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Some results on derivations of MV-algebras

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Abstract. In this paper, we review some of their related properties of derivations on MV-algebras and give some characterizations of additive derivations. Then we prove that the fixed point set of Boolean additive derivations and that of their adjoint derivations are isomorphic. In particular, we prove that every MV-algebra is isomorphic to the direct product of the fixed point set of Boolean additive derivations and that of their adjoint derivations. Finally we show that every Boolean algebra is isomorphic to the algebra of all Boolean additive (implicative) derivations. These results also give the negative answers to two open problems, which were proposed in [Fuzzy Sets and Systems, **303**(2016), 97-113] and [Information Sciences, **178**(2008), 307-316].

§1 Introduction

MV-algebras were introduced by Chang for the purpose of providing an algebraic proof of the completeness theorem of infinite-valued propositional logics [2]. In the present paper, the infinite-valued logic refers to that proposed by Łukasiewicz and Tarski [14] with truth values in the interval [0,1] of real numbers. Thus, in a certain sense, MV-algebras stand in relation to multiple-valued logic as Boolean algebras do to classical logic. Moreover, Chang [3] established a bijective correspondence between the linearly ordered MV-algebras and the linearly ordered abelian ℓ-groups with a strong unit and used this result in order to obtain an algebraic proof for the completeness theorem of Łukasiewicz propositional logic in another way. For a detailed consideration of MV-algebras and their related results, we refer to [2-4,17-22].

The notion of derivations, introduced from the analytic theory, is helpful for studying algebraic structures and properties in algebraic systems. In 1957, Posner [15] introduced the notion of derivations in a prime ring $(R, +, \cdot)$, which is a map $d : R \to R$ satisfying the following two conditions:

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$$\text{(i) } d(x+y) = d(x) + d(y), \text{ (ii) } d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$$

for all $x, y \in R$. Subsequently, a number of research articles have appeared on derivations in the theory of rings and references there in [1,5,12]. Inspired by derivations on rings, Jun et al [13] applied the notion of derivations to BCI-algebras and gave some characterizations of p-semisimple BCI-algebras. In the past few years, Xin [24] introduced the concept of derivations in a lattice $(L, \wedge, \vee)$, which is a map $d : L \to L$ satisfying the two conditions:

$$\text{(i) } d(x \vee y) = d(x) \vee d(y), \text{ (ii) } d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$$

for all $x, y \in L$, and characterized modular lattices and distributive lattices by isotone derivations; Ghorbain et al [10] introduced the notions of additive derivations on an MV-algebra $(L, \oplus, *, 0)$, which is a map $d : L \to L$ satisfying the following two conditions:

$$\text{(i) } d(x \oplus y) = d(x) \oplus d(y), \text{ (ii) } d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y))$$

for any $x, y \in L$, and proved that an additive derivation of a linearly ordered MV-algebra is isotone. In order to get the general algebraic results of derivation on t-norm based logical algebras, He [11] investigated derivations on residuated lattices and characterized Heyting algebras in terms of derivations, and proved that the fixed point set of principal ideal derivations and that of their adjoint derivations are lattice isomorphic.

It always been known that ideals play a central role in studying logical algebras, and so the relationship between derivations and ideals is an important research topic to study. For example, Xin proved the fixed point set of a lattice derivation is an ideal in lattices and proposed an open problem related to them as follow: **(OP1) for any lattice ideal I of a lattice L, whether there exists a derivation d such that $Fix_d(L) = I$.** They gave the positive answer to the (OP1) under certain conditions [23, Theorem 4.13]. Inspired by this, He further clarified the relationship between lattice ideal and derivations in residuated lattices, and proposed another open problem is similar to that of (OP1), that is, **(OP2) for any lattice ideal I of a general residuated lattice L, whether there exists a derivation d such that** $Fix_d(L) = I$. He also gave the positive answer to the (OP2) under certain conditions in Heyting algebras in [24, Theorem 4.14]. Unfortunately, none of the above-mentioned open problems have been completely solved so far.

In this paper, we will further study the derivations of MV-algebras. One of our aims is to obtain some representations and characterizations of MV-algebras and Boolean algebras via derivations. In particular, we will obtain the following main results:

(1) Every MV-algebra is isomorphic to the direct product of the fixed point set of Boolean additive derivations and that of their adjoint derivations, which shows that Boolean derivations coincide with direct product decompositions of MV-algebras (See **Theorem 5.16** and **Corollary 5.17**).

(2) The fixed point set of Boolean additive derivations and that of their adjoint derivations in MV-algebras are **isomorphic**. Indeed, this result essentially goes a step further of the following result in [24, Theorem 4.10] that the fixed point set of principal ideal derivations and that of their adjoint derivations are **lattice isomorphic**. (See **Theorem 5.15**)

(3) Every Boolean algebra is isomorphic to the algebra of all Boolean additive (implicative) derivations. Indeed, this result essentially go a step further of the following result in [24, Theorem 3.29] that every distributive lattice is isomorphic to the algebra of all principle derivations in distributive lattices. (See **Theorem 5.21** and **Theorem 5.22**)

The other aim of us is to further study the relationship between derivations and ideals of MV-algebras. Indeed, we will obtain the following main results:

(4) We also give a negative answer to the above two open problems (OP1) and (OP2). (See **Remark 4.7**)

The paper is organized as follows: In Section 2, we review some basic definitions and results about MV-algebras. In Section 3, we further study derivations in MV-algebras. In Section 4, we further clarify the relationship between ideal and derivations in MV-algebras and give a negative answer to open problems in [11,24]. In Section 5, we obtain some representations and characterizations of MV-algebras and Boolean algebras via Boolean derivations and their adjoint derivations.

§2 Preliminary

In this section, we summarize some definitions and results about MV-algebras, which will be used in the following sections.

An algebra $(L, \oplus, ^*, 0)$ of type $(2, 1, 0)$ is called an *MV-algebra* if it satisfies the following conditions:

(1) $(L, \oplus, 0)$ is a commutative monoid,

(2) $(x^*)^* = x$,

(3) $0^* \oplus x = 0^*$,

(4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,

for any $x, y \in L$.

We shall adopt the usual conventions for MV-algebras: * operation is more binding than $\oplus$. On each MV-algebra L, we define the constant 1 and the operations $\odot$, $\ominus$, $\rightarrow$ as follows:

$$1 = 0^*,\ x \odot y = (x^* \oplus y^*)^*,\ x \ominus y = x \odot y^* \text{ and } x \rightarrow y = x^* \oplus y$$

for any $x, y \in L$. We define $x \leq y$ if and only if $x^* \oplus y = 1$. It follows that $\leq$ is a partial order, called the *natural order* of L. The natural order determines a lattices structure, in which,

$$x \vee y = (x \odot y^*) \oplus y, \quad x \wedge y = x \odot (x^* \oplus y)$$

for any $x, y \in L$. The structure $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. We say that the MV-algebra L is linearly ordered if the lattice $(L, \wedge, \vee, 0, 1)$ is linearly ordered. An MV-algebra is a Boolean algebra if its satisfies the additional equation $x \oplus x = x$ (or $x \odot x = x$) for any $x \in L$, and denote by $B(L) = \{x \in L \mid x \oplus x = x\}$ be the set of all idempotent elements of L. As MV-algebras form a variety, the notions of homomorphism, subalgebra are just the particular cases of the corresponding universal algebraic notions [2,3,4].

Example 2.1. ([3]) Let $L = [0, 1]$ be the real unit interval. If we define

$$x \oplus y = \min\{1, x+y\}, \quad x^* = 1 - x$$

for any $x, y \in L$, then $(L, \oplus, ^*, 0)$ is an MV-algebra. Also, for each number $n \geq 2$, then n-element set

$$S_n = \{0, \tfrac{1}{n-1}, \tfrac{2}{n-1}, \cdots, \tfrac{n-2}{n-1}, 1\}$$

is a subalgebra of an MV-algebra L.

Proposition 2.2. ([3,4]) *In any MV-algebra L, the following properties hold: for all $x, y, z \in L$,*

(1) $x \oplus x^* = 1$,
(2) $x \odot x^* = 0$,
(3) $x \leq y$ *if and only if* $x \to y = 1$,
(4) $x \odot y \leq x \wedge y$,
(5) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,
(6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(7) $x \leq y$ *implies* $x \odot z \leq y \odot z$,
(8) $x \vee y = (x \to y) \to y = (y \to x) \to x$,
(9) $x \leq y \to x$,
(10) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
(11) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,
(12) $x \odot y \leq z$ *if and only if* $x \leq y \to z$,
(13) $x \ominus y \leq z$ *if and only if* $x \leq y \oplus z$.

Proposition 2.3. ([10]) *Let L be an MV-algebra and $e \in B(L)$. Then the following properties hold: for any $x, y \in L$,*

(1) $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$,
(2) $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$,
(3) $e \wedge (x \oplus y) = (e \wedge x) \oplus (e \wedge y)$,
(4) $e \vee (x \oplus y) = (e \vee x) \oplus (e \vee y)$,
(5) $e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)]$,
(6) $e \to (x \to y) = (e \to x) \to (e \to y)$.

Let L be an MV-algebra. A nonempty subset I of L is called an *ideal* of L if it satisfies: (1) $x, y \in I$ implies $x \oplus y \in I$; (2) $x \in I$, $y \in L$ and $y \leq x$ imply $y \in I$. An ideal I of L is *proper* if $I \neq L$. A proper ideal I of L is called a prime ideal if for any $x, y \in L$ such that $x \wedge y \in I$, then $x \in I$ or $y \in I$. A nonempty subset I of L is called a *lattice ideal* of L if it satisfies: (i) for all $x, y \in I$, $x \vee y \in I$; (ii) for all $x, y \in L$, if $x \in I$ and $y \leq x$, then $y \in I$, that is, a lattice ideal of an MV-algebra L is the notion of ideal in the underlying lattice.

For any nonempty subset X of L, the smallest lattice ideal containing X is called the lattice ideal generated by X. The lattice ideal generated by X will be denoted by $(X]$. In particular, if $X = \{t\}$, we write $(t]$ for $(\{t\}]$, $(t]$ is called a principal lattice ideal of L. It is easy to check that $(t] = \downarrow t = \{y \in L | y \leq t\}$. From Principle of Duality, we can define the lattice filter and the principal lattice filter of an MV-algebra L.

Let I be an ideal of an MV-algebra L. We define a binary relation θ_I on L as follows: for any $x, y \in L$, $(x, y) \in \theta_I$ if and only if $(x \ominus y) \oplus (y \ominus x) \in I$. Then, θ_I is a congruence relation on L. Thus, the binary relation $\leq$ on L/I which is defined by $[x] \leq [y]$, if and only if $x \to y \in I$, is an order relation on L/I. For any $x \in L$, let $[x]_I$ be the equivalence class $[x]_{\theta_I}$ and $L/I = L/\theta_I = \{[x]_I | x \in L\}$. Then L/I becomes an MV-algebra with the natural operations induced from those of L ([6,7,16]).

Definition 2.4. ([9]) Given ordered sets E, F and order-preserving mappings $f : E \longrightarrow F$ and $g : F \longrightarrow E$, we say that the pair (f, g) establishes a *Galois connection* between E and F if $fg \geq id_F$ and $gf \leq id_E$.

§3 Some derivations of MV-algebras

In this section, we further study derivations on MV-algebras and give some characterizations of additive derivations.

Definition 3.1. ([10]) Let L be an MV-algebra. A map $d : L \longrightarrow L$ is called a *derivation* on L if it satisfies the following condition: for any $x, y \in L$,

$$d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y)).$$

Example 3.2. ([10]) Let $L = \{0, a, b, c, d, 1\}$ and operations $\oplus$ and $*$ be defined as follows:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	d	c	1	1
b	b	d	b	1	d	1
c	c	c	1	c	1	1
d	d	1	d	1	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(\{0, a, b, c, d, 1\}, \oplus, *, 0)$ is an MV-algebra. Define a map $d : L \longrightarrow L$ by

$$d(x) = \begin{cases} 0, & x = 0, a, c \\ b, & x = b, d, 1. \end{cases}$$

It is verified that d is a derivation on an MV-algebra L.

Proposition 3.3. ([10]) *Let L be an MV-algebra and d be a derivation on L. Then we have: for any $x \in L$,*

(1) $d(0) = 0$,
(2) $d(1) \in B(L)$,
(3) $d(x) \odot x^* = x \odot d(x^*) = 0$,
(4) $d(x) \leq x$,
(5) $d(x) = d(x) \oplus (x \odot d(1))$.

Definition 3.4. ([10]) Let L be an MV-algebra and d be a derivation on L.

(1) d is called an *isotone derivation* provided that $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in L$,

(2) d is called an *additive derivation* provided that $d(x \oplus y) = d(x) \oplus d(y)$ for all $x, y \in L$.

Example 3.5. ([10]) Let S_4 be the MV-algebra in Example 2.1. Define a map $d : S_4 \longrightarrow S_4$ by

$$d(x) = \begin{cases} 0, & x = 0, \frac{1}{3}, 1 \\ \frac{1}{3}, & x = \frac{2}{3}. \end{cases}$$

Then d is a derivation on S_4, but it is not an additive derivation on L, since

$$d(\tfrac{1}{3} + \tfrac{2}{3}) = d(1) = 0 \neq \tfrac{1}{3} = d(\tfrac{1}{3}) + d(\tfrac{2}{3}).$$

Moreover, d is not an isotone derivation on L, since

$$\tfrac{2}{3} \leq 1,\ d(\tfrac{2}{3}) = \tfrac{1}{3} \geq 0 = d(1).$$

Example 3.6. Let S_n be the MV-algebra in Example 2.1. Define a map $d : S_n \to S_n$ as follows: for all $x \in S_n$,

$$d(x) = \begin{cases} \frac{1}{n-1}, & x = 1 \\ \frac{1}{n-1} \odot x, & x \neq 1 \end{cases}$$

It is verified that d is not only an additive, but also an isotone derivation on S_n.

Proposition 3.7. *Let L be an MV-algebra and d be an additive derivation on L. Then we have: for any $x, y \in L$,*

(1) *d is an isotone derivation,*

(2) *$d(x) = d(1) \odot x$,*

(3) *$d(d(x)) = d(x)$,*

(4) *$d(x) \in B(L)$,*

(5) *$d(d(x) \to d(y)) = d(x \to y)$,*

(6) *$Fix_d(L) = d(L)$, where $Fix_d(L) = \{x \in L | d(x) = x\}$,*

(7) *if $d(L) = L$, then $d = id_L$,*

(8) *$Ker(d)$ is an ideal of L, where $Ker(d) = \{x \in L | d(x) = 0\}$.*

Proof. (1) If $x \leq y$, then $y = x \vee y = x \oplus (x^* \odot y)$. By Definition 3.4(2), we have

$$d(y) = d(x \oplus (x^* \odot y)) = d(x) \oplus d(x^* \odot y) \geq d(x),$$

which implies $d(x) \leq d(y)$.

(5) It follows from Propositions 2.3(5), 3.3(2) and (2) that

$$d(d(x) \to d(y)) = d(1) \odot [(d(1) \odot x) \to (d(1) \odot y)] = d(1) \odot (x \to y) = d(x \to y),$$

which implies $d(d(x) \to d(y)) = d(x \to y)$ for any $x, y \in L$. □

As a consequence of Propositions 3.3 and 3.7, we have the following fact.

Theorem 3.8. *Let L be an MV-algebra and d be a derivation on L. Then the following statements are equivalent: for any $x, y \in L$,*

(1) *d is an additive derivation,*

(2) *d is an isotone derivation,*

(3) $d(x) \leq d(1)$,

(4) $d(x) = d(1) \odot x$,

(5) $d(x \odot y) = d(x) \odot y = x \odot d(y)$,

(6) $d(x \wedge y) = d(x) \wedge d(y)$,

(7) $d(x \vee y) = d(x) \vee d(y)$,

(8) $d(x \odot y) = d(x) \odot d(y)$,

(9) $d(x) \leq y$ *if and only if* $d(x) \leq d(y)$,

(10) $d(x) \to d(y) = d(x) \to y$.

Proof. (1) $\Rightarrow$ (2) It follows from Proposition 3.7(1)

(2) $\Rightarrow$ (3) It is straightforward.

(3) $\Rightarrow$ (4) It follows from Proposition 3.7(2).

(4) $\Rightarrow$ (1) From Propositions 2.3(3),3.3(2) and Proposition 3.7(2), we have

$$d(x \oplus y) = d(1) \odot (x \oplus y) = (d(1) \odot x) \oplus (d(1) \odot y) = d(x) \oplus d(y).$$

(4) $\Rightarrow$ (5) From (4), we have

$$d(x \odot y) = d(1) \odot x \odot y = x \odot (d(1) \odot y) = x \odot d(y).$$

(5) $\Rightarrow$ (4) Taking $y = 1$ in (5), we have $d(x) = d(1) \odot x$.

(4) $\Rightarrow$ (6) From (4) and Proposition 3.3(2), we have

$$d(x \wedge y) = d(1) \odot (x \wedge y) = d(1) \wedge (x \wedge y) = d(1) \wedge d(1) \wedge x \wedge y = d(x) \wedge d(y).$$

(4) $\Rightarrow$ (7) From (4) and Proposition 3.3(2), we have

$$d(x\vee y) = d(1)\odot(x\vee y) = d(1)\wedge(x\vee y) = (d(1)\wedge x)\vee(d(1)\wedge y) = (d(1)\odot x)\vee(d(1)\odot y) = d(x)\vee d(y).$$

(7) $\Rightarrow$ (2) Let $x \leq y$, we have $x \vee y = y$. Then it follows from (7), we have

$$d(y) = d(x \vee y) = d(x) \vee d(y) \geq d(x).$$

(4) $\Rightarrow$ (8) From (4) and Proposition 3.3(2), we have

$$d(x \odot y) = d(1) \odot (x \odot y) = d(1) \odot d(1) \odot x \odot y = d(x) \odot d(y).$$

(6) $\Rightarrow$ (4), (7) $\Rightarrow$ (4), (8) $\Rightarrow$ (4) are straightforward.

(2) $\Rightarrow$ (9) If $d(x) \leq y$, then $d(d(x)) \leq d(y)$, and hence by Proposition 3.7(3), we have $d(x) \leq d(y)$. Conversely, if $d(x) \leq d(y)$, from Proposition 3.3(4), then $d(x) \leq d(y) \leq y$ for all $x, y \in L$.

(9) $\Rightarrow$ (2) If $x \leq y$, then $d(x) \leq x \leq y$, and hence $d(x) \leq d(y)$ for all $x, y \in L$.

(2) $\Rightarrow$ (10) Assume that d is an isotone derivation on L. From $d(y) \leq y$ for any $y \in L$, it follows that $d(x) \to d(y) \leq d(x) \to y$. On the other hand, let $t \leq d(x) \to y$ for all $t \in L$, we can obtain $d(x) \odot t \leq y$. Since d is an isotone derivation, we have $d(d(x) \odot t) \leq d(y)$ for all

$x, y, t \in L$. From $d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y))$, we get $d(x) \odot y \leq d(x \odot y)$ for all $x, y \in L$. It follows $d(d(x)) \odot t \leq d(d(x) \odot t)$. By Proposition 3.7(3), we have $d(x) \odot t \leq d(d(x) \odot t) \leq d(y)$. Hence $t \leq d(x) \to d(y)$ for all $t \in L$, which implies $d(x) \to y \leq d(x) \to d(y)$ for all $x, y \in L$. Therefore, we obtain $d(x) \to d(y) = d(x) \to y$ for any $x, y \in L$.

(10) $\Rightarrow$ (2) Assume that $d(x) \to d(y) = d(x) \to y$ for all $x, y \in L$. For any $x, y \in L$, let $x \leq y$, by Proposition 3.3(4), we have $d(x) \odot 1 = d(x) \leq x \leq y$. It follows that $1 \leq d(x) \to y = d(x) \to d(y)$, which implies $d(x) \leq d(y)$ for any $x, y \in L$. □

Remark 3.9. (1) Theorem 3.8 shows that isotone derivations are equivalent to additive derivations on MV-algebras.

(2) Example 3.2 shows that an additive derivation is not a homomorphism on an MV-algebra in general, since $d(a^*) = b \neq 1 = (d(a))^*$.

(3) Example 3.2 shows that the fixed point set of an additive derivation d is not a subalgebra of an MV-algebra L in general, since $0^* = 1 \notin \{0, b\} = Fix_d(L)$.

(4) Theorem 3.8(4) shows that every additive derivation d on an MV-algebra L is completely defined by the image $d(1)$ of the 1.

Theorem 3.10. *Let d be an additive derivation on MV-algebra L. Then*

$$(Fix_d(L), \oplus, \neg, 0)$$

is an MV-algebra, where $\neg x = d(x^*) = (x \oplus (d(1))^*)^*$ for any $x \in Fix_d(L)$.

Proof. It follows from Definition 3.4(2) that $Fix_d(L)$ is closed under $\oplus$.

Also, for all $x \in Fix_d(L)$, from Propositions 3.3(2) and 3.7(2), we have

$$d(\neg x) = d(d(x^*)) = d(1) \odot d(1) \odot x^* = d(1) \odot x^* = d(x^*) = \neg x,$$

which implies that $Fix_d(L)$ is closed under $\neg$.

(MV1) It follows from Definition 3.4(2).

(MV2) From Theorem 3.8(3), we have

$$\neg\neg x = (\neg x \oplus (d(1))^*)^* = (x \oplus (d(1))^*) \odot d(1) = x \wedge d(1) = d(x) \wedge d(1) = x,$$

which implies $\neg\neg x = x$ for all $x \in Fix_d(L)$.

(MV3) From Theorem 3.8(3) and Proposition 3.3(2), we have

$$x \oplus \neg 0 = x \oplus d(1) = d(x) \oplus d(1) = d(x) \vee d(1) = d(1),$$

which implies $x \oplus \neg 0 = d(1)$ for all $x \in Fix_d(L)$.

(MV4) From Theorem 3.8(4), we have

$$\begin{aligned}
\neg(\neg x \oplus y) \oplus y &= (\neg x \oplus y \oplus (d(1))^*)^* \oplus y \\
&= ((x \oplus (d(1))^*)^* \oplus y \oplus (d(1))^*)^* \oplus y \\
&= ((x \oplus (d(1))^*) \odot d(1) \odot y^*) \oplus y \\
&= ((x \wedge d(1)) \odot y^*) \oplus y \\
&= x \vee y,
\end{aligned}$$

which implies $\neg(\neg x \oplus y) \oplus y = x \vee y$ for any $x, y \in Fix_d(L)$. □

Proposition 3.11. *Let L be an MV-algebra and d an additive derivation on L. Then we have the following properties:*

(1) $d : L \longrightarrow Fix_d(L)$ *is a surjective homomorphism,*

(2) $\bar{d} : L/Ker(d) \longrightarrow Fix_d(L)$ *is an isomorphism.*

Proof. It follows from Theorem 3.8 and Proposition 3.10.

(2) It is verified that $Ker(d)$ is an ideal of L. If $x \sim_{Ker(d)} y$, then $(x \ominus y) \oplus (y \ominus x) \in Ker(d)$, and hence

$$d((x \ominus y) \oplus (y \ominus x)) = d(x \ominus y) \oplus d(y \ominus x) = 0.$$

Then it follows from Theorem 3.8(5) that $d(x \ominus y) = d(x \odot y^*) = d(x) \odot y^* = 0$, which implies $d(x) \leq y$. Similarity, we can prove $d(y) \leq x$. Then by Theorem 3.8(10), $d(x) = d(y)$. Thus, $\bar{d}$ is well defined. Moreover, it follows from (1) that $\bar{d} : L/Ker(d) \longrightarrow Fix_d(L)$ is an isomorphism. □

Corollary 3.12. Let L be an MV-algebra and $d : L \to L$ be a map on L such that $d(L) \subseteq B(L)$. Then the following statements are equivalent: for any $x, y \in L$,

(1) d is an additive derivation on L,

(2) $d(x) = d(1) \odot x$,

(3) $d(x \odot y) = d(x) \odot y = x \odot d(y)$.

Proof. (1) ⇒ (2) It follows from Proposition 3.7(1).

(2) ⇒ (3) The proof is similar to that of Theorem 3.8 (4) ⇒ (5).

(3) ⇒ (1) The proof is similar to that of Theorem 3.8 (5) ⇒ (1). □

§4 Solution to open problems related to derivations

In this section, we further discuss the relationship between ideals and additive derivations on MV-algebras, and give a negative answer to the open problems (OP1) and (OP2).

Example 4.1. Let S_3 be the MV-algebra in Example 2.1. Now, we define a map $d : S_3 \longrightarrow S_3$ by

$$d(x) = \begin{cases} 0, & x = 0, 1 \\ \frac{1}{2}, & x = \frac{1}{2}. \end{cases}$$

Then d is a derivation on S_3, while is not an additive derivation ([9]). Also, $Fix_d(L) = \{0, \frac{1}{2}\}$ is not an ideal of L since $\frac{1}{2} \oplus \frac{1}{2} = 1 \notin Fix_d(L)$.

Proposition 4.2. *Let L be an MV-algebra and d be an additive derivation on L. Then $Fix_d(L)$ is an ideal of L.*

Proof. (1) It follows from Definition 3.4(2) that $Fix_d(L)$ is closed under $\oplus$.

(2) Let $x \leq y$ and $y \in Fix_d(L)$. Then

$$\begin{aligned} d(x) = d(x \wedge y) &= d((x \oplus y^*) \odot y) \\ &= (d(x \oplus y^*) \odot y) \oplus ((x \oplus y^*) \odot d(y)) \\ &= (d(x \oplus y^*) \odot y) \oplus ((x \oplus y^*) \odot y) \\ &= (d(x \oplus y^*) \odot y) \oplus (x \wedge y) \\ &= (d(x \oplus y^*) \odot y) \oplus x, \end{aligned}$$

which implies $x \leq d(x)$, and hence $x \in Fix_d(L)$. □

However, the converse of Proposition 4.2 may not hold in general.

Example 4.3. Let L be the MV-algebra in Example 3.5. Then $Fix_d(L) = \{0\}$ is an ideal of L, but d is not an additive derivation on L.

Inspired by Proposition 4.2, we naturally ask that **whether there exists an additive derivation d such that $Fix_d(L) = I$ for given ideal I in an MV-algebra L.** For the similar question regard to the lattice and residuated lattice, Xin gives the positive answer under certain conditions as follows in [11,23].

Proposition 4.4. *Let L be a lattice (Heyting algebra) and I be a non-void prime ideal of L. Then there exists a derivation d such that $Fix_d(L) = I$.*

Proof. Let I be a non-void prime ideal of L. Then there exists a map $d : L \longrightarrow L$ defined by

$$d(x) = \begin{cases} x, & x \in I \\ x \wedge t, & x \in L \backslash I, \text{ where } t \in I, \end{cases}$$

is a derivation satisfying $Fix_d(L) = I$. Indeed, if $x, y \in I$, then we can see that

$$d(x \wedge y) = x \wedge y = (x \wedge y) \vee (x \wedge y) = (dx \wedge y) \vee (x \wedge dy).$$

If $x \in I, y \in L\ I$, then $x \wedge y \leq x$ and so $x \wedge y \in I$. Hence $d(x \wedge y) = x \wedge y$, $(dx \wedge y) \vee (x \wedge dy) = (x \wedge y) \vee (x \wedge y \wedge a) = x \wedge y$, which shows that $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$.

If $x, y \in L\ I$, then $x \wedge y \in L\ I$ since I is prime. Hence $d(x \wedge y) = x \wedge y \wedge a$, $(dx \wedge y) \vee (x \wedge dy) = (x \wedge a \wedge y) \vee (x \wedge y \wedge a) = x \wedge y \wedge a$. By the above argument, we can see that d is a derivation. Clearly $Fix_d(L) = I$. □

The following example shows that $t \in I$ in Proposition 4.4 is necessary.

Example 4.5. Let $L = \{0, a, b, c, 1\}$, where $0 \leq a \leq b, c \leq 1$. Define operation $\rightarrow$ on L as follows:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

It is verified that $(\{0, a, b, c, 1\}, \wedge, \vee, \rightarrow, 0, 1)$ is a Heyting algebra and $I = \{0, a, b\}$ is a non-void prime ideal of L. Then by Proposition 4.4 there exists a map $d : L \rightarrow L$ defined by

$$d(x) = \begin{cases} x, & x \in I \\ x \wedge c, & x \in L \backslash I, \end{cases}$$

is a derivation on L. But $Fix_d(L) = \{0, a, b, c\} \neq I$.

The following example shows that the condition "prime ideal" in Proposition 4.4 is necessary.

Example 4.6. ([22]) Let $L = [0, 1]$ and $I = (0, 1)$. Then $(L, \leq)$ is a lattice and I is an ideal of L, but it is not prime, where $\leq$ is the ordinary order. Moreover, we can see that there is not any isotone derivation d such that $Fix_d(L) = I$.

The following remark give a negative answer to the two open problems (OP1) and (OP2).

Remark 4.7. The answer to the above open problems are easily seen to be negative even for Boolean algebras. In particular, let L be the finite-cofinite Boolean algebra on an infinite set and I the (prime) ideal of finite sets. If $I = Fix_d(L)$ for some additive derivation d, which is equivalent to the isotone lattice derivation on L, then by Theorem 3.8(9) for each finite set X and infinite set Y we have $X \subseteq d(Y)$ if and only if $X \subseteq Y$, which implies that $d(X)$ have to be the smallest finite set below Y, but no such set exists. It is well known that Boolean algebras are a subclass of MV-algebras, residuated lattices and lattices, and additive derivations are a special classes of derivations and lattice derivations. Then we give a negative answer to the two open problems (OP1) and (OP2).

§5 Representations of Boolean algebras based on derivations

In this section, we give some representations of Boolean algebras via two special kinds of derivations, which are defined as follows:

(1) $d_a : L \rightarrow L$ such that $d_a(x) = a \odot x$, for all $x \in L$,

(2) $g_a : L \rightarrow L$ such that $g_a(x) = a \oplus x$, for all $x \in L$.

Example 5.1. Let S_3 be the MV-algebra and d be a derivation in Example 4.1. Then d is not an additive derivation on S_3, since $d(\frac{1}{2} \oplus \frac{1}{2}) = d(1) = 0 \neq 1 = d(\frac{1}{2}) \oplus d(\frac{1}{2})$.

Theorem 5.2. *Let L be an MV-algebra. Then the following statements are equivalent:*

(1) L is a Boolean algebra,

(2) d_a is an additive derivation on L, for all $a \in L$.

Proof. (1) $\Rightarrow$ (2) If L is a Boolean algebra, by Proposition 2.2, then

$$d_a(x \odot y) = a \odot (x \odot y) = (a \odot x) \odot y = d_a(x) \odot y,$$
$$d_a(x \oplus y) = (x \oplus y) \odot a = (x \vee y) \odot a = (x \odot a) \vee (y \odot a) = d_a(x) \vee d_a(y) = d_a(x) \oplus d_a(y),$$

for any $x, y \in L$, which implies that d_a is an additive derivation on L.

(2) $\Rightarrow$ (1) If d_a is an additive derivation on MV-algebra L, then

$$a = d_a(1) = d_a(1 \odot 1) = (d_a(1) \odot 1) \oplus (1 \odot d_a(1)) = a \oplus a,$$

for any $a \in L$, which implies $L \subseteq B(L)$. Thus, L is a Boolean algebra. $\square$

Remark 5.3. Theorem 5.2 shows that d_a is an additive derivation on a Boolean skeleton of an MV-algebra. Hence we call d_a a Boolean additive derivation on an MV-algebra. We denote the set of all Boolean additive derivations on MV-algebras by $D(L)$, that is, $D(L) = \{d_a | a \in B(L)\}$.

Now, we introduce the adjoint derivation of Boolean additive derivations on MV-algebras.

Definition 5.4. Let L be an MV-algebra. A map $g : L \to L$ is called an *implication derivation* on L if it preserves $\to$ and satisfies the following condition: for any $x, y \in L$,

$$g(x \to y) = (g(x) \to y) \oplus (x \to g(y)).$$

Example 5.5. Let $L = \{0, a, b, 1\}$, where $0 \leq a, b \leq 1$. Define operations $\oplus$ and $*$ as follows:

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(\{0, a, b, 1\}, \oplus, *, 0)$ is an MV-algebra. Define a map $g : L \longrightarrow L$ by

$$g(x) = \begin{cases} a, & x = 0, a \\ 1, & x = b, 1. \end{cases}$$

It is verified that g is an implicative derivation on L, while g is not a homorphism on L, since $g(0) \neq 0$.

Theorem 5.6. *Let L be an MV-algebra. Then the following statements are equivalent:*

(1) L is a Boolean algebra,

(2) g_a is an implicative derivation on L, for all $a \in L$.

Proof. (1) $\Rightarrow$ (2) If L is a Boolean algebra and $a \in L$, then

$$g_a(x \to y) = g_a(x) \to g_a(y).$$

In particular, by Proposition 2.3(6), we have

$$a^* \to (x \to y) = (a^* \to x) \to (a^* \to y),$$

which is equivalent to

$$a \oplus (x \to y) = (a \oplus x) \to (a \oplus y),$$

which implies $g_a(x \to y) = g_a(x) \to g_a(y)$ for any $a \in L$.

Then by Proposition 2.3, we have

$$\begin{aligned} g_a(x \to y) &= a \oplus (x \to y) \\ &= x^* \oplus (a \oplus y) \\ &= x^* \oplus g_a(y) \\ &= ((g_a(x))^* \vee x^*) \oplus (g_a(y) \vee y) \\ &= ((g_a(x))^* \oplus x^*) \oplus (g_a(y) \oplus y) \\ &= ((g_a(x))^* \oplus y) \oplus (x^* \oplus g_a(y)) \\ &= (g_a(x) \to y) \oplus (x \to g_a(y). \end{aligned}$$

which implies that g_a is an implicative derivation on L.

(2) $\Rightarrow$ (1) If g_a is an implicative derivation on MV-algebra L, then

$$g_a(x \to y) = g_a(x) \to g_a(y)$$

for all $x, y \in L$. Taking $x = a^*$ and $y = a^* \odot a^*$ in the above equation, we have

$$\begin{aligned} g_a(x \to y) &= a \oplus (x \to y) \\ &= a^* \to (a^* \to a^* \odot a^*) \\ &= a^* \odot a^* \to a^* \odot a^* \\ &= 1, \end{aligned}$$

and

$$g_a(x) \to g_a(y) = (a^* \to a^*) \to (a^* \to a^* \odot a^*) = a^* \to a^* \odot a^*,$$

which implies $a^* \leq a^* \odot a^*$, and hence $a^* = a^* \odot a^*$, that is, $a \oplus a = a$. □

Remark 5.7. Theorem 5.6 shows that g_a is an implicative derivation on a Boolean skeleton of an MV-algebra. Hence we call g_a a Boolean implicative derivation on an MV-algebra. We denote the set of all Boolean implicative derivations on MV-algebras by $G(L)$, that is, $G(L) = \{g_a | a \in B(L)\}$.

Definition 5.8. A Boolean additive derivation μ_a is called *residuated* if there exists a Boolean implicative derivation ν_a such that the pair (μ_a, ν_a) forms a Galois connection.

The Boolean implicative derivation ν_a is called the *adjoint derivation* of the Boolean additive derivation μ_a.

If the Boolean additive derivation μ_a is residuated, with the properties of residuated map, then μ_a and it's adjoint derivation must be isotone. In particular, if the Boolean additive derivation μ_a has the adjoint derivation ν_a, then the adjoint of μ_a unique. Thus we shall denote this unique ν_a by μ_a^*.

Example 5.9. Let L be the MV-algebra in Example 5.5. Define two maps ν_a and μ_a on L as follows:

$$\mu_a = \begin{cases} 0, & x = 0, b \\ a, & x = a, 1 \end{cases}, \ \nu_a = \begin{cases} a, & x = 0, a \\ 1, & x = b, 1. \end{cases}$$

Then ν_a and μ_a are Boolean implicative and additive derivations on L, respectively, and (ν_a, μ_a) forms a Galois connection on L.

Theorem 5.10. *Let L be an MV-algebra and $a \in B(L)$. Then the map g_{a^*} is the adjoint derivation of the map d_a on L.*

Proof. Theorems 5.2 and 5.6 show that d_a and g_{a^*} are a Boolean additive derivation and a Boolean implicative derivation on L, respectively.

(1) If $x \le y$, then

$$g_{a^*}(x) = a \to x \le a \to y = g_{a^*}(y),\ d_a(x) = a \oplus x \le a \oplus y = d_a(y),$$

which implies g_{a^*} and d_a are isotone.

(2) $d_a(x) = a \odot x \le y$ if and only if $x \le a \to y = g_{a^*}(y)$.

Thus (d_a, g_{a^*}) forms a Galois connection on L. □

As a corollary of Theorem 3.10 and Theorem 5.10, we have the following result.

Corollary 5.11. Let L be an MV-algebra and $a \in B(L)$. Then $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ is an MV-algebra, where $\neg_1 x = d_a(x^*)$ for all $x \in L$.

Corollary 5.12. Let L be an MV-algebra and $a \in B(L)$. Then $(Fix_{d_{a^*}}(L), \oplus, \neg_2, 0, a^*)$ is an MV-algebra, where $\neg_2 x = d_{a^*}(x^*)$ for all $x \in L$.

Corollary 5.13. Let L be an MV-algebra and $a \in B(L)$. Then $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$ is an MV-algebra, where $x^{\circ_1} = g_a(x^*)$ for all $x \in L$.

Corollary 5.14. Let L be an MV-algebra and $a \in B(L)$. Then $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$ is an MV-algebra, where $x^{\circ_2} = g_{a^*}(x^*)$ for all $x \in L$.

We have seen in [11, Theorem 4.10] that the fixed point set of principal ideal derivations and that of their adjoint derivations in residuated lattices are **lattice isomorphic**. The strong version result for MV-algebras is as follows, i.e., the fixed point set of Boolean additive derivations and that of their adjoint derivations in MV-algebras are **isomorphic**.

Theorem 5.15. *Let L be an MV-algebra and $a \in B(L)$. Then $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ and $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$ are isomorphic.*

Proof. Let f:$Fix_{d_a}(L) \longrightarrow Fix_{g_{a^*}}(L)$ be defined by

$$f(x) = a^* \oplus x = a \to x,$$

for all $x \in Fix_{d_a}(L)$, $a \in L$. Then f is well defined.

(1) If $f(x) = f(y)$, then $a^* \oplus x = a^* \oplus y$, that is, $x = a \odot x$, $y = a \odot y$ for all $x, y \in Fix_{d_a}(L)$. Now, if $a \odot x \le x$, then $x \le a \to x = a^* \oplus x$, and hence $x \le a^* \oplus y = a \to y$, that is $x \le y$. If $x \le a \odot x$, then $y \le x$. So $x = y$, which implies that f is injective.

(2) If $x \in Fix_{g_{a^*}}(L)$, then $x = g_{a^*}(x) = a^* \oplus x$. So

$$f(a \odot x) = a^* \oplus (a \odot x) = a^* \oplus (a \odot (a^* \oplus x)) = a^* \oplus x = x,$$

which implies that f is surjective.

(3) From Propositions 2.2 and 2.3, we have

$$f(x \oplus y) = a^* \oplus (x \oplus y) = a^* \oplus a^* \oplus x \oplus y = (a^* \oplus x) \oplus (a^* \oplus y) = f(x) \oplus f(y),$$
$$f(\neg_1 x) = a^* \oplus (a \odot x^*) = a^* \oplus x = a^* \oplus (a^* \oplus x) = (a^* \oplus x)^{\circ_2} = (f(x))^{\circ_2},$$

which implies that f is an homomorphism.

It is easily to checked that $f^{-1}(x) = a \odot x$ is also an homomorphism.

Therefore, $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ and $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$ are isomorphic. □

The next result gives a representation of MV-algebra via Boolean derivations.

Theorem 5.16. *Every MV-algebra L is isomorphic to the direct product $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ and $(Fix_{d_{a^*}}(L), \oplus, \neg_2, 0, a^*)$, where $a \in B(L)$.*

Proof. Let $\tau : L \longrightarrow Fix_{d_a}(L) \times Fix_{d_{a^*}}(L)$ be defined by

$$\tau(x) = (x \odot a, x \ominus a),$$

for all $x, a \in L$. Then it follows from Theorem 5.2 that τ is well defined.

(1) It follows from Corollaries 3.12,5.11 and 5.12 that τ is a surjective homomorphism from L to $Fix_{d_a}(L) \times Fix_{d_{a^*}}(L)$.

(2) If $x_1 \in Fix_{d_a}(L)$ and $x_2 \in Fix_{d_{a^*}}(L)$, then for $x = x_1 \vee x_2$,

$$\tau(x) = (x_1, x_2).$$

Since $(L, \wedge, \vee)$ is a distributive lattice,

$$x = (x \wedge a) \vee (x \wedge a^*)$$

for all $x \in L$. Thus τ is injective.

(3) It is easy to verify that $\tau^{-1}(x, y) = x \vee y$ for all $(x, y) \in Fix_{d_a}(L) \times Fix_{d_{a^*}}(L)$, is also an MV-homomorphism from $Fix_{d_a}(L) \times Fix_{d_{a^*}}(L)$ to L. □

As a consequence of Corollaries 5.13 and 5.14, we have the following result.

Corollary 5.17. Every MV-algebra L is isomorphic to the direct product $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$ and $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$, where $a \in B(L)$.

Theorem 5.18. *Let L be an MV-algebra. Then the following statements are equivalent:*

(1) *L is a Boolean algebra,*

(2) *for any $a \in L$, $Fix_{d_a}(L) = (a]$,*

(3) *for any $a \in L$, $Fix_{g_a}(L) = [a)$.*

Proof. (1) $\Rightarrow$ (2) If L is a Boolean algebra, then $d_a(a) = a \odot a = a$, which implies $a \in Fix_{d_a}(L)$. Then it follows from Proposition 4.2 that $Fix_{d_a}(L)$ is an ideal of L, that is, for all $x \in L$, if $x \leq a$, then $x \in Fix_{d_a}(L)$. So $(a] \subseteq Fix_{d_a}(L)$. Conversely, if $x \in Fix_{d_a}(L)$, then $d_a(x) = a \odot x = a \wedge x = x$, which implies that $x \leq a$, that is, $x \in (a]$, and hence $Fix_{d_a}(L) \subseteq (a]$.

(2) $\Rightarrow$ (1) Suppose that $Fix_{d_a}(L) = (a]$ for all $a \in L$. Notice that $a \in (a]$, we have $a \in Fix_{d_a}(L)$. Then $d_a(a) = a$, that is, $a \odot a = a$ for all $a \in L$. So L is a Boolean algebra.

(1) $\Leftrightarrow$ (3) The proof is similar to that of (1) $\Leftrightarrow$ (2). □

The following theorems give a representation of Boolean algebras in terms of Boolean derivations. Namely, every Boolean algebra is isomorphic to the algebra of all Boolean derivations.

Theorem 5.19. *Let L be an MV-algebra. Then $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$ is a Boolean algebra, where*

$$(d_a \sqcup d_b)x = (d_a x) \vee (d_b x),\ (d_a \sqcap d_b)x = (d_a x) \wedge (d_b x),\ (d_a)^\star x = d_{a^*} x,$$

for any $d_a, d_b \in D(L), x \in L$.

Proof. (1) We show that $(D(L), \sqcup, \sqcap, d_0, d_1)$ is a bounded lattice with d_0 as the smallest element and d_1 as the greatest element.

For all $d_a, d_b \in D(L)$ and $x \in L$, we have

$$\begin{aligned}(d_a \sqcap d_b)(x) &= (d_a(x)) \wedge (d_b(x)) \\ &= (a \odot x) \wedge (b \odot x) \\ &= (a \wedge b) \odot x \\ &= d_{a\wedge b}(x),\end{aligned}$$

which implies $d_a \sqcap d_b \in D(L)$.

Also, we have

$$\begin{aligned}(d_a \sqcup d_b)(x) &= (d_a(x) \vee d_b(x)) \\ &= (a \odot x) \vee (b \odot x) \\ &= (a \vee b) \odot x \\ &= d_{a\vee b}(x),\end{aligned}$$

which implies $d_a \sqcup d_b \in D(L)$.

Moreover, for all $d_a \in D(L)$ and $x \in L$, we have

$$(d_a \sqcap d_0)(x) = d_a(x) \wedge d_0(x) = 0 = d_0(x)$$
$$(d_a \sqcup d_1)(x) = d_a(x) \vee d_1(x) = x = d_1(x),$$

which implies that d_0 and d_1 are the smallest element and greatest element in $D(L)$, respectively.

(2) We show that $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$ is a Boolean algebra.

For all $d_a \in D(L)$ and $x \in L$, we have

$$(d_a)^\star(x) = d_{a^*}(x) = a^* \odot x = d_{a^*}(x),$$

which implies $(d_a)^\star(x) = d_{a^*}(x)$. Then follow from $a^* \in B(L)$ that $d_a{}^\star \in D(L)$.

Also, we have

$$(d_a \sqcup (d_a)^\star)(x) = (d_a)(x) \vee d_{a^*}(x) = (a \odot x) \vee (a^* \odot x) = (a \vee a^*) \odot x = x = d_1(x),$$
$$(d_a \sqcap (d_a)^\star)(x) = (d_a)(x) \wedge d_{a^*}(x) = (a \wedge x) \wedge (a^* \wedge x) = (a \wedge a^*) \wedge x = 0 = d_0(x),$$

which implies $d_a \sqcup (d_a)^\star = d_1$, $d_a \sqcap (d_a)^\star = d_0$. $\square$

Theorem 5.20. *Let L be an MV-algebra. Then $(G(L), \cap, \cup, \bullet, g_0, g_1)$ is a Boolean algebra, where*

$$(g_a \cup g_b)x = (g_a x) \vee (g_b x), (g_a \cap g_b)x = (g_a x) \wedge (g_b x), (g_a)^\bullet x = g_{a^*} x,$$

for any $g_a, g_b \in G(L), x \in L$.

Proof. The proof is similar to that of Theorem 5.19. □

We have seen in [23, Theorem 3.29] that every distributive lattice is isomorphic to the algebra of all principle derivations. The corresponding results for Boolean algebra are as follows.

Theorem 5.21. *Every Boolean algebra* $(L, \wedge, \vee, *, 0, 1)$ *is isomorphic to* $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$.

Proof. Let $\phi : L \longrightarrow D(L)$ be defined by

$$\phi(a)(x) = a \wedge x,$$

for all $a, x \in L$. Then easily verified that ϕ is well defined, surjective and injective.

Also, for any $a, b \in L$, we have

$$\begin{aligned}
\phi(a \wedge b) = d_{a \wedge b} = (a \wedge b) \wedge x = (a \wedge x) \wedge (b \wedge x) = d_a \sqcap d_b = \phi(a) \sqcap \phi(b),\\
\phi(a \vee b) = d_{a \vee b} = (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = d_a \sqcup d_b = \phi(a) \sqcup \phi(b),\\
\phi(a^*) = d_{a^*} = a^* \wedge x = x \ominus a = (x \ominus a) \vee (x \ominus x) = (x \wedge a) \ominus x = (\phi(a))^\star.
\end{aligned}$$

which implies that ϕ is a homomorphism.

Thus $(L, \wedge, \vee, *, 0, 1)$ and $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$ are isomorphic. □

Theorem 5.22. *Every Boolean algebra* $(L, \wedge, \vee, *, 0, 1)$ *is isomorphic to* $(G(L), \cap, \cup, \bullet, g_0, g_1)$.

Proof. The proof is similar to that of Theorem 5.21. □

§6 Concluding remarks

The notion of derivations gives a tool for studying structures and properties in algebraic systems. In the paper, we obtain that the fixed point set of additive derivations is still an MV-algebra, and show that the fixed point set of Boolean additive derivations and that of their adjoint derivations are isomorphic. Then we obtain that some representation and characterization of MV-algebras via Boolean derivations and their adjoint derivations. This results also give negative answers to two open problems.

In [8], B. Gerla introduced a pair of semirings $(L, \vee, \odot, 0, 1)$ and $(L, \wedge, \oplus, 0, 1)$ on MV-algebra $(L, \oplus, *, 0)$ such that * is semirings isomorphism between the above two semirings. Hence in future, we will use the other operations to define different derivations on an MV-algebra and we will obtain their properties. Also, we will study the relationship between them. We hope that the above work would serve as a foundation for further on study the structure of various derivations.

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