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Estimates on the eigenvalues of complex nonlocal Sturm-Liouville problems

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Abstract. The present paper deals with the eigenvalues of complex nonlocal Sturm-Liouville boundary value problems. The bounds of the real and imaginary parts of eigenvalues for the nonlocal Sturm-Liouville differential equation involving complex nonlocal potential terms associated with nonlocal boundary conditions are obtained in terms of the integrable conditions of coefficients and the real part of the eigenvalues.

§1 Introduction

Consider the nonlocal Sturm-Liouville differential equation

$$-y''(x) + q(x)y(x) + \int_0^1 K(x,t)y(t)\mathrm{d}t = \lambda w(x)y(x) \quad \text{in} \quad L^2_w[0,1] \tag{1.1}$$

associated to suitable boundary value conditions, where λ is the spectral parameter, q is the local potential, $K(x,t)$ is the nonlocal potential, $w(x) > 0$ a.e. $x \in [0,1]$ is the weight function and $L^2_w[0,1]$ is the weighted Hilbert space consisting of all Lebesgue measurable, complex-valued functions f on $[0,1]$ satisfying $\int_0^1 w|f|^2 < \infty$ with the inner product $(f,g)_w = \int_0^1 wf\overline{g}$ and the norm $\|f\|^2_w = \int_0^1 w|f|^2$.

The nonlocal differential equation (1.1) occurs in certain physical models, particularly in quantum mechanics, diffusion processes, point interactions and voltage-driven electrical systems([2,3,15,29]). In the case where $q \equiv 0$, $w \equiv 1$ and $K(x,t) = v(x)u(t)$, $v,\ u \in C([0,1],\mathbb{R})$ in (1.1), the authors in [8] investigated the reality of eigenvalues with Dirichlet boundary conditions. For the case

$$K(x,t) = v(x)\delta(t-c) + \overline{v(t)}\delta(x-c),\ c \in [0,1], \tag{1.2}$$

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where $v \in L^2([0,1],\mathbb{C})$, δ is Dirac's distribution. Nizhnik and Albeverio in [3,19,20] studied the inverse spectral problems and the isospectral nonlocal potentials with the boundary condition

$$y(0) = 0,\ y'(1) + \int_0^1 v(t)y(t)\mathrm{d}t = 0,$$

$$y(0) = y(1) = 0,\ y'(x_0+0) - y'(x_0-0) - \int_0^1 v(t)y(t)\mathrm{d}t = 0,\ x_0 \in (0,1)$$

and the periodic boundary condition

$$y(0) = y(1),\ y'(1) - y'(0) + \int_0^1 v(t)y(t)\mathrm{d}t = 0,$$

respectively. For some other applications and a survey of nonlocal theory, we refer to [6,7,9] and references cited therein.

It is well known that equation (1.1) is formally self-adjoint if and only if the coefficients are real-valued functions. If the potential function q and the weight function w are complex valued, even in the case of non-local term $K(x,t) \equiv 0$, the equation (1.1) is formally non-self-adjoint, and hence non-real eigenvalues may exist. For example, if we set $\operatorname{Im} q > 0, K(x,t) \equiv 0$ in the equation (1.1) with Dirichlet boundary conditions, then it can be transferred to a strictly dissipative operator, and therefore, non-real eigenvalues exist.

In the case of complex local problems (i.e., the non-local term $K(x,t) \equiv 0$, $\operatorname{Im} q \neq 0$), some sufficient conditions were given in [28] to guarantee the eigenvalues of the problem (1.1) with different self-adjoint boundary conditions to be simple. Moreover, for the finiteness of eigenvalues of the local problem with complex-valued potential on half-line or whole line was investigated in [5,16]. The classification results for non-self-adjoint, complex coefficients and non-symmetric local Sturm-Liouville problems have been studied in [23,25–27,30]. For other research topics of complex differential operators such as essential spectra and expansion of eigenfunctions et al. we mention [1,4,10,13,17,24] and references cited therein.

Determining a priori bounds of non-real eigenvalues is an interesting problem in Sturm-Liouville theory. Recently, the estimates on the upper bound have been solved for the local indefinite Sturm-Liouville problem, i.e., $K(x,t) = 0$, w changes its sign on $[0,1]$ in (1.1) with self-adjoint boundary conditions in [14,21,31]. The estimates on the bounds of eigenvalues for the complex local Sturm-Liouville problems have been studied by the Rayleigh-Ritz method for $w \equiv 1$, $q > 0$ in [11] and for the general case in [12].

In this paper, we will consider complex nonlocal Sturm-Liouville problems (1.1) under some suitable boundary conditions. The main results are the bounds of the real and imaginary parts of eigenvalues for this complex nonlocal problem, in which the methods are partly inspired by Qi et al. [21,31].

The arrangement of this paper is as follows: in Section 2, we derive the complex nonlocal Sturm-Liouville problem associated to nonlocal boundary condition through the Dirac's distribution in (1.2), then the lower bounds of the real parts and the upper bounds of the imaginary part of eigenvalues for the problem (3.1) (see the below) in terms of the coefficients and the real part of eigenvalue are obtained in Section 3 (see Theorem 3.1 and Theorem 3.2).

§2 The complex nonlocal Sturm-Liouville problems

Let the nonlocal potential $K(x,t)$ in (1.1) be given in the form

$$K(x,t) = v(x)\delta(t-\alpha) + \overline{v(t)}\delta(x-\alpha), \quad \alpha \in [0,1], \tag{2.1}$$

where $v \in L^1([0,1], \mathbb{C}\backslash\mathbb{R})$ and δ is Dirac's distribution. For every continuous function f on $[0,1]$, the Dirac delta distribution at point α is defined by

$$\int_0^1 \delta(x-\alpha)f(x)\mathrm{d}x = \begin{cases} f(\alpha), & \alpha \in [0,1], \\ 0, & \alpha \notin [0,1]. \end{cases} \tag{2.2}$$

By a solution of (1.1) we mean a function $y \in AC[0,1]$ such that $y' \in AC([0,\alpha) \cup (\alpha, 1])$, $y'(\alpha \pm 0)$ exist, and the equation holds almost everywhere. For $\alpha = 1$, we use $y'(1)$ instead of $y'(1+0)$. It follows from (2.1), (2.2) and the continuity of the solution y that equation (1.1) takes the form

$$-y''(x) + q(x)y(x) + v(x)y(\alpha) + \delta(x-\alpha)\int_0^1 \overline{v(t)}y(t)\mathrm{d}t = \lambda w(x)y(x) \tag{2.3}$$

for a.e. $x \in [0,1]$. Therefore, for $x \in [0,1]$ and $x \neq \alpha$, the equation has the form

$$-y''(x) + q(x)y(x) + v(x)y(\alpha) = \lambda w(x)y(x) \quad \text{a.e. } x \in [0,1].$$

Integrating both sides of (2.3) on the interval $[\alpha-\varepsilon, \alpha+\varepsilon]$ for arbitrary $\varepsilon > 0$, then

$$y'(\alpha-\varepsilon) - y'(\alpha+\varepsilon) + \int_0^1 \overline{v(t)}y(t)\mathrm{d}t = \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \left((\lambda w(x) - q(x))y(x) - v(x)y(\alpha)\right)\mathrm{d}x.$$

Let $\varepsilon \to 0$, one sees that $y'(\alpha-0) - y'(\alpha+0) + \int_0^1 \overline{v(x)}y(x)\mathrm{d}x = 0$. Then y satisfies

$$\begin{cases} -y''(x) + q(x)y(x) + v(x)y(\alpha) = \lambda w(x)y(x) \text{ a.e. } x \in [0,1], \ x \neq \alpha, \\ y'(\alpha-0) - y'(\alpha+0) + \displaystyle\int_0^1 \overline{v(x)}y(x)\mathrm{d}x = 0. \end{cases} \tag{2.4}$$

If the boundary condition is given in the form $y(0) = 0,\ y'(1) = 0$ for (1.1) and let $\alpha = 1$, then from (2.4) we have that the nonlocal eigenvalue problem takes the form

$$\begin{cases} -y''(x) + q(x)y(x) + v(x)y(1) = \lambda w(x)y(x) \quad \text{in} \quad L^2_w[0,1], \\ y(0) = 0, \quad y'(1-0) + \displaystyle\int_0^1 \overline{v(x)}y(x)\mathrm{d}x = 0. \end{cases}$$

For simplicity, we denote by $\|\cdot\|_p$, $\|\cdot\|_c$ and $y'(1)$ instead of the form of $L^p[0,1]$, the maximum norm of $C[0,1]$ and $y'(1-0)$ in the following discussion, respectively.

§3 Estimate on the bounds of non-real eigenvalues

Consider the complex nonlocal Sturm-Liouville problems

$$\begin{cases} \tau y := -y''(x) + q(x)y(x) + v(x)y(1) = \lambda w(x)y(x), \\ \mathcal{B}y = 0: \ y(0) = 0, \quad y'(1) + \displaystyle\int_0^1 \overline{v(x)}y(x)\mathrm{d}x = 0, \end{cases} \tag{3.1}$$

where $q,\ v,\ w$ satisfy the conditions

$$\begin{aligned} & q, v \in L^1([0,1], \mathbb{C}\backslash\mathbb{R}),\ w \in L^1([0,1], \mathbb{R}),\ q = q_1 + iq_2,\ i = \sqrt{-1}, \\ & w(x) > 0 \text{ a.e. } x \in [0,1],\ q_k^- = -\min\{q_k, 0\},\ q_k^+ = \max\{q_k, 0\},\ k = 1, 2. \end{aligned} \tag{3.2}$$

Since $w(x) > 0$ a.e. on $[0,1]$, we can choose $\theta > 0$ and $\eta > 0$ such that

$$\Theta(\theta) = \{x \in [0,1] :\ w(x) < \theta\},\ M(\theta) = \text{mes } \Theta, \tag{3.3}$$

$$\Pi(\eta) = \{x \in [0,1] :\ w^2(x) < \eta\},\ N(\eta) = \text{mes } \Pi. \tag{3.4}$$

If q, v, w are real valued in (3.1), the operator $T := \frac{1}{w}\tau$ associated to this nonlocal problem is self-adjoint in the Hilbert space $(L_w^2, (\cdot,\cdot)_w)$ and the spectrum consists of real eigenvalues, which are bounded from below (see [22]). Since the coefficients q and v are complex-valued functions, unlike the self-adjoint operator T, the problem (3.1) is non-self-adjoint and has non-real eigenvalues. Therefore, we will give the estimate results on the non-real eigenvalues of the problem (3.1) in the following.

Theorem 3.1. *Let* (3.2), (3.3) *hold and* $\Gamma_{q_1^-,v} = 2\left(1 + 4(\|q_1^-\|_1 + 4\|v\|_1^2)\right)$.

(i) If λ is a non-real eigenvalue of problem (3.1) *with* $\operatorname{Re}\lambda \le 0$, *then it holds that*

$$\begin{aligned} \operatorname{Re}\lambda &\ge -\frac{2}{\theta}\Big(1 + \|q_1^+\|_1 + 2\Gamma_{q_1^-,v}(1 + \|v\|_1)\Big), \\ |\operatorname{Im}\lambda| &\le \frac{2}{\theta}\Big(1 + \|q_2^+\|_1 + \Gamma_{q_1^-,v}\Big), \end{aligned} \tag{3.5}$$

where θ satisfies $2\Gamma_{q_1^-,v}M(\theta) < 1$.

(ii) If λ is a non-real eigenvalue of problem (3.1) *with* $\operatorname{Re}\lambda > 0$, *then*

$$|\operatorname{Im}\lambda| \le \frac{2}{\theta}\Big(1 + \|q_2^+\|_1 + \Gamma_{q_1^-,v} + 8\operatorname{Re}\lambda\|w\|_1\Big), \tag{3.6}$$

where θ satisfies $2(\Gamma_{q_1^-,v} + 8\operatorname{Re}\lambda\|w\|_1)M(\theta) < 1$.

If $w \in AC[0,1]$ and $w' \in L^2[0,1]$, where $AC[0,1]$ denotes the locally absolutely continuous functions on $[0,1]$, then we have the following results.

Theorem 3.2. *Let* (3.2) *and* (3.4) *hold. Suppose that* $w \in AC[0,1]$, $w' \in L^2[0,1]$, $\Lambda = (\int_0^1 |w'|^2)^{1/2}$ *and* $\Gamma_{q_1^-,v} = 2\left(1 + 4(\|q_1^-\|_1 + 4\|v\|_1^2)\right)$.

(i) If λ is a non-real eigenvalue of problem (3.1) *with* $\operatorname{Re}\lambda \le 0$, *then*

$$\begin{aligned} \operatorname{Re}\lambda &\ge -\frac{2}{\eta}\left\{\|w\|_c\Big(1 + \|q_1^+\|_1 + 2\Gamma_{q_1^-,v}(1 + \|v\|_1)\Big) + \Lambda\Gamma_{q_1^-,v}\right\}, \\ |\operatorname{Im}\lambda| &\le \frac{2}{\eta}\left\{\|w\|_c\Big(1 + \|q_2^+\|_1 + \Gamma_{q_1^-,v}(1 + 2\|v\|_1)\Big) + \Lambda\Gamma_{q_1^-,v}\right\}, \end{aligned} \tag{3.7}$$

where η satisfies $2\Gamma_{q_1^-,v}N(\eta) < 1$.

(ii) If λ is a non-real eigenvalue of problem (3.1) *with* $\operatorname{Re}\lambda > 0$, *then*

$$\begin{aligned} |\operatorname{Im}\lambda| \le \frac{2}{\eta}\Big\{&\|w\|_c\Big(1 + \|q_2^+\|_1 + (\Gamma_{q_1^-,v} + 8\operatorname{Re}\lambda\|w\|_1)(1 + 2\|v\|_1)\Big) \\ &+ \Lambda(\Gamma_{q_1^-,v} + 8\operatorname{Re}\lambda\|w\|_1)\Big\}, \end{aligned} \tag{3.8}$$

where η satisfies $2(\Gamma_{q_1^-,v} + 8\operatorname{Re}\lambda\|w\|_1)N(\eta) < 1$.

In order to prove Theorem 3.1 and 3.2, let ψ be an eigenfunction of (3.1) corresponding to the eigenvalue λ. That is $\mathcal{B}\psi = 0$ and

$$-\psi'' + q\psi + v\psi(1) = \lambda w\psi. \tag{3.9}$$

Since the problem (3.1) is a linear system and ψ is continuous, we can choose ψ satisfying $\int_0^1 |\psi(x)|^2 \mathrm{d}x = 1$ in the following discussion. Firstly, we introduce some concepts and prepare some lemmas (cf. [18]). Let f be a real-valued function defined on the closed, bounded interval $[a,b]$ and $\triangle: \ a = x_0 < x_1 < \cdots < x_n - 1 < x_n = b$ be a partition of $[a,b]$. We define the *variation* of f with respect to $\triangle$ by

$$\mathrm{Var}_{\triangle} = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

and the *total variation* of f on $[a,b]$ by

$$\bigvee_a^b(f) = \sup\{\mathrm{Var}_{\triangle} : \triangle \text{ is an any partition of } [a,b]\}.$$

A real-valued function f is said to be of *bounded variation* on the closed and bounded interval $[a,b]$ if $\bigvee_a^b(f) < \infty$.

Lemma 3.3. (*cf. [14, Lemma 2] and [18, Lemma 5.2.2, p246]) Let g be of bounded variation over all of $[a,b]$, that is, g satisfies the inequality $\int_a^x |dg(x)| < \infty$. Then for all $x \in (a,b]$ and for every $\delta > 0$ there exists a $\rho = \rho(\delta, x) > 0$ such that*

$$\int_a^x |f(t)|^2 |dg(t)| \le \rho(\delta,x) \int_a^x |f(t)|^2 dt + \delta \int_a^x |f'(t)|^2 dt, \tag{3.10}$$

where

$$\rho(\delta,x) = \frac{1}{x-a} + \frac{c}{\delta}, \quad c = \int_a^b |dg(x)|.$$

Lemma 3.4. *Let q_1^-, q_1^+, v be defined in (3.2). Then*

$$\int_0^1 \left|q_1^- + 4\|v\|_1 |v|\right| |\psi|^2 \le \left(1 + \frac{\|q_1^-\|_1 + 4\|v\|_1^2}{\delta}\right) \int_0^1 |\psi|^2 + \delta \int_0^1 |\psi'|^2. \tag{3.11}$$

The similar conclusion holds for

$$\int_0^1 q_k^+ |\psi|^2 \le \left(1 + \|q_k^+\|_1\right) \int_0^1 |\psi|^2 + \int_0^1 |\psi'|^2, \ k = 1, 2. \tag{3.12}$$

Proof. Replacing $f(t)$ and $g(t)$ by $\psi(t)$ and $\int_0^t (q_1^-(x) + 4\|v\|_1 |v(x)|)\mathrm{d}x$ in Lemma 3.3, respectively. Then

$$\begin{aligned}\int_0^x |\mathrm{d}g(t)| &= \int_0^x \left|\mathrm{d}\left(\int_0^t (q_1^-(x) + 4\|v\|_1|v(x)|)\mathrm{d}x\right)\right| \\ &= \int_0^x \left|q_1^-(t) + 4\|v\|_1 |v(t)|\right| \mathrm{d}t \le \|q_1^-\|_1 + 4\|v\|_1^2 < \infty.\end{aligned}$$

Using this result in (3.10), one sees that (3.11) holds immediately. □

The following lemma is the estimate of $\|\psi'\|_2$.

Lemma 3.5. *Let λ and ψ be defined as above with $\operatorname{Re}\lambda \le 0$. Then*

$$\|\psi'\|_2^2 \le \Gamma_{q_1^-,v}, \ \Gamma_{q_1^-,v} = 2\left(1 + 4(\|q_1^-\|_1 + 4\|v\|_1^2)\right).$$

Proof. Multiplying both sides of (3.9) by $\overline{\psi}$ and integrating by parts over the interval $[0,1]$, then according to $\mathcal{B}\psi = 0$ we have

$$\lambda \int_0^1 w|\psi|^2 = \int_0^1 |\psi'|^2 + \int_0^1 q|\psi|^2 + 2\,\mathrm{Re}\left(\int_0^1 v\overline{\psi}\psi(1)\right). \tag{3.13}$$

Separating the real parts yields

$$\mathrm{Re}\,\lambda \int_0^1 w|\psi|^2 = \int_0^1 |\psi'|^2 + \int_0^1 q_1|\psi|^2 + 2\,\mathrm{Re}\left(\int_0^1 v\overline{\psi}\psi(1)\right). \tag{3.14}$$

From (3.14), $w > 0$ and $\mathrm{Re}\,\lambda \le 0$, one sees that

$$\int_0^1 |\psi'|^2 + \int_0^1 q_1|\psi|^2 + 2\,\mathrm{Re}\left(\int_0^1 v\overline{\psi}\psi(1)\right) \le 0. \tag{3.15}$$

Noting that $\psi(x) = \int_0^x \psi'(t)\mathrm{d}t$ by $\psi(0) = 0$ and setting $\delta = 1/4$ in (3.11), we get

$$\begin{aligned}
&\int_0^1 q_1^-|\psi|^2 + 2\int_0^1 |v||\overline{\psi}||\psi(1)| \\
&\le \int_0^1 q_1^-|\psi|^2 + 2\left(\int_0^1 |v|\right)^{1/2}\left(\int_0^1 |v||\psi|^2\right)^{1/2}\left(\int_0^1 |\psi'|^2\right)^{1/2} \\
&\le \int_0^1 \left(q_1^- + 4\|v\|_1|v|\right)|\psi|^2 + \frac{1}{4}\int_0^1 |\psi'|^2 \\
&\le \left(1 + 4(\|q_1^-\|_1 + 4\|v\|_1^2)\right)\int_0^1 |\psi|^2 + \frac{1}{2}\int_0^1 |\psi'|^2.
\end{aligned}$$

This together with (3.15) and $\int_0^1 |\psi|^2 = 1$ yields that $\int_0^1 |\psi'|^2 \le \Gamma_{q_1^-,v}$.

The proof of Theorem 3.1. From $\psi(0) = 0$ one sees that $\psi(x) = \int_0^x \psi'(t)\mathrm{d}t$, by Cauchy-Schwarz inequality and Lemma 3.5, we have

$$|\psi(x)|^2 = \left|\int_0^x \psi'(t)\mathrm{d}t\right|^2 \le x\int_0^x |\psi'(t)|^2\mathrm{d}t \le \int_0^1 |\psi'(x)|^2\mathrm{d}x \le \Gamma_{q_1^-,v}. \tag{3.16}$$

It follows from $w(x) > 0$ a.e. on $[0,1]$ and the definition of $\Theta(\theta)$, $M(\theta)$ in (3.3) that $M(\theta) \to 0,\ \theta \to 0$, so we can choose $\theta > 0$ such that

$$2\Gamma_{q_1^-,v}M(\theta) < 1, \text{ where } \Gamma_{q_1^-,v} \text{ is defined in Lemma 3.5.} \tag{3.17}$$

Then

$$\begin{aligned}
\int_0^1 w(x)|\psi(x)|^2\mathrm{d}x &\ge \int_{[0,1]\setminus\Theta(\theta)} w(x)|\psi(x)|^2\mathrm{d}x \\
&\ge \theta\left(\int_0^1 |\psi(x)|^2\mathrm{d}x - \int_{\Theta(\theta)} |\psi(x)|^2\mathrm{d}x\right) \\
&\ge \theta\left(1 - M(\theta)\Gamma_{q_1^-,v}\right) \ge \theta/2.
\end{aligned} \tag{3.18}$$

By (3.12), (3.14), $\psi(x) = \int_0^x \psi'(t)\mathrm{d}t$ and Schwary inequality, one sees that

$$\begin{aligned}
|\mathrm{Re}\,\lambda|\int_0^1 w|\psi|^2 &= \left|\int_0^1 |\psi'|^2 + \int_0^1 q_1|\psi|^2 + 2\,\mathrm{Re}\left(\int_0^1 v\overline{\psi}\psi(1)\right)\right| \\
&\le \int_0^1 |\psi'|^2 + \int_0^1 q_1^+|\psi|^2 + 2\int_0^1 |v||\overline{\psi}||\psi(1)|
\end{aligned}$$

$$\leq \Gamma_{q_1^-,v} + \left(1+\|q_1^+\|_1\right)\int_0^1 |\psi|^2 + \int_0^1 |\psi'|^2 + 2\|v\|_1\int_0^1 |\psi'|^2.$$

This together with (3.18) yields that

$$|\operatorname{Re}\lambda|\frac{\theta}{2} \leq |\operatorname{Re}\lambda|\int_0^1 w|\psi|^2 \leq 1+\|q_1^+\|_1 + 2\Gamma_{q_1^-,v}(1+\|v\|_1). \tag{3.19}$$

Separating the imaginary parts of (3.13) yields

$$\operatorname{Im}\lambda\int_0^1 w|\psi|^2 = \int_0^1 q_2|\psi|^2.$$

It follows from $q_2^+ = \max\{q_2, 0\}$, (3.18) and (3.12) that

$$|\operatorname{Im}\lambda|\frac{\theta}{2} \leq \int_0^1 w|\psi|^2 \leq \int_0^1 q_2^+|\psi|^2 \leq 1+\|q_2^+\|_1 + \Gamma_{q_1^-,v}. \tag{3.20}$$

So the inequalities in (3.5) follow from (3.19) and (3.20) immediately.

Now, if λ is an eigenvalue of (3.1) with $\operatorname{Re}\lambda > 0$, then we consider the eigenvalue problem

$$-y'' + (q-\operatorname{Re}\lambda w)y + vy(1) = \lambda wy, \quad \mathcal{B}y = 0. \tag{3.21}$$

It can be easily verified that $\lambda - \operatorname{Re}\lambda$ is also an eigenvalue of (3.21). Clearly, $\operatorname{Re}(\lambda-\operatorname{Re}\lambda) = 0$, and hence

$$0 = \operatorname{Re}(\lambda-\operatorname{Re}\lambda)\int_0^1 w|\psi|^2 = \int_0^1 |\psi'|^2 + \int_0^1 (q_1-\operatorname{Re}\lambda w)|\psi|^2 + 2\operatorname{Re}\left(\int_0^1 v\overline{\psi}\psi(1)\right). \tag{3.22}$$

Similar to Lemma 3.4 we have

$$\int_0^1 \left(q_1^- + \operatorname{Re}\lambda w + 4\|v\|_1|v|\right)|\psi|^2 \leq 1 + 4(\|q_1^-\|_1 + \operatorname{Re}\lambda\|w\|_1 + 4\|v\|_1^2) + \frac{1}{4}\int_0^1 |\psi'|^2,$$

which together with (3.22) imply that

$$\int_0^1 |\psi'|^2 \leq 2\left(1+4(\|q_1^-\|_1 + \operatorname{Re}\lambda\|w\|_1 + 4\|v\|_1^2)\right) = \Gamma_{q_1^-,v} + 8\operatorname{Re}\lambda\|w\|_1.$$

Then from $\psi(x) = \int_0^x \psi'(t)\mathrm{d}t$ and Schwarz inequality, we get

$$|\psi(x)|^2 = \left|\int_0^x \psi'(t)\mathrm{d}t\right|^2 \leq \int_0^1 |\psi'(x)|^2\mathrm{d}x \leq \Gamma_{q_1^-,v} + 8\operatorname{Re}\lambda\|w\|_1.$$

According to the choice of θ in (3.17), one sees that there exists λ satisfying $2(\Gamma_{q_1^-,v}+8\operatorname{Re}\lambda\|w\|_1)$ $M(\lambda) < 1$, such that

$$\begin{aligned}|\operatorname{Im}\lambda|\frac{\lambda}{2} &\leq |\operatorname{Im}\lambda|\int_0^1 w|\psi|^2 = |\operatorname{Im}(\lambda-\operatorname{Re}\lambda)|\int_0^1 w|\psi|^2 = \int_0^1 q_2|\psi|^2\\ &\leq \int_0^1 q_2^+|\psi|^2 \leq 1+\|q_2^+\|_1 + \int_0^1 |\psi'|^2 \leq 1+\|q_2^+\|_1 + \Gamma_{q_1^-,v} + 8\operatorname{Re}\lambda\|w\|_1.\end{aligned}$$

So the inequality in (3.8) holds. The proof of Theorem 3.1 is completed. □

The proof of Theorem 3.2. Multiplying both sides of (3.9) by $w\overline{\psi}$ and integrating by parts on $[0,1]$, then from $\mathcal{B}\psi = 0$ we have

$$\lambda\int_0^1 w^2|\psi|^2 = \int_0^1 w|\psi'|^2 + \int_0^1 wq|\psi|^2 + \int_0^1 w'\psi'\overline{\psi} + \int_0^1 wv\overline{\psi}\psi(1) + \int_0^1 w(1)\overline{v}\psi\overline{\psi(1)}.$$

Separating the real and imaginary parts we get

$$\operatorname{Re}\lambda\int_0^1 w^2|\psi|^2 = \int_0^1 \left(w|\psi'|^2 + wq_1|\psi|^2\right) + \int_0^1 \operatorname{Re}\left(w'\psi'\overline{\psi} + wv\overline{\psi}\psi(1) + w(1)\overline{v}\psi\overline{\psi(1)}\right), \tag{3.23}$$

$$\operatorname{Im}\lambda\int_0^1 w^2|\psi|^2=\int_0^1 wq_2|\psi|^2+\int_0^1 \operatorname{Im}\left(w'\psi'\overline{\psi}+wv\overline{\psi}\psi(1)+w(1)\overline{v}\psi\overline{\psi(1)}\right). \tag{3.24}$$

Using $q,v\in L^1[0,1]$, $\psi(x)=\int_0^x\psi'(t)\mathrm{d}t$, (3.12) and Lemma 3.5, one sees that

$$\begin{aligned}&\left|\int_0^1\left(w|\psi'|^2+wq_1|\psi|^2+wv\overline{\psi}\psi(1)+w(1)\overline{v}\psi\overline{\psi(1)}\right)\right|\\ &\leq \|w\|_c\int_0^1|\psi'|^2+\|w\|_c\int_0^1 q_1^+|\psi|^2+2\|w\|_c\int_0^1|v|\int_0^1|\psi'|^2\\ &\leq \|w\|_c\left(1+\|q_1^+\|_1+2\Gamma_{q_1^-,v}(1+\|v\|_1)\right).\end{aligned} \tag{3.25}$$

It follows from $w'\in L^2[0,1]$, $\Lambda=(\int_0^1|w'|^2)^{1/2}$, Lemma 3.5 and Schwarz inequality that

$$\left|\int_0^1 w'\psi'\overline{\psi}\right|\leq\left(\int_0^1|w'|^2\right)^{1/2}\left(\int_0^1|\psi'|^2\right)^{1/2}\left(\int_0^1|\psi'|^2\right)^{1/2}\leq\Lambda\Gamma_{q_1^-,v}. \tag{3.26}$$

The facts $q_2^+=\max\{q_2,0\}$ and (3.12) lead to

$$\begin{aligned}&\left|\int_0^1\left(wq_2|\psi|^2+wv\overline{\psi}\psi(1)+w(1)v\psi\overline{\psi(1)}\right)\right|\\ &\leq \|w\|_c\int_0^1 q_2^+|\psi|^2+2\|w\|_c\|v\|_1\int_0^1|\psi'|^2\\ &\leq \|w\|_c\left(1+\|q_2^+\|_1+\Gamma_{q_1^-,v}(1+2\|v\|_1)\right).\end{aligned} \tag{3.27}$$

Since $w^2(x)>0$ a.e. on $[0,1]$, recall the definition of $N(\eta)$ in (3.4), we can choose $\eta>0$ such that $2\Gamma_{q_1^-,v}N(\eta)<1$, then (3.3) and (3.16) lead to

$$\begin{aligned}&\int_0^1 w^2(x)|\psi(x)|^2\mathrm{d}x\geq\int_{[0,1]\setminus\Pi(\eta)}w^2(x)|\psi(x)|^2\mathrm{d}x\\ &\geq\eta\left(\int_0^1|\psi(x)|^2\mathrm{d}x-\int_{\Pi(\eta)}|\psi(x)|^2\mathrm{d}x\right)\geq\eta\left(1-N(\eta)\Gamma_{q_1^-,v}\right)\geq\eta/2,\end{aligned} \tag{3.28}$$

which together with (3.23)-(3.28) gives (3.7).

Furthermore, if λ is an eigenvalue of (3.1) with $\operatorname{Re}\lambda>0$, then (3.21) holds. With a similar argument in the proof of Theorem 3.1, it follows from (3.21), (3.24), (3.26), (3.27) and (3.28) that (3.8) holds. This completes the proof of Theorem 3.2. □

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