k = \gamma^{m_k}\mu_k$, with m_k being the smallest nonnegative integer m satisfying

$$\left\langle F(w^k - \gamma^{m_k}\mu_k e(w^k, \mu_k)), e(w^k, \mu_k)\right\rangle \ge \frac{\sigma}{\mu_k}\left\|e(w^k, \mu_k)\right\|^2. \tag{7}$$

Step 3. Compute

$$x^{k+1} = P_{C\cap H_k^1 \cap H_k^2}(x^0), \tag{8}$$

where

$$H_k^1 = \left\{x \in \Re^N | \left\|y^k - x\right\|^2 \le \left\|w^k - x\right\|^2\right\},$$
$$H_k^2 = \left\{x \in \Re^N | \left\langle x - x^k, x^0 - x^k\right\rangle \le 0\right\}.$$

Set $k = k + 1$ and go to Step 1.

Before establishing the global convergence of Algorithm 3.1, we first give the following lemmas.

Lemma 3.2 *There exists a nonnegative number m satisfying* (7), *for all $k \ge 0$.*

proof. Suppose that, for some k, (7) is not satisfied for any integer m , that is,

$$\left\langle F(w^k - \gamma^m\mu_k e(w^k, \mu_k)), e(w^k, \mu_k)\right\rangle \le \frac{\sigma}{\mu_k}\left\|e(w^k, \mu_k)\right\|^2. \tag{9}$$

By the definition of $e(w^k, \mu_k)$, and Lemma 2.1 we know that

$$\left\langle P_C(w^k - \mu_k F(w^k)) - (w^k - \mu_k F(w^k)), w^k - P_C(w^k - \mu_k F(w^k))\right\rangle \ge 0.$$

Then

$$\left\langle F(w^k), e(w^k, \mu_k)\right\rangle \ge \frac{1}{\mu_k}\left\|e(w^k, \mu_k)\right\|^2 > 0. \tag{10}$$

Since $\gamma \in (0, 1)$ and $\varepsilon < \mu_k \le \min\left\{\frac{2}{\rho(A^TA)} - \varepsilon, 1\right\}$, from (9) we get

$$\lim_{m\to\infty}(w^k - \gamma^m\mu_k e(w^k, \mu_k)) = w^k.$$

Hence,

$$\left\langle F(w^k), e(w^k, \mu_k)\right\rangle \le \frac{\sigma}{\mu_k}\left\|e(w^k, \mu_k)\right\|^2 < \frac{1}{\mu_k}\left\|e(w^k, \mu_k)\right\|^2. \tag{11}$$

But (11) contradicts (10) because $\left\|e(w^k, \mu_k)\right\| \ge 0$. Hence, (7) is satisfied for some integer m.□

Lemma 3.3 *If the solution set $\Gamma \ne \emptyset$, then $\Gamma \subset H_k^1 \cap C$ for all $k \ge 0$.*

proof. Let $x^* \in \Gamma$, then

$$\begin{aligned}
\left\|z^k - x^*\right\|^2 &= \left\|P_C(w^k - \mu_k A^T(I-P_Q)Aw^k) - x^*\right\|^2 \\
&\le \left\|w^k - x^* - \mu_k A^T(I-P_Q)Aw^k\right\|^2 \\
&= \left\|w^k - x^*\right\|^2 - 2\mu_k \left\langle w^k - x^*, A^T(I-P_Q)Aw^k\right\rangle + {\mu_k}^2\left\|A^T(I-P_Q)Aw^k\right\|^2 \\
&= \left\|w^k - x^*\right\|^2 - 2\mu_k \left\langle A(w^k - x^*), (I-P_Q)Aw^k\right\rangle + {\mu_k}^2\left\|A^T(I-P_Q)Aw^k\right\|^2 \\
&= \left\|w^k - x^*\right\|^2 - 2\mu_k \left\langle Aw^k - P_QAw^k + P_QAw^k - Ax^*, (I-P_Q)Aw^k\right\rangle \\
&\quad + {\mu_k}^2\left\|A^T(I-P_Q)Aw^k\right\|^2 \\
&= \left\|w^k - x^*\right\|^2 - 2\mu_k\left\|(I-P_Q)Aw^k\right\|^2 \\
&\quad - 2\mu_k \left\langle P_QAw^k - Ax^*, (I-P_Q)Aw^k\right\rangle + {\mu_k}^2\left\|A^T(I-P_Q)Aw^k\right\|^2.
\end{aligned}$$

From Lemma 2.1, we obtain

$$\begin{aligned}
\left\|z^k - x^*\right\|^2 &\le \left\|w^k - x^*\right\|^2 - \mu_k\left\{2\left\|(I-P_Q)Aw^k\right\|^2 - \mu_k\left\|A^T(I-P_Q)Aw^k\right\|^2\right\} \\
&\le \left\|w^k - x^*\right\|^2 - \mu_k\left(\frac{2}{\rho(A^TA)} - \mu_k\right)\left\|A^T(I-P_Q)Aw^k\right\|^2 \qquad (12) \\
&\le \left\|w^k - x^*\right\|^2. \qquad (13)
\end{aligned}$$

Also, it follows from the definition of y^k that

$$\begin{aligned}
\left\|y^k - x^*\right\|^2 &= \left\|w^k - x^* - \eta_k(w^k - z^k)\right\|^2 \\
&= \left\|w^k - x^*\right\|^2 - 2\eta_k\left\langle w^k - x^*, w^k - z^k\right\rangle + {\eta_k}^2\left\|w^k - z^k\right\|^2,
\end{aligned}$$

which can be rewritten as

$$\left\langle w^k - x^*, w^k - z^k\right\rangle = -\frac{1}{2\eta_k}\left\{\left\|y^k - x^*\right\|^2 - \left\|w^k - x^*\right\|^2 - {\eta_k}^2\left\|w^k - z^k\right\|^2\right\}. \qquad (14)$$

On the other hand, by the definition of y^k, we have

$$\begin{aligned}
\left\|y^k - x^*\right\|^2 &= \left\|w^k - x^* - \eta_k(w^k - z^k)\right\|^2 \\
&= \left\|z^k - x^* + (1-\eta_k)(w^k - z^k)\right\|^2 \\
&= \left\|z^k - x^*\right\|^2 + 2(1-\eta_k)\left\langle z^k - x^*, w^k - z^k\right\rangle + (1-\eta_k)^2\left\|w^k - z^k\right\|^2 \\
&= \left\|z^k - x^*\right\|^2 + 2(1-\eta_k)\left\langle z^k - w^k + w^k - x^*, w^k - z^k\right\rangle + (1-\eta_k)^2\left\|w^k - z^k\right\|^2 \\
&= \left\|z^k - x^*\right\|^2 + 2(1-\eta_k)\left\langle w^k - x^*, w^k - z^k\right\rangle + ({\eta_k}^2 - 1)\left\|w^k - z^k\right\|^2. \qquad (15)
\end{aligned}$$

Since $\left\|z^k - x^*\right\|^2 \le \left\|w^k - x^*\right\|^2$, then by (14) and (15) we get

$$\begin{aligned}
\frac{1}{\eta_k}\left\|y^k - x^*\right\|^2 &= \left\|z^k - x^*\right\|^2 + (\frac{1}{\eta_k} - 1)\left\|w^k - x^*\right\|^2 + (\eta_k - 1)\left\|w^k - z^k\right\|^2 \\
&\le \left\|w^k - x^*\right\|^2 + (\frac{1}{\eta_k} - 1)\left\|w^k - x^*\right\|^2 = \frac{1}{\eta_k}\left\|w^k - x^*\right\|^2.
\end{aligned}$$

Hence

$$\left\|y^k - x^*\right\|^2 \le \left\|w^k - x^*\right\|^2,$$

which implies $x^* \in H_k^1$. Moreover, it is easy to see that $\Gamma \subset H_k^1 \cap C$, $\forall k \ge 0$. □

The following lemma says that if the solution set is nonempty, then $\Gamma \subset H_k^1 \cap H_k^2 \cap C$ and thus $H_k^1 \cap H_k^2 \cap C$is a nonempty set.

Lemma 3.4 [9, Lemma 3.4] *If the solution set* $\Gamma \neq \emptyset$*, then* $\Gamma \subset H_k^1 \cap H_k^2 \cap C$ *for all* $k \geq 0$*.*

For the case that the solution set is empty, we have that $H_k^1 \cap H_k^2 \cap C$ is also nonempty from the following lemma, which implies the feasibility of Algorithm 3.1.

Lemma 3.5 *Suppose that* $\Gamma = \emptyset$*, then* $H_k^1 \cap H_k^2 \cap C \neq \emptyset$ *for all* $k \geq 0$*.*

Lemma 3.6 *Let* $\{x^k\}$ *be a sequence generated by algorithm 3.1. Then*

(i) $\lim_{k\to\infty} \|x^k - w^k\| = 0$;

(ii) $\lim_{k\to\infty} \|x^k - y^k\| = 0$;

(iii) $\lim_{k\to\infty} \|x^k - z^k\| = 0$;

(iv) $\lim_{k\to\infty} \|A^T(I - P_Q)Aw^k\| = 0$.

proof. (i) From [9, theorem 3.1], we know that $\{x^k\}$ is a bounded sequence and convergent, which implies that

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{16}$$

By the definition of w^k, it follows that ,

$$\begin{aligned}\|x^k - w^k\| &= \|x^k - P_C[x^k + t_k(x^k - x^{k-1})]\| \\ &\leq \|x^k - [x^k + t_k(x^k - x^{k-1})]\| \\ &= |t_k| \cdot \|x^k - x^{k-1}\|.\end{aligned}$$

Therefore, from the selection of parameter t_k and (16), we have

$$\lim_{k\to\infty} \|x^k - w^k\| = 0. \tag{17}$$

(ii) From (16) and (17), we obtain

$$\begin{aligned}\|x^{k+1} - w^k\| &= \|x^{k+1} - x^k + x^k - w^k\| \\ &\leq \|x^{k+1} - x^k\| + \|x^k - w^k\| \to 0 \ as\ k \to \infty,\end{aligned}$$

which with $x^{k+1} \in H_k^1$ implies that

$$\lim_{k\to\infty} \|x^{k+1} - y^k\| = 0.$$

Again, since $\|x^k - y^k\| \leq \|x^k - x^{k+1}\| + \|x^{k+1} - y^k\|$, it then follows that

$$\lim_{k\to\infty} \|x^k - y^k\| = 0. \tag{18}$$

(iii) Since $\|w^k - y^k\| \leq \|w^k - x^k\| + \|x^k - y^k\|$, then form (17) and (18) we have

$$\lim_{k\to\infty} \|w^k - y^k\| = 0. \tag{19}$$

From the definition of y^k, we get

$$w^k - y^k = \eta_k e(w^k, \mu_k) = \eta_k \left(w^k - z^k\right).$$

Then, note that from the construction of η_k and (19), we have

$$\left\|w^k - z^k\right\| = \frac{1}{|\eta_k|}\left\|w^k - y^k\right\| \to 0 \ as \ k \to \infty. \tag{20}$$

Hence, from (17) and (20), we get

$$\left\|x^k - z^k\right\| \le \left\|x^k - w^k\right\| + \left\|w^k - z^k\right\| \to 0 \ as \ k \to \infty.$$

(iv) Observe that by (12)

$$\begin{aligned}\mu_k\left(\frac{2}{\rho\left(A^TA\right)} - \mu_k\right)\left\|A^T(I-P_Q)Aw^k\right\|^2 &\le \left\|w^k - x^*\right\|^2 - \left\|z^k - x^*\right\|^2 \\ &= \left\|w^k - z^k\right\|^2 + 2\left\langle z^k - x^*, w^k - z^k\right\rangle \\ &\le 2\left\langle z^k - x^*, w^k - z^k\right\rangle.\end{aligned}$$

From (i), we know that $\{x^k\}$ is a convergent and bounded sequence. By the definition of w^k, the sequence $\{w^k - x^*\}$ is bounded, for $\forall x^* \in \Gamma$. And since $\left\|z^k - x^*\right\|^2 \le \left\|w^k - x^*\right\|^2$, we conclude that $\{z^k - x^*\}$ is bounded. Then, from (20), it follows that

$$\lim_{k\to\infty}\mu_k\left(\frac{2}{\rho\left(A^TA\right)} - \mu_k\right)\left\|A^T(I-P_Q)Aw^k\right\|^2 = 0. \tag{21}$$

This together with the fact that $\lim_{k\to\infty}\mu_k\left(\frac{2}{\rho(A^TA)} - \mu_k\right) \ne 0$ further implies

$$\lim_{\mathrm{k}\to\infty}\left\|A^T(I-P_Q)Aw^k\right\| = 0. \tag{22}$$

□

We now prove our main convergence result.

Theorem 3.1 *Suppose the solution set* Γ *is nonempty, then the sequence* $\{x^k\}$ *generated by Algorithm 3.1 is bounded, and all its cluster points belong to the solution set. Moreover, the sequence* $\{x^k\}$ *globally converges to a solution* x^* *such that* $x^* = P_\Gamma(x^0)$.

proof. We have already kown in (i) that $\lim_{\mathrm{k}\to\infty}\left\|x^k - x^0\right\|$ exists.

Now, we show that $x^k \to \bar{x} \in \Gamma$. Let $m, n \in N$, since

$$x^n = P_{H^1_{n-1}\cap H^2_{n-1}\cap C}(x^0),$$

then

$$\|x^m - x^n\|^2 \le \left\|x^m - x^0\right\|^2 - \left\|x^n - x^0\right\|^2.$$

Hence $\lim_{m,n\to\infty}\|x^m - x^n\|^2=0$. Thus $\left\{x^k\right\}_{k\ge2}$ is a Cauchy sequence in C. Since C is nonempty convex sets in $\Re^N$, it implies that there exists $\bar{x} \in C$ such that $x^k \to \bar{x} \ as \ k \to \infty$. More so, since $\left\|x^k - w^k\right\| \to 0$, then $w^k \to \bar{x}$ and by the linearity of A, we have $Aw^k \to A\bar{x}$.

Also from (22), we have

$$\lim_{\mathrm{k}\to\infty}\left\|A^T(I-P_Q)Aw^k\right\|^2 = \left\|A^T(I-P_Q)A\overline{x}\right\|^2=0.$$

Since the projected residual function

$$e(x^k) = x^k - P_C(x^k - A^T(I-P_Q)Ax^k),$$

then, we have

$$e(\bar{x}) = \lim_{k\to\infty} e(x^k) = \bar{x} - P_C(\bar{x} - A^T(I-P_Q)A\bar{x})=\bar{x} - P_C(\bar{x})=0.$$

Thus $\bar{x}$ is a solution of problem (1).

Now, we prove that the sequence $\{x^k\}$ converges to a point contained in Γ.

Let $x^* = P_\Gamma(x^0)$. Since $x^* \in \Gamma$, by Lemma 3.4 we have

$$x^* \in H^1_{k_j-1} \cap H^2_{k_j-1} \cap C, \forall j.$$

So, by the iterative sequence of Algorithm 3.1 we have

$$\left\|x^{k_j} - x^0\right\| \leq \left\|x^* - x^0\right\|.$$

Thus

$$\begin{aligned}\left\|x^{k_j} - x^*\right\|^2 &= \left\|x^{k_j} - x^0 + x^0 - x^*\right\|^2 \\ &= \left\|x^{k_j} - x^0\right\|^2 + \left\|x^0 - x^*\right\|^2 + 2\left\langle x^{k_j} - x^0, x^0 - x^*\right\rangle \\ &\leq \left\|x^* - x^0\right\|^2 + \left\|x^0 - x^*\right\|^2 + 2\left\langle x^{k_j} - x^0, x^0 - x^*\right\rangle.\end{aligned}$$

Letting $j \to \infty$, we have

$$\begin{aligned}\|\bar{x} - x^*\|^2 &\leq 2\left\|x^0 - x^*\right\|^2 + 2\left\langle \bar{x} - x^0, x^0 - x^*\right\rangle \\ &= 2\left\langle \bar{x} - x^*, x^0 - x^*\right\rangle \leq 0,\end{aligned}$$

where the last inequality is due to Lemma 2.1 and the fact that $x^* = P_\Gamma(x^0)$ and $\bar{x} \in \Gamma$. So,

$$\bar{x} = x^* = P_\Gamma(x^0).$$

Thus, the sequence $\{x^k\}$ has a unique cluster point $P_\Gamma(x^0)$, which shows the global convergence of $\{x^k\}$. □

Now, we put another line search method in the Algorithm 3.1 to get a new algorithm.

Algorithm 3.2

Step 0. Choose arbitrary initial points $x^0, x^1 \in \Re^N$, and parameters $\eta_0 > 0$, $t_k \in (0,1)$, $\gamma \in (0,1)$, $\sigma \in (0,1)$, $\varepsilon \in (0,1)$, and $\theta > 1$, and set $k = 0$.

Step 1. Assuming x^{k-1}, x^k have been constructed, compute

$$\mathrm{w}^k = P_C[x^k + t_k(x^k - x^{k-1})],$$

$$z^k = P_C[w^k - \mu_k F(w^k)],$$

where μ_k is a positive number satisfying $\varepsilon < \mu_k \leq \min\left\{\frac{2}{\rho(A^TA)} - \varepsilon, 1\right\}$. We have $e(w^k, \mu_k) = w^k - z^k$. If $e(w^k, \mu_k) = 0$, then stop;

Step 2. Compute

$$y^k = (1 - \delta_k)w^k + \delta_k(z^k - \eta_k e(w^k, \mu_k)), \tag{23}$$

where $\varepsilon < \delta_k < \frac{1}{(1+\eta_k)}$ and $\eta_k = \gamma^{m_k}\mu_k$ with m_k being the smallest nonnegative integer m satisfying

$$\left\langle F(w^k - \gamma^{m_k}\mu_k e(w^k, \mu_k)), e(w^k, \mu_k)\right\rangle \geq \frac{\sigma}{\mu_k}\left\|e(w^k, \mu_k)\right\|^2.$$

Step 3. Compute

$$x^{k+1} = P_{C \cap H^1_k \cap H^2_k}(x^0),$$

where

$$H^1_k = \left\{x \in \Re^N | \left\|y^k - x\right\|^2 \leq \left\|w^k - x\right\|^2\right\},$$

$$H_k^2 = \left\{x \in \Re^N \,|\, \left\langle x - x^k, x^0 - x^k \right\rangle \le 0\right\}.$$

Set $k = k + 1$ and go to Step 1.

Theorem 3.2 *Let $\{x^k\}$ be a sequence generated by Algorithm 3.2, If $\Gamma \neq \emptyset$, then $\{x^k\}$ globally converges to a solution x^* such that $x^* = P_\Gamma(x^0)$.*

proof. The proof of Theorem 3.2 is similar to that of Theorem 3.1, so we provide only a sketch. Let $x^* \in \Gamma$, from the definition of y^k, we obtain

$$\begin{aligned}\left\|y^k - x^*\right\|^2 &= \left\|(1-\delta_k)w^k + \delta_k(z^k - \eta_k e(w^k, \mu_k)) - x^*\right\|^2 \\ &= \left\|w^k - x^* - (1+\eta_k)\delta_k(w^k - z^k)\right\|^2 \\ &= \left\|w^k - x^*\right\|^2 - 2(1+\eta_k)\delta_k \left\langle w^k - x^*, w^k - z^k\right\rangle + (1+\eta_k)^2{\delta_k}^2\left\|w^k - z^k\right\|^2, \end{aligned} \tag{24}$$

which can be written as

$$\left\langle w^k - x^*, w^k - z^k\right\rangle = -\frac{1}{2(1+\eta_k)\delta_k}\left\{\left\|y^k - x^*\right\|^2 - \left\|w^k - x^*\right\|^2 - (1+\eta_k)^2{\delta_k}^2\left\|w^k - z^k\right\|^2\right\}. \tag{25}$$

On the other hand, by the definition of y^k, we can get

$$\begin{aligned}\left\|y^k - x^*\right\|^2 &= \left\|z^k - x^* - ((1+\eta_k)\delta_k - 1)(w^k - z^k)\right\|^2 \\ &= \left\|z^k - x^*\right\|^2 - 2((1+\eta_k)\delta_k - 1)\left\langle z^k - x^*, w^k - z^k\right\rangle \\ &\quad + ((1+\eta_k)\delta_k - 1)^2\left\|w^k - z^k\right\|^2 \\ &= \left\|z^k - x^*\right\|^2 - 2((1+\eta_k)\delta_k - 1)\left\langle w^k - x^*, w^k - z^k\right\rangle \\ &\quad + ((1+\eta_k)^2{\delta_k}^2 - 1)\left\|w^k - z^k\right\|^2. \end{aligned} \tag{26}$$

Since $\left\|z^k - x^*\right\|^2 \le \left\|w^k - x^*\right\|^2$ and the choice of δ_k, then from (25) and (26) we have

$$\begin{aligned}\frac{1}{(1+\eta_k)\delta_k}\left\|y^k - x^*\right\|^2 &\le \frac{1}{(1+\eta_k)\delta_k}\left\|w^k - x^*\right\|^2 + ((1+\eta_k)\delta_k - 1)\left\|w^k - z^k\right\|^2 \\ &\le \frac{1}{(1+\eta_k)\delta_k}\left\|w^k - x^*\right\|^2. \end{aligned} \tag{27}$$

This implies that

$$\left\|y^k - x^*\right\|^2 \le \left\|w^k - x^*\right\|^2.$$

Then we have $x^* \in H_k^1$, therefore $\Gamma \subset H_k^1 \cap C$.
We know that the sequence $\left\{w^k - x^*\right\}$ is bounded, for $\forall x^* \in \Gamma$. Since $\left\|z^k - x^*\right\|^2 \le \left\|w^k - x^*\right\|^2$, we have

$$\begin{aligned}\left\|w^k - z^k\right\| &\le \left\|w^k - x^*\right\| + \left\|z^k - x^*\right\|, \\ &\le 2\left\|w^k - x^*\right\|, \end{aligned} \tag{28}$$

which implies that $\left\{w^k - z^k\right\}$ is a bounded sequence. Then using similar arguments in obtaining (24), one can show that

$$\left\|x^k - y^k\right\|^2 = \left\|x^k - w^k\right\|^2 + 2(1+\eta_k)\delta_k\left\langle x^k - w^k, w^k - z^k\right\rangle + (1+\eta_k)^2{\delta_k}^2\left\|w^k - z^k\right\|^2,$$

which can be written as

$$\left\|w^k - z^k\right\|^2 = -\frac{1}{(1+\eta_k)^2{\delta_k}^2}\left\{\left\|x^k - y^k\right\|^2 - \left\|x^k - w^k\right\|^2 - 2(1+\eta_k)\delta_k\left\langle x^k - w^k, w^k - z^k\right\rangle\right\}.$$

From (17) and (18), we have

$$\lim_{k\to\infty} \left\| w^k - z^k \right\| = 0. \tag{29}$$

Note also that

$$\left\| x^k - z^k \right\|^2 \le \left\| x^k - w^k \right\|^2 + \left\| w^k - z^k \right\|^2.$$

Therefore by (17) and (28), we get $\lim_{k\to\infty} \left\| x^k - z^k \right\| = 0$.

The rest of the convergence proof is identical to that of Theorem 3.1. □

Remark 3.1 In the two algorithms, a projection from $\Re^N$ onto the intersection $C \cap H_k^1 \cap H_k^2$ needs to be computed, that is, procedure $x^{k+1} = P_{C\cap H_k^1 \cap H_k^2}(x^0)$ at each iteration. Surely, if the domain set C has a special structure such as a box or a ball, then the next iteration x^{k+1} can easily be computed. If the domain set C is defined by a set of linear (in) equalities, then computing the projection is equivalent to solving a strictly convex quadratic optimization problem.

§4 Numerical experiments

In this section, we present two numerical examples to compare the performance of our algorithms with the genaral CQ algorithm. Throughout the computational experiments, the parameters are set as $\gamma = 0.7$, $\sigma = 0.6$, $\theta = 1.5$, $\eta_0 = 0.3$, $\beta = 1$. We define the error as $\frac{\|x_{k+1}-x_k\|_2^2}{\|x_2-x_1\|_2^2}$, and use $\frac{\|x_{k+1}-x_k\|_2^2}{\|x_2-x_1\|_2^2} < 10^{-5}$ as the stopping criterion. The implementations are done in MATLAB to solve the following examples.

Example 4.1 Let $C=\left\{x \in \Re^3 \left| x_1 + x_2^2 + 2x_3 \le 0 \right.\right\}$, $Q = \left\{x \in \Re^3 \left| {x_1}^2 + x_2 - x_3 \le 0 \right.\right\}$. $A =$ ones(3). Find $x \in C$ with $Ax \in Q$.

The numerical results are given in Table 1. To make it explicit, we also measure the performance of the algorithms by plotting the curve of error. The corresponding results are reported in Figure 1 and Figure 2. In the table, k denotes the number of iterations, s denotes the computing time, and x^* denotes the approximate solution.

Table 1. Results for Example 4.1 (Case $t_k = 0.5$).

	$x^0 = (0,1,2)'$	$x^0 = (-2,-1,3)'$	$x^0 = (-3,1,2)'$
Algorithm 3.1	$k = 13; s = 0.0156;$ $x^* = (-2.9164, -0.5533, -3.2353)'$	$k = 24; s = 0.0156;$ $x^* = (-4.0049, -0.7343, -4.9438)'$	$k = 12; s = 0.0156;$ $x^* = (-2.8158, -1.0362, -4.9884)'$
Algorithm 3.2	$k = 14; s = 0.0156;$ $x^* = (0.5817, 0.2605, -2.4136)'$	$k = 13; s = 0.0156;$ $x^* = (3.6451, -1.2376, -8.0093)'$	$k = 21; s = 0.0156;$ $x^* = (3.6838, -1.8361, -6.6465)'$
CQ Algorithm	$k = 211; s = 0.0313;$ $x^* = (-0.0023, 0.0015, 0.0011)'$	$k = 170; s = 0.0313;$ $x^* = (-0.0056, 0.0033, 0.0027)'$	$k = 208; s = 0.0625;$ $x^* = (-0.0065, 0.0035, 0.0032)'$

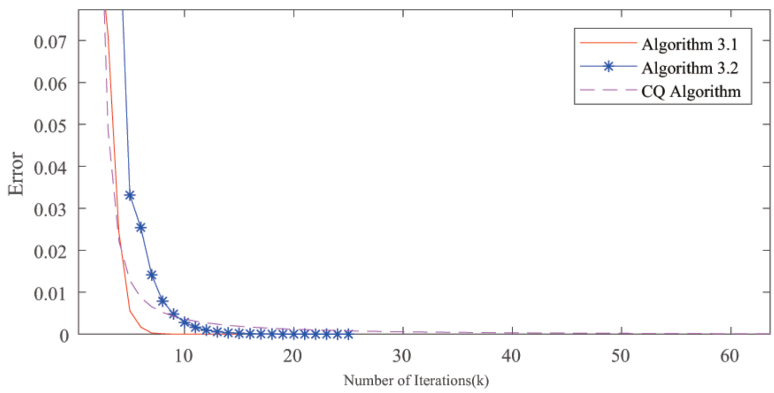

Figure 1. The performance of our algorithms and the CQ algorithm (Case $x^0 = (-1, 2, 4)'$ and $t_k = 0.5$).

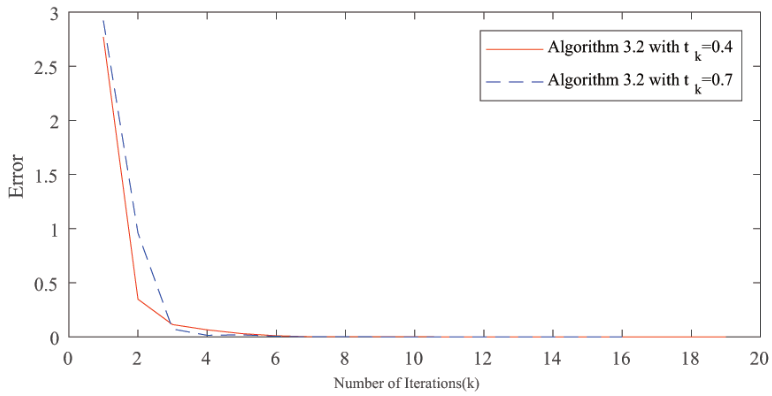

Figure 2. The performance of Algorithms 3.1 (Case $x^0 = (-5, 2, -3)'$).

In the view of Table 1 and Figure 1, it is easy to observe that our algorithms have better performance than the general CQ Algorithm. It appears that our algorithms need fewer iterations and converge more quickly than the general CQ Algorithm. In addition, by observing Figure 2, we find that the magnitude of inertial term has a certain effect on the number of iterations.

Example 4.2 Let $A = (a_{ij})_{M\times N}$, $a_{ij} \in (0, 1)$ be a random matrix, M,N be two positive integers. $C = \{x \in \Re^N | \sum_{l=1}^{N} {x_l}^2 \leq r^2\}$, $Q = \{x \in \Re^M | x \leq b\}$. To ensure the existence of the solution of the problem, the vector b is generated by using the following way: Given a random N-dimensional negative vector (each component is negative) $z \in C$, $r = \|z\|$, taking $b = Az$. Find $x \in C$ with $Ax \in Q$.

The numerical results of Example 4.2 can be seen from Table 2. In the table, k denotes the number of iterations, s denotes the computing time.

Table 2. Results for Example 4.2.

M, N	t_k	Algorithm 3.1	Algorithm 3.2	CQ Algorithm
M=20, N=10 $x^0 = (1, 1, 1, 0, 0, \cdots, 0)'$	0.2	$k = 63; s = 0.0938$	$k = 67; s = 0.0938$	$k = 308; s = 0.1250$
	0.4	$k = 61; s = 0.0625$	$k = 47; s = 0.0625$	
	0.6	$k = 59; s = 0.0625$	$k = 30; s = 0.0625$	
M=100, N=90 $x^0 = (1, 1, 1, 1, 1, 0, \cdots, 0)'$	0.1	$k = 44; s = 0.1875$	$k = 23; s = 0.1250$	$k = 167; s = 0.3438$
	0.2	$k = 43; s = 0.1875$	$k = 19; s = 0.1250$	
	0.4	$k = 42; s = 0.1875$	$k = 16; s = 0.0938$	

As can be seen from the numerical results in Table 2, in the case of higher dimensions, our algorithms are still effective, and they still converge faster than the general CQ algorithm.

§5 Some concluding remarks

This paper presentes two hybrid inertial CQ projection algorithms with different rules of stepsize selection for solving SFP. Based on the hybrid CQ projection algorithm, we add the inertial term in the first projection step to accelerate the convergence of the algorithm. The main difference between Algorithm 3.1 and Algorithm 3.2 is the projection region obtained by different line-search methods in the second projection step. According to our convergence theory and also confirmed by numerical simulations, we can see that our proposed algorithms have better convergence properties than the general CQ algorithm.

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