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Complement of the reduced non-zero component graph of free semimodules

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Abstract. Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$ with identity having invariant basis number property with a basis $\alpha = \{\alpha_1, \ldots, \alpha_k\}$. The complement $\overline{\Gamma^*}(\mathbb{M})$ of the reduced non-zero component graph $\Gamma^*(\mathbb{M})$ of $\mathbb{M}$, is the simple undirected graph with $V = \mathbb{M}^* \setminus \{\sum\limits_{i=1}^{k} c_i\alpha_i : c_i \neq 0\ \forall\ i\}$ as the vertex set and such that there is an edge between two distinct vertices $a = \sum\limits_{i=1}^{k} a_i\alpha_i$ and $b = \sum\limits_{i=1}^{k} b_i\alpha_i$ if and only if there exists no i such that both a_i, b_i are non-zero. In this paper, we show that the graph $\overline{\Gamma^*}(\mathbb{M})$ is connected and find its domination number, clique number and chromatic number. In the case of finite semirings, we determine the degree of each vertex, order, size, vertex connectivity and girth of $\overline{\Gamma^*}(\mathbb{M})$. Also, we give a necessary and sufficient condition for $\overline{\Gamma^*}(\mathbb{M})$ to be Eulerian or Hamiltonian or planar.

§1 Introduction

In recent years, the interplay between an algebraic structure and a graph structure is studied by many researchers. One of the important graph constructed from finite groups is the Cayley graph. Cayley graphs have been well studied as they are used as an underlying network for routing problems in parallel computing. In fact, Cayley graphs are vertex transitive and regular. Another important graph construction from commutative rings is the zero-divisor graph. Actually, the concept of a graph from a commutative ring was introduced by Beck [6] and later modified and named as the zero-divisor graph by Anderson and Livingston [3]. In these attempts, researcher defines a graph whose vertices are a set of elements or a set of ideals in the ring and edges are defined with respect to an algebraic condition on the elements of

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the vertex set. Certain well-studied classes of graphs from commutative rings are zero-divisor graph, total graph, annihilating graph, comaximal graph, unit graph, Cayley graph, Jacobson graph, generalized total graph, Cayley sum graph and trace graph of matrices. One can translate some algebraic properties of commutative rings to graph theoretic language and then the geometric properties of graphs help to explore some interesting results related to commutative rings [1, 2, 4, 5, 13–15, 17, 19]. Similar to these graphs, Das [10, 11] has introduced and investigated a graph called the non-zero component graph of a finite dimensional vector space. Recently, it was generalized for semimodules by Bhuniya and Maity [7] and named the graph as the reduced non-zero component graph $\Gamma^*(\mathbb{V})$. In this paper, we study the complement of the reduced non-zero component graph [7, 10] of a finitely generated free semimodule $\mathbb{M}$ over a semiring $\mathbb{S}$ with identity having invariant basis number property. A semiring $\mathbb{S}$ is said to have invariant basis number property if any two bases of a finitely generated free semimodule over $\mathbb{S}$ have the same cardinality. If $\mathbb{S}$ is a semiring having invariant basis number property, then by Corollary 3.1 [20], it follows that every vector of a finitely generated free semimodule $\mathbb{M}$ over $\mathbb{S}$ can be expressed uniquely in terms of basis elements. If $\{\alpha_1, \ldots, \alpha_k\}$ is a basis of a semimodule $\mathbb{M}$ over a semiring $\mathbb{S}$, then every element $a \in \mathbb{M}$ can be expressed uniquely as $a = a_1\alpha_1 + \cdots + a_k\alpha_k$, where $a_i \in \mathbb{S}$. We call a_i as the ith component of a. One may refer to Golan [12] for basic notions and results on semirings and semimodules.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. The *complement* $\overline{G}$ of G is the graph whose vertex set is $V(G)$ and two vertices u, v are adjacent in $\overline{G}$ if and only if they are not adjacent in G. For a subset $A \subseteq V(G)$, $\langle A \rangle$ denotes the subgraph of G induced by A. For undefined graph theoretical terms, one may refer to [8, 21]. Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$ with identity having invariant basis number property and $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ is a basis for $\mathbb{M}$. Then the reduced non-zero component graph $\Gamma^*(\mathbb{M})$ [7] of $\mathbb{M}$ with respect to the basis α, is the graph with vertex set $V = \mathbb{M}^* \setminus \{\sum_{i=1}^{k} c_i\alpha_i : c_i \neq 0 \ \forall\ i\}$ and two distinct vertices a and $b \in V$ are adjacent if there exists i such that both a_i, b_i are non-zero. The complement $\overline{\Gamma^*}(\mathbb{M})$ of $\Gamma^*(\mathbb{M})$ is the graph with vertex set $\mathbb{M}^* \setminus \{\sum_{i=1}^{k} c_i\alpha_i : c_i \neq 0 \ \forall\ i\}$ and two distinct vertices $a = \sum_{i=1}^{k} a_i\alpha_i$ and $b = \sum_{i=1}^{k} b_i\alpha_i$ are adjacent if and only if there exists no i such that both a_i, b_i are non-zero.

Throughout this paper, by $\mathbb{S}$, we mean a commutative semiring $\mathbb{S}$ with additive identity 0 and multiplicative identity 1. Also, by a semimodule $\mathbb{M}$, we always mean $\mathbb{M}$ is finitely generated free over $\mathbb{S}$ with invariant basis number property. We take $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ as a basis where $k = \dim_{\mathbb{S}}(\mathbb{M})$ or $(rank_{\mathbb{S}}(\mathbb{M}))$. We state below certain results which are used in this paper.

Theorem 1.1. ([21, Whitney, Theorem 4.1.9]) *If G is a simple graph, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Theorem 1.2. ([21, Proposition 7.2.1]) *Every Hamiltonian graph is 2-connected.*

Theorem 1.3. ([21, Proposition 7.2.3]) *If G has a Hamiltonian cycle, then for each nonempty set $A \subseteq V$, the graph $G - A$ has at most $|A|$ components.*

Theorem 1.4. ([9, Strong Perfect Graph Theorem]) *A graph G is perfect if and only if neither G nor its complement $\overline{G}$ contains an odd cycle of length at least 5 as an induced subgraph.*

§2 Basic properties of $\overline{\Gamma^*}(\mathbb{M})$

In this section, we obtain some basic properties of the graph $\overline{\Gamma^*}(\mathbb{M})$ like diameter, girth, connectedness and domination number. After obtaining these parameters of $\overline{\Gamma^*}(\mathbb{M})$, we prove that $\overline{\Gamma^*}(\mathbb{M})$ is weakly perfect. The following concerns about $\overline{\Gamma^*}(\mathbb{M}_\alpha)$ and $\overline{\Gamma^*}(\mathbb{M}_\beta)$ with respect to two bases α and β of $\mathbb{M}$ of equal cardinality.

Theorem 2.1. *Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$ with two bases $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_k\}$ of $\mathbb{M}$. Then the graphs $\overline{\Gamma^*}(\mathbb{M}_\alpha)$ and $\overline{\Gamma^*}(\mathbb{M}_\beta)$ are isomorphic.*

Proof. Define $\Phi : \mathbb{M} \longrightarrow \mathbb{M}$ by $\Phi(c_1\alpha_1 + \cdots + c_k\alpha_k) = c_1\beta_1 + \cdots + c_k\beta_k$. Clearly Φ is an $\mathbb{S}$-semimodule isomorphism on $\mathbb{M}$ such that $\Phi(\alpha_i) = \beta_i$ for all $i \in \{1, 2, \ldots, k\}$. One can check that the restriction map $\Phi' : V(\overline{\Gamma^*}(\mathbb{M}_\alpha)) \longrightarrow V(\overline{\Gamma^*}(\mathbb{M}_\beta))$ of Φ on $\mathbb{M}^* \setminus \{\sum_{i=1}^{k} c_i\alpha_i : c_i \neq 0 \ \forall\ i\}$ induces a graph isomorphism. □

In view of Theorem 2.1, properties of $\overline{\Gamma^*}(\mathbb{M})$ are independent on the choice of the basis for $\mathbb{M}$. So one can take any basis of $\mathbb{M}$ to study $\overline{\Gamma^*}(\mathbb{M})$ henceforth, we take standard basis of $\mathbb{M}$ to study $\overline{\Gamma^*}(\mathbb{M})$.

Note 2.2. If $\dim(\mathbb{M}) = 1$, then $\overline{\Gamma^*}(\mathbb{M})$ is null graph. Further if $\dim(\mathbb{M}) = 2$ and $|\mathbb{S}| = 2$, then $\overline{\Gamma^*}(\mathbb{M})$ is K_2.

In the following results, we prove that $\overline{\Gamma^*}(\mathbb{M})$ is connected in the remaining cases.

Lemma 2.3. *Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$. If $\dim(\mathbb{M}) = 2$ and $|\mathbb{S}| \geq 3$, then $\overline{\Gamma^*}(\mathbb{M})$ is connected and diam$(\overline{\Gamma^*}(\mathbb{M})) = 2$.*

Proof. Let a and b be two distinct vertices in $\overline{\Gamma^*}(\mathbb{M})$. If a and b are adjacent in $\overline{\Gamma^*}(\mathbb{M})$, then $d(a, b) = 1$. Suppose a and b are not adjacent. Since $\dim(\mathbb{M}) = 2$, either of the following is true for the components of a and b: (i) $a_1 \neq 0$, $a_2 = 0$, $b_1 \neq 0$ and $b_2 = 0$; (ii) $a_1 = 0$, $a_2 \neq 0$, $b_1 = 0$ and $b_2 \neq 0$.

Suppose (i) is true. Since $|\mathbb{S}| \geq 3$, there exists a vertex $c \in \overline{\Gamma^*}(\mathbb{M})$ with $c_1 = 0$ and $c_2 \neq 0$. Then, $a - c - b$ is a path in $\overline{\Gamma^*}(\mathbb{M})$ and so $d(a, b) = 2$. Similar fact is true in the case of (ii). □

Theorem 2.4. *Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$. If $\dim(\mathbb{M}) \geq 3$, then $\overline{\Gamma^*}(\mathbb{M})$ is connected and diam$(\overline{\Gamma^*}(\mathbb{M})) = 3$.*

Proof. Let $\dim(\mathbb{M}) = k$. Let a and b be two distinct vertices in $\overline{\Gamma^*}(\mathbb{M})$. If a and b are adjacent in $\overline{\Gamma^*}(\mathbb{M})$, then $d(a, b) = 1$. Otherwise, there exists at least one $i \in \{1, \ldots, k\}$ such that $a_i, b_i \neq 0$. Since $a, b \in V(\overline{\Gamma^*}(\mathbb{M}))$, there exist j and ℓ in $\{1, \ldots, k\}$ such that a_j and b_ℓ are zero.

If $j = \ell$, then we take a vertex c such that $c_j \neq 0$ and $c_i = 0$ for all $1 \leq i \neq j \leq k$. Then $a - c - b$ is a path in $\overline{\Gamma^*}(\mathbb{M})$ and hence $d(a, b) = 2$.

If $j \neq \ell$, then we take vertices c and d with $c_j \neq 0$, $c_i = 0$ for all $1 \leq i \neq j \leq k$ and $d_\ell \neq 0$, $d_i = 0$ for all $1 \leq i \neq \ell \leq k$. Hence $a - c - d - b$ is a path in $\overline{\Gamma^*}(\mathbb{M})$ and so $d(a, b) = 3$. Therefore $diam(\overline{\Gamma^*}(\mathbb{M})) = 3$. □

Now we characterize when $\overline{\Gamma^*}(\mathbb{M})$ is a complete bipartite or a complete graph.

Theorem 2.5. *Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$. Then $\overline{\Gamma^*}(\mathbb{M})$ is complete bipartite if and only if* $\dim(\mathbb{M}) = 2$.

Proof. Assume that $\overline{\Gamma^*}(\mathbb{M})$ is complete bipartite and so $\overline{\Gamma^*}(\mathbb{M})$ does not contain an odd cycle. Suppose that $\dim(\mathbb{M}) > 2$. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{M}$ such that $\alpha_1, \alpha_2, \alpha_3$ are in a basis of $\mathbb{M}$. It is easy to observe that the induced subgraph $\langle\{\alpha_1, \alpha_2, \alpha_3\}\rangle$ of $\overline{\Gamma^*}(\mathbb{M})$ is K_3, which is a contradiction to $\overline{\Gamma^*}(\mathbb{M})$ does not contain an odd cycle. Therefore $\dim(\mathbb{M}) = 2$.

Conversely, assume that $\dim(\mathbb{M}) = 2$ with $\{\alpha_1, \alpha_2\}$ as a basis of $\mathbb{M}$. Since $\dim(\mathbb{M}) = 2$, the vertices of $\overline{\Gamma^*}(\mathbb{M})$ is of the form $\{a\alpha_1 + b\alpha_2 \in \mathbb{M} : a = 0 \text{ or } b = 0\}$. Note that $H_1 = \{a\alpha_1 \in \mathbb{M} : a \neq 0\}$ and $H_2 = \{b\alpha_2 \in \mathbb{M} : b \neq 0\}$ are independent sets and every vertex in H_1 is adjacent to every vertex in H_2. Hence $\overline{\Gamma^*}(\mathbb{M})$ is a complete bipartite graph. □

Theorem 2.6. *Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$. Then $\overline{\Gamma^*}(\mathbb{M})$ is complete if and only if* $\dim(\mathbb{M}) = 2$ *and* $|\mathbb{S}| = 2$.

Proof. Assume that $\overline{\Gamma^*}(\mathbb{M})$ is complete. From this $\Gamma^*(\mathbb{M})$ is totally disconnected. Suppose that $\dim(\mathbb{M}) \geq 3$. By [7, Theorem 3.1], $\Gamma^*(\mathbb{M})$ is connected and $diam(\Gamma^*(\mathbb{M})) = 2$, which is a contradiction. Therefore $\dim(\mathbb{M}) \leq 2$. By Note 2.2, we have $\dim(\mathbb{M}) = 2$. Let $\{\alpha_1, \alpha_2\}$ be a basis of $\mathbb{M}$. If $|\mathbb{S}| \geq 3$, then there exists $a_1 \in \mathbb{S}, a_1 \neq 0$, $a_1 \neq 1$ and so α_1 is not adjacent to $a_1\alpha_1$ in $\overline{\Gamma^*}(\mathbb{M})$ which is a contradiction. Hence $|\mathbb{S}| = 2$.

Conversely, assume that $\dim(\mathbb{M}) = 2$ and $|\mathbb{S}| = 2$. Then $\overline{\Gamma^*}(\mathbb{M})$ is K_2 as observed in Note 2.2. □

Next, we obtain the girth of $\overline{\Gamma^*}(\mathbb{M})$.

Theorem 2.7. *Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$ of dimension k. Then*

$$gr(\overline{\Gamma^*}(\mathbb{M})) = \begin{cases} 3 & if \quad k \geq 3; \\ 4 & if \quad k = 2 \text{ and } |\mathbb{S}| \geq 3; \\ \infty & if \quad k = 2 \text{ and } |\mathbb{S}| = 2. \end{cases}$$

Proof. **Case 1.** Let $k \geq 3$. Assume that $A = \{\alpha_1, \ldots, \alpha_k\}$ be a basis for $\mathbb{M}$ with $|A| = k \geq 3$. The subgraph induced by A is complete and so $\langle A\rangle$ is K_k with $k \geq 3$. Hence $gr(\overline{\Gamma^*}(\mathbb{M})) = 3$.

Case 2. Let $k = 2$ and $|\mathbb{S}| \geq 3$. By Theorem 2.5, $\overline{\Gamma^*}(\mathbb{M})$ is complete bipartite and hence $gr(\overline{\Gamma^*}(\mathbb{M})) = 4$.

Case 3. If $k = 2$ and $|\mathbb{S}| = 2$, then as observed in Note 2.2, $\overline{\Gamma^*}(\mathbb{M})$ is K_2 and hence $gr(\overline{\Gamma^*}(\mathbb{M})) = \infty$. □

Next we are interested in the domination number $\gamma(\overline{\Gamma^*}(\mathbb{M}))$.

Remark 2.8. (i) If $\dim(\mathbb{M}) = 2$ and $|\mathbb{S}| = 2$, then by Note 2.2, $\gamma(\overline{\Gamma^*}(\mathbb{M})) = 1$.
(ii) If $\dim(\mathbb{M}) = 2$ and $|\mathbb{S}| \geq 3$, then by Theorem 2.5, $\overline{\Gamma^*}(\mathbb{M})$ is complete bipartite. This implies that $\gamma(\overline{\Gamma^*}(\mathbb{M})) = 2$.

Theorem 2.9. *Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$ with $\dim(\mathbb{M}) \geq 3$. Then $\gamma(\overline{\Gamma^*}(\mathbb{M})) = \dim(\mathbb{M})$.*

Proof. Let $\dim(\mathbb{M}) = k$ and $A = \{\alpha_1, \ldots, \alpha_k\}$ be a basis for $\mathbb{M}$. For any $a \in V(\overline{\Gamma^*}(\mathbb{M})) \setminus A$, there exist $i, j, i \neq j$ such that $a_i \neq 0$ and $a_j = 0$. Hence a is adjacent to α_j. Hence A is a dominating set of $\overline{\Gamma^*}(\mathbb{M})$. Suppose $A' \subset A$ and A' is a dominating set of $\overline{\Gamma^*}(\mathbb{M})$. Note that $|A'| < |A| = k$ and so $\exists\ \alpha_i \in A$ and $\alpha_i \notin A'$. It is clear that any vertex in $\{\sum\limits_{j=1}^{k} c_j\alpha_j\ :\ \exists$ only one i such that $c_i = 0$ and $c_j \neq 0\ \forall\ 1 \leq j \neq i \leq k\}$ is not dominated by A'. Therefore A' is not a dominating set. Hence A is a minimal dominating set of $\overline{\Gamma^*}(\mathbb{M})$. To complete the proof it is enough to prove that no set with less than k elements is a dominating set.

Assume that there exists a dominating set of cardinality less than k. Let $A'' = \{a_1, \ldots, a_{k-1}\}$ be a dominating set. Now, one can partition the vertex set of $\overline{\Gamma^*}(\mathbb{M})$ into three sets namely H_1, H_2, H_3, where

$$H_1 = \{\sum_{j=1}^{k} c_j\alpha_j\ :\ \exists \text{ only one } i \text{ such that } c_i \neq 0 \text{ and } c_j = 0\ \forall\ 1 \leq j \neq i \leq k\},$$
$$H_2 = \{\sum_{j=1}^{k} c_j\alpha_j\ :\ \exists \text{ only one } i \text{ such that } c_i = 0 \text{ and } c_j \neq 0\ \forall\ 1 \leq j \neq i \leq k\}$$

and $H_3 = V(\overline{\Gamma^*}(\mathbb{M})) \setminus (H_1 \cup H_2)$.

Note that the elements in H_2 are dominated only by the elements of H_1 and also H_2 is an independent set in $\overline{\Gamma^*}(\mathbb{M}) - \langle H_1 \rangle$. Clearly $|H_1| \geq k$ and $|H_2| \geq k$. Next, we partition H_1 into k-independent sets each one containing vertices with non-zero components in the same position. Now, choose a subset H_2' of H_2 containing k-elements from H_2 such that these k elements are not having zero in the same component. By the adjacency of $\overline{\Gamma^*}(\mathbb{M})$, there is a bijection between elements of H_2' and k independent sets of H_1. Therefore at least k-elements from H_1 are needed for dominating elements of H_2. Hence A'' is not a dominating set. □

Theorem 2.10. *Let $\mathbb{M}$ be a finitely generated free semimodule over a semiring $\mathbb{S}$ with $\dim(\mathbb{M}) = k$. Then $\omega(\overline{\Gamma^*}(\mathbb{M})) = k = \chi(\overline{\Gamma^*}(\mathbb{M}))$, i.e., $\overline{\Gamma^*}(\mathbb{M})$ is weakly perfect.*

Proof. Let $A = \{\alpha_1, \ldots, \alpha_k\}$ be a basis for $\mathbb{M}$. Clearly $\langle A \rangle$ is a complete subgraph of $\overline{\Gamma^*}(\mathbb{M})$. Suppose there exists $a \in V(\overline{\Gamma^*}(\mathbb{M}))$, $a \neq \alpha_i$ for every i and $\langle A \cup \{a\} \rangle$ is a clique of $\overline{\Gamma^*}(\mathbb{M})$. Let $a = \sum\limits_{i=1}^{k} a_i\alpha_i$. Since a is adjacent to α_i for every i, we have that $a_i = 0$ for every $1 \leq i \leq k$. This implies that $a \notin V(\overline{\Gamma^*}(\mathbb{M}))$, which is a contradiction. Thus, A is a maximal clique of size k. Suppose $A' = \{x_1, x_2, \ldots, x_{k+1}\}$ be a clique of size $k+1$. Then anyone of the x_i and x_j must have non-zero in the same component. Hence x_i is not adjacent to x_j which is a contradiction.

Thus $\omega(\overline{\Gamma^*}(\mathbb{M})) = k$. We know that for any graph G, $\omega(G) \leq \chi(G)$. So, we have $k \leq \chi(\overline{\Gamma^*}(\mathbb{M}))$. Assign color 1 to all the vertices having first component as non-zero and color 2 to all the vertices having first component as zero and 2nd component as non-zero. Continuing in this way, we assign color k to all the vertices having only k-th component as non-zero. By this way of coloring, one can see that the vertices having same color can never be adjacent. Thus, we get a proper coloring for $\overline{\Gamma^*}(\mathbb{M})$ and hence $\chi(\overline{\Gamma^*}(\mathbb{M})) \leq k$. □

§3 $\overline{\Gamma^*}(\mathbb{M})$ over finite semirings

In this section, we discuss some basic properties of $\overline{\Gamma^*}(\mathbb{M})$ where $\mathbb{M}$ is a finitely generated free semimodule over a finite semiring $\mathbb{S}$. More specifically, we obtain the degree of vertices in $\overline{\Gamma^*}(\mathbb{M})$.

Theorem 3.1. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring $\mathbb{S}$ with q elements and $\overline{\Gamma^*}(\mathbb{M})$ be its associated graph with respect to the basis $\{\alpha_1, \ldots, \alpha_k\}$. Then, the degree of the vertex $c_1\alpha_{i_1} + \cdots + c_r\alpha_{i_r}$, where c_i's are non-zero, is $q^{k-r} - 1$.*

Proof. Note that the vertex α_{i_1} is adjacent to the vertices with $c_2\alpha_{i_2} + \cdots + c_r\alpha_{i_r}$, where c_i's are non-zero. Therefore $deg(\alpha_{i_1}) = q^{k-1} - 1$. The vertices of the form $\alpha_{i_1} + \alpha_{i_2}$ are adjacent to vertices of the form $c_3\alpha_{i_3} + \cdots + c_r\alpha_{i_r}$, where c_i's are non-zero. Therefore $deg(\alpha_{i_1} + \alpha_{i_2}) = q^{k-2} - 1$. Proceeding in this way, we get that $deg(\alpha_{i_1} + \cdots + \alpha_{i_r}) = q^{k-r} - 1$. It is easy to verify that the set of all vertices adjacent to $\alpha_{i_1} + \cdots + \alpha_{i_r}$ is same as the set of all vertices adjacent to $c_1\alpha_{i_1} + \cdots + c_r\alpha_{i_r}$ i.e., $N(\alpha_{i_1} + \cdots + \alpha_{i_r}) = N(c_1\alpha_{i_1} + \cdots + c_r\alpha_{i_r})$ where c_i's are non-zero. Hence $deg(c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_r\alpha_{i_r}) = q^{k-r} - 1$, where c_i's are non-zero. □

From Theorem 3.1, we have the following characterization for $\overline{\Gamma^*}(\mathbb{M})$ to be Eulerian.

Corollary 3.2. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring $\mathbb{S}$ with $|\mathbb{S}| = q$. Then $\overline{\Gamma^*}(\mathbb{M})$ is Eulerian if and only if q is odd.*

Theorem 3.3. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring $\mathbb{S}$ with $\dim(\mathbb{M}) = k$ and $|\mathbb{S}| = q$. Then the number of vertices of $\overline{\Gamma^*}(\mathbb{M})$ is $(q^k - 1) - (q-1)^k$ and the number of edges (m) of $\overline{\Gamma^*}(\mathbb{M})$ is $\frac{(2q-1)^k - 2q^k + 1}{2}$.*

Proof. Note that $A = \{\sum_{i=1}^{k} c_i\alpha_i : c_i \neq 0 \ \forall\ i\}$ contains $(q-1)^k$ elements and so number of vertices in $\overline{\Gamma^*}(\mathbb{M}) = |\mathbb{M}^*| - |A| = (q^k - 1) - (q-1)^k$. By Theorem 3.1, the degree of the vertex $c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_r\alpha_{i_r}$, where c_i's are non-zero is $q^{k-r} - 1$. Now, there are $\binom{k}{r}(q-1)^r$ vertices with exactly r components as non-zero in its basic representation. Since the sum of degrees of all vertices in $\overline{\Gamma^*}(\mathbb{M})$ is $2m$, we have

$$\begin{aligned}
2m &= \sum_{r=1}^{k-1} \binom{k}{r}(q-1)^r(q^{k-r} - 1) \\
&= \sum_{r=1}^{k-1} \binom{k}{r}(q-1)^r q^{k-r} - \sum_{r=1}^{k-1} \binom{k}{r}(q-1)^r \\
&= (q + q - 1)^k - q^k - (q-1)^k - [(q - 1 + 1)^k - 1 - (q-1)^k]
\end{aligned}$$

$= (2q-1)^k - q^k - (q-1)^k - q^k + 1 + (q-1)^k$
$= (2q-1)^k - 2q^k + 1$
and hence $m = \frac{(2q-1)^k - 2q^k + 1}{2}$. □

In the following result, we obtain the minimum and maximum degrees of $\overline{\Gamma^*}(\mathbb{M})$.

Theorem 3.4. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring $\mathbb{S}$ with $\dim(\mathbb{M}) = k$ and $|\mathbb{S}| = q$. Then the minimum degree $\delta(\overline{\Gamma^*}(\mathbb{M})) = q-1$ and the maximum degree $\Delta(\overline{\Gamma^*}(\mathbb{M})) = q^{k-1} - 1$.*

Proof. From Theorem 3.1, the degree of the vertex $c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_r\alpha_{i_r}$, where c_i's are non-zero is $q^{k-r} - 1$. Thus the minimum degree corresponds to $r = k-1$ and hence $\delta(\overline{\Gamma^*}(\mathbb{M})) = q-1$ where as the maximum degree corresponds to $r = 1$ and hence $\Delta(\overline{\Gamma^*}(\mathbb{M})) = q^{k-1} - 1$. □

Lemma 3.5. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring $\mathbb{S}$ with $\dim(\mathbb{M}) = k$ and $|\mathbb{S}| = q$. Then the following statements hold:*

(i) *The number of vertices of maximum degree in $\overline{\Gamma^*}(\mathbb{M})$ is $k(q-1)$;*

(ii) *The number of vertices of minimum degree in $\overline{\Gamma^*}(\mathbb{M})$ is $\binom{k}{k-1}(q-1)^{k-1}$.*

Proof. (i) Note that each vertex with exactly one component as non-zero is of maximum degree. Since $|\mathbb{S}| = q$ and $\mathbb{M}$ is k-dimensional, we get the number of vertices of maximum degree in $\overline{\Gamma^*}(\mathbb{M})$ is $k(q-1)$.

(ii) Each vertex with exactly $k-1$ components as non-zero is of minimum degree and the number of vertices of minimum degree in $\overline{\Gamma^*}(\mathbb{M})$ is $\binom{k}{k-1}(q-1)^{k-1}$. □

Next, we find the vertex connectivity and edge connectivity of $\overline{\Gamma^*}(\mathbb{M})$.

Theorem 3.6. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring $\mathbb{S}$ with $\dim(\mathbb{M}) = k$ and $|\mathbb{S}| = q$. Then $\kappa(\overline{\Gamma^*}(\mathbb{M})) = q-1$.*

Proof. Consider the case $k = 2$. By Theorem 2.5, $\overline{\Gamma^*}(\mathbb{M})$ is complete bipartite. Hence in this case $\overline{\Gamma^*}(\mathbb{M}) \cong K_{q-1,q-1}$. Furthermore, $\overline{\Gamma^*}(\mathbb{M}) = \{c_1\alpha_1 + c_2\alpha_2 \ : \ \text{either } c_1 = 0 \text{ (or) } c_2 = 0\} = A_1 \cup A_2$, where $A_1 = \{c_1\alpha_1 \ : \ c_1 \neq 0\}$ and $A_2 = \{c_2\alpha_2 \ : \ c_2 \neq 0\}$. Clearly $|A_1| = q-1$ and $|A_2| = q-1$. Hence $\kappa(\overline{\Gamma^*}(\mathbb{M})) = \kappa(K_{q-1,q-1}) = q-1$ if $k = 2$.

Consider the case $k \geq 3$. By Theorem 3.4, $\delta(\overline{\Gamma^*}(\mathbb{M})) = q-1$. In fact, every vertex in the set $X = \{\sum\limits_{i=1}^{k} c_i\alpha_i \ : \ \exists \text{ only one } j \text{ such that } c_j = 0 \text{ and } c_i \neq 0 \ \forall\, i \neq j \text{ and } 1 \leq j \leq k\}$ is a minimum degree vertex. For a fixed j, consider the set $A_j = \{c_j\alpha_j \ : \ c_j \neq 0\}$. Clearly $|A_j| = q-1$ for every j.

Now we claim that A_j is a minimal vertex cut of $\overline{\Gamma^*}(\mathbb{M})$. For, let $B_j = \{\sum\limits_{i=1}^{k} c_i\alpha_i \ : \ c_j = 0 \text{ and } c_i \neq 0 \ \forall\, i \neq j\}$. Clearly $|B_j| = (q-1)^{k-1}$. One can see that all vertices in $B_j \subset X$ are of minimum degree and so $N(v) = A_j$ for every $v \in B_j$. Hence every vertex in B_j is isolated in $\overline{\Gamma^*}(\mathbb{M}) - A_j$. Hence $A_j (1 \leq j \leq k)$ is a vertex cut with $q-1$ vertices of $\overline{\Gamma^*}(\mathbb{M})$.

To prove the minimality of A_j, let $H_j = H_{1j} \cup H_{2j}$, where $H_{1j} = (\bigcup\limits_{i=1}^{k} A_i) \setminus A_j$ and $H_{2j} = \{\sum\limits_{i=1}^{k} c_i\alpha_i \ :\ \text{at least two } c_i\text{'s are non zero}\} \setminus B_j$.

Claim. $\langle H_j \rangle$ is connected.

Let $a, b \in H_{1j}$. Then $a = a_{t_1}\alpha_{t_1}$ where $a_{t_1} \neq 0, 1 \leq t_1 \leq k$ and $t_1 \neq j$ and $b = b_{t_2}\alpha_{t_2}$ where $b_{t_2} \neq 0, 1 \leq t_2 \leq k$ and $t_2 \neq j$. If $t_1 \neq t_2$, then by adjacency in $\overline{\Gamma^*}(\mathbb{M})$, we get a and b are adjacent. If $t_1 = t_2$, since $k \geq 3$, there exists $d \in H_{1j}$ such that $d = d_s\alpha_s$ where $s \neq t_1, j$ and $1 \leq s \leq k$. Therefore $a - d - b$ is a path in $\overline{\Gamma^*}(\mathbb{M})$. Hence $\langle H_{1j} \rangle$ is connected.

Next, we want to prove that every vertex in H_{2j} is adjacent to at least one vertex in H_{1j}. For, let $e \in H_{2j}$. By the choice of vertex set in $\overline{\Gamma^*}(\mathbb{M})$, one can choose $e_s = 0$ for some $s \neq j$. Consider a vertex $u \in H_{1j}$ with $u_s \neq 0$. By adjacency in $\overline{\Gamma^*}(\mathbb{M})$, we get that e is adjacent to u. Hence $\langle H_j \rangle$ is connected.

Note that $\overline{\Gamma^*}(\mathbb{M}) - A_j = \langle H_j \cup B_j \rangle$ is a disconnected graph and has $(q-1)^{k-1} + 1$ connected components. Suppose $A_j^{'} \subset A_j$. Then there exists a vertex $d = d_j\alpha_j \in A_j$ such that $d_j \neq 0$ in $\overline{\Gamma^*}(\mathbb{M}) - A_j^{'}$. Clearly d is adjacent to all vertices in H_{1j} and B_j. Hence $\overline{\Gamma^*}(\mathbb{M}) - A_j^{'}$ is connected and hence $A_j^{'}$ is not a vertex cut. Therefore A_j is a minimal vertex cut of $\overline{\Gamma^*}(\mathbb{M})$.

Suppose there exists a subset A such that $|A| \leq q - 2$. To conclude the proof, it is enough to show that $\overline{\Gamma^*}(\mathbb{M}) - A$ is connected. For let $H = \bigcup\limits_{i=1}^{k} A_i$, where $A_i = \{c_i\alpha_i \ :\ c_i \neq 0\}$. Since each A_i contains $q-1$ vertices, $\langle H \setminus A \rangle$ is a complete multi-partite graph and hence connected. Let $a \in V(\overline{\Gamma^*}(\mathbb{M})) \setminus (H \cup A)$. By choice of vertex set for $\overline{\Gamma^*}(\mathbb{M})$, one can choose $a_t = 0$ for some $1 \leq t \leq k$. Choose a vertex $b \in H \setminus A$ with $b_t \neq 0$. Clearly a is adjacent to b. Thus $\overline{\Gamma^*}(\mathbb{M}) - A$ is connected. Hence A_j is a vertex cut with minimum cardinality in $\overline{\Gamma^*}(\mathbb{M})$. □

From Theorems 1.1, 3.4 and 3.6, we have the following corollary.

Corollary 3.7. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring $\mathbb{S}$ with q elements and $\dim(\mathbb{M}) = k$. Then the edge connectivity of $\overline{\Gamma^*}(\mathbb{M})$ is $\lambda(\overline{\Gamma^*}(\mathbb{M})) = q - 1$.*

Next we obtain a necessary and sufficient condition for $\overline{\Gamma^*}(\mathbb{M})$ to be Hamiltonian.

Theorem 3.8. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring $\mathbb{S}$ with $\dim(\mathbb{M}) = k$ and $|\mathbb{S}| = q$. Then $\overline{\Gamma^*}(\mathbb{M})$ is Hamilitonian if and only if $k = 2$ and $q \geq 3$.*

Proof. Assume that $\overline{\Gamma^*}(\mathbb{M})$ is Hamiltonian. Suppose $q = 2$. By Theorem 3.6, $\kappa(\overline{\Gamma^*}(\mathbb{M})) = 1$. This implies $\overline{\Gamma^*}(\mathbb{M})$ is not 2-connected, which is a contradiction to Theorem 1.2. Therefore $q \geq 3$.

Assume that $k \geq 3$. Let A be a minimum vertex cut of $\overline{\Gamma^*}(\mathbb{M})$. By Theorem 3.6, $|A| = q-1$ and as observed in the proof of Theorem 3.6, the number of connected components of $\overline{\Gamma^*}(\mathbb{M}) - A$ is $(q-1)^{k-1} + 1 > q - 1 = |A|$. This is a contradiction to Theorem 1.3. Thus $k = 2$ and $q \geq 3$.

Conversely, assume that $k = 2$ and $q \geq 3$. By Theorem 2.5, $\overline{\Gamma^*}(\mathbb{M})$ is complete bipartite and $\overline{\Gamma^*}(\mathbb{M})$ is $K_{q-1,q-1}$. Hence $\overline{\Gamma^*}(\mathbb{M})$ is Hamiltonian. □

§4 Perfectness of $\overline{\Gamma^*}(\mathbb{M})$

Throughout this section, we assume that $\mathbb{M}$ is finite dimensional with dimension k over a semiring $\mathbb{S}$. It was shown in the Theorem 2.10 that $\overline{\Gamma^*}(\mathbb{M})$ is weakly perfect, i.e. $\omega(\overline{\Gamma^*}(\mathbb{M})) = \chi(\overline{\Gamma^*}(\mathbb{M})) = \dim(\mathbb{M})$. In this section, we characterize k for which $\overline{\Gamma^*}(\mathbb{M})$ is perfect.

Theorem 4.1. *Let* $\mathbb{M}$ *be a finitely generated free semimodule over a semiring* $\mathbb{S}$ *with* $\dim(\mathbb{M}) = k$*. Then* $\overline{\Gamma^*}(\mathbb{M})$ *is perfect if and only if* $k \leq 4$.

Proof. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis of $\mathbb{M}$. Assume that $\overline{\Gamma^*}(\mathbb{M})$ is perfect. Suppose that $k \geq 5$. Let $x_1 = \alpha_1 + \alpha_2,\ x_2 = \alpha_3 + \alpha_4,\ x_3 = \alpha_1 + \alpha_5,\ x_4 = \alpha_2 + \alpha_3$, and $x_5 = \alpha_4 + \alpha_5$ in $V(\overline{\Gamma^*}(\mathbb{M}))$. Then the subgraph induced by $\Omega = \{x_1, x_2, x_3, x_4, x_5\}$ is C_5 in $\overline{\Gamma^*}(\mathbb{M})$. By Theorem 1.4, $\overline{\Gamma^*}(\mathbb{M})$ is not perfect, which is a contradiction. Therefore $k \leq 4$.

Conversely assume that $k \leq 4$.

Claim 1. $\overline{\Gamma^*}(\mathbb{M})$ does not contain any induced odd cycle of length greater than or equal to 5.

Case 1.1. If $k = 2$, then by Theorem 2.5, $\overline{\Gamma^*}(\mathbb{M})$ is complete bipartite and hence $\overline{\Gamma^*}(\mathbb{M})$ has no induced cycle of odd length.

Case 1.2. Assume that $k = 3$. Let $C:\ x_1 - x_2 - \cdots - x_{2\ell+1} - x_1$ be an induced cycle with $\ell \geq 2$ where $x_i = x_{i1}\alpha_1 + x_{i2}\alpha_2 + x_{i3}\alpha_3$, for $1 \leq i \leq 2\ell + 1$ in $\overline{\Gamma^*}(\mathbb{M})$.

Without loss of generality, let $x_{11} \neq 0$ and $x_{13} = 0$. Since x_1 is adjacent to x_2, we have $x_{21} = 0$.

Subcase 1.2.1. Suppose that $x_{12} \neq 0$. Since x_1 is adjacent to x_2, then $x_{22} = 0$. This implies that $x_1 = x_{11}\alpha_1 + x_{12}\alpha_2$ and $x_2 = x_{23}\alpha_3$ with $x_{23} \neq 0$. If $x_{43} = 0$, then x_2x_4 is an edge and so it is a chord. If $x_{43} \neq 0$, then $x_{53} = 0$ and so the edge x_2x_5 is a chord.

Subcase 1.2.2. Suppose that $x_{12} = 0$. Then $x_1 = x_{11}\alpha_1$ with $x_{11} \neq 0$. If $x_{31} = 0$, then x_1x_3 is a chord. Otherwise $x_{31} \neq 0$. Since x_3 is adjacent to x_4, we have $x_{41} = 0$ and in this case there is a chord between x_1 and x_4.

Case 1.3. Assume that $k = 4$. Consider an induced cycle $C:\ x_1 - x_2 - \cdots - x_{2\ell+1} - x_1$ with $\ell \geq 2$ where $x_i = x_{i1}\alpha_1 + x_{i2}\alpha_2 + x_{i3}\alpha_3 + x_{i4}\alpha_4$, for $1 \leq i \leq 2\ell+1$ in $\overline{\Gamma^*}(\mathbb{M})$. Without loss of generality, let $x_{11} \neq 0$ and $x_{14} = 0$. Since x_1 is adjacent to x_2, we get that $x_{21} = 0$.

Subcase 1.3.1. Suppose that $x_{12} \neq 0$ and $x_{13} \neq 0$. Now we have $x_1 = x_{11}\alpha_1 + x_{12}\alpha_2 + x_{13}\alpha_3$. As argued in the Subcase 1.2.1, we get a chord in C.

Subcase 1.3.2. Suppose that $x_{12} = 0$ and $x_{13} = 0$. Now the basic representation of x_1 is $x_{11}\alpha_1$. As argued in the Subcase 1.2.2, we get a chord in C.

Subcase 1.3.3 Suppose that $x_{12} \neq 0$ and $x_{13} = 0$. Now the vertex x_1 is of the form $x_{11}\alpha_1 + x_{12}\alpha_2$. Since x_1 is adjacent to x_2, we get that $x_{22} = 0$.

If $x_{23} = 0$, then $x_{24} \neq 0$. Now the vertex $x_2 = x_{24}\alpha_4$. Suppose $x_{44} = 0$, then the edge between x_4 and x_2 is a chord. Suppose not, since x_4 is adjacent to x_5, then $x_{54} = 0$ and in this case the edge between x_5 and x_2 is a chord.

If $x_{23} \neq 0$ and $x_{24} \neq 0$, then $x_2 = x_{23}\alpha_3 + x_{24}\alpha_4$. Since x_2 is adjacent to x_3, we have that $x_{33} = x_{34} = 0$.

Suppose that $x_{31} \neq 0$ and $x_{32} \neq 0$. Then $x_{41} = x_{42} = 0$ and so the edge between x_4 and x_1 is a chord.

Suppose $x_{31} \neq 0$ and $x_{32} = 0$. If $x_{51} = 0$, then the edge between x_5 and x_3 is a chord. If not, $x_{51} \neq 0$, and so x_5 is not adjacent to x_1. Since x_5 is adjacent to x_6, we get that $x_{61} = 0$ and so the edge between x_6 and x_3 is a chord.

Claim 2. $\Gamma^*(\mathbb{M})$ does not contain any induced odd cycle of length greater than or equal to 5.

Case 2.1. If $k = 2$, then by Theorem 2.5, $\overline{\Gamma^*}(\mathbb{M})$ is complete bipartite. Thus $\Gamma^*(\mathbb{M})$ is a disconnected graph with two connected components each of which is a complete graph and hence $\Gamma^*(\mathbb{M})$ has no induced cycle of odd length greater than or equal to 5.

Case 2.2. Assume that $k = 3$. For, let $C: \; x_1 - x_2 - \cdots - x_{2\ell+1} - x_1$ be an induced cycle with $\ell \geq 2$ in $\Gamma^*(\mathbb{M})$. Since x_1 is adjacent to x_2 and $x_{2\ell+1}$ is adjacent to x_1, there exist i and j such that x_{1i} and x_{2i} are non-zero and $x_{(2\ell+1)j}$ and x_{1j} are non-zero. If $i = j$, then the edge x_2x_{2l+1} is a chord. Suppose that $i \neq j$. Since x_2 is adjacent to x_3, then there exists t such that x_{2t} and x_{3t} are non-zero.

Suppose that $t = i$ or $t = j$, then the edge x_1x_3 is a chord. Suppose that $t \neq i$ and $t \neq j$. Since x_3 is adjacent to x_4, then there exists s such that x_{3s} and x_{4s} are non-zero. Since $k = 3$, either $s = i$ or $s = j$ or $s = t$. If $s = i$ or $s = j$, then the edge x_4x_1 is a chord. If $s = t$, then the edge x_4x_2 is a chord.

Case 2.3. Assume that $k = 4$. Let $C: \; x_1 - x_2 - \cdots - x_{2\ell+1} - x_1$ be an induced cycle with $\ell \geq 2$ in $\Gamma^*(\mathbb{M})$. In similar to the arguments in the Case 2.2, we get that $s \neq i, \; s \neq j$ and $s \neq t$. Since x_4 is adjacent to x_5, then there exists r such that x_{4r} and x_{5r} are non-zero. Since $k = 4$, either $r = i$ or $r = j$ or $r = t$ or $r = s$. Suppose $r = i$ or $r = j$, then the edge x_4x_1 is a chord. If $r = t$ or $r = s$, then the edge x_5x_3 is a chord.

From the above assertions, $\overline{\Gamma^*}(\mathbb{M})$ and $\Gamma^*(\mathbb{M})$ contain no induced cycle of odd length greater than or equal to 5 when $k \leq 4$. By Theorem 1.4, we get that $\overline{\Gamma^*}(\mathbb{M})$ is perfect if $k \leq 4$. □

§5 Genus of $\overline{\Gamma^*}(\mathbb{M})$

One of the most important topological properties of a graph is the genus of graphs. The problem of finding the genus of a graph associated with commutative rings and vector spaces have been studied in [16, 18]. In this section, we characterize the planar and toroidal nature of $\overline{\Gamma^*}(\mathbb{M})$. For details on the notion of embedding a graph in a surface, one can refer [22]. We state below certain results related to genus of graphs.

Theorem 5.1. ([8, Kuratowski]) *A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Lemma 5.2. ([22, Theorem 6.37]) $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ *if $m, n \geq 2$.*

Lemma 5.3. ([22, Theorem 6.38]) $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ *if $n \geq 3$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.*

Lemma 5.4. ([22, Corollary 6.14]) *If G is a connected graph with $n \geq 3$ vertices and m edges, then $g(G) \geq \frac{m}{6} - \frac{n}{2} + 1$. Furthermore, equality holds if and only if a triangular imbedding can be found for G.*

Theorem 5.5. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring with* $\dim(\mathbb{M}) = k$ *and* $|\mathbb{S}| = q$. *Then $\overline{\Gamma^*}(\mathbb{M})$ is planar if and only if ($k = 2$ and $q \leq 3$) or ($k = 3$ and $q = 2$).*

Proof. Let $\{\alpha_1, \ldots, \alpha_k\}$ be the basis of $\overline{\Gamma^*}(\mathbb{M})$. Assume that $\overline{\Gamma^*}(\mathbb{M})$ is planar.

Claim 1. $k \leq 3$. Suppose $k \geq 4$. Let $x_1 = \alpha_1$, $x_2 = \alpha_2$, $x_3 = \alpha_3$, $x_4 = \alpha_4$, $x_5 = \alpha_1 + \alpha_2$ and $x_6 = \alpha_3 + \alpha_4$. Then $\Omega = \{x_i \ : \ 1 \leq i \leq 6\}$ is a subset of the vertex set of $\overline{\Gamma^*}(\mathbb{M})$ and the subgraph induced by Ω contains $K_{3,3}$, which is a contradiction to $\overline{\Gamma^*}(\mathbb{M})$ is planar. Hence $k \leq 3$.

Claim 2. If $k = 3$, then q must be 2. Suppose $q \geq 3$. Let $x_1 = \alpha_1$, $x_2 = \alpha_2$, $x_3 = \alpha_3$, $x_4 = a_1\alpha_1$, $x_5 = a_1\alpha_2$, $x_6 = \alpha_2 + \alpha_3$ and $x_7 = \alpha_1 + \alpha_3$. Then $\Omega = \{x_i \ : \ 1 \leq i \leq 7\}$ is a subset of the vertex set of $\overline{\Gamma^*}(\mathbb{M})$ and the subgraph of $\overline{\Gamma^*}(\mathbb{M})$ induced by Ω contains a subdivision of K_5 as a subgraph, which is a contradiction. Hence $q = 2$.

Claim 3. If $k = 2$, then $q \leq 3$. Suppose $q \geq 4$. Since $k = 2$, by Theorem 2.5, $\overline{\Gamma^*}(\mathbb{M}) = K_{q-1,q-1}$ with $q \geq 4$. Therefore $K_{3,3}$ is a subgraph of $\overline{\Gamma^*}(\mathbb{M})$, which is a contradiction. Hence $q \leq 3$.

Conversely, assume that ($k = 2$ and $q \leq 3$) or ($k = 3$ and $q = 2$).

Case 1. If $k = 2$ and $q = 2$, then $\overline{\Gamma^*}(\mathbb{M}) = K_{1,1}$, a planar graph.

Case 2. If $k = 2$ and $q = 3$, then by Theorem 2.5, $\overline{\Gamma^*}(\mathbb{M}) = K_{2,2}$, a planar graph.

Case 3. Assume that $k = 3$ and $q = 2$. A planar embedding of $\overline{\Gamma^*}(\mathbb{M})$ is given in Figure 1. □

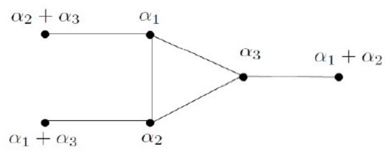

Figure 1. Planar Embedding of $\overline{\Gamma^*}(\mathbb{M})$.

Next, we characterize all finitely generated free semimodules $\mathbb{M}$ over a finite semirings for which $\overline{\Gamma^*}(\mathbb{M})$ is toroidal.

Theorem 5.6. *Let $\mathbb{M}$ be a finitely generated free semimodule over a finite semiring $\mathbb{S}$ with* $\dim(\mathbb{M}) = k \geq 2$ *and* $|\mathbb{S}| = q$. *Then $\overline{\Gamma^*}(\mathbb{M})$ is toroidal if and only if either of the following is true:*
(a) $k = 2$ *and* $q = 4$; *(b)* $k = 2$ *and* $q = 5$; *(c)* $k = 3$ *and* $q = 3$; *(d)* $k = 4$ *and* $q = 2$.

Proof. Let $\{\alpha_1, \ldots, \alpha_k\}(k \geq 2)$ be a basis of $\overline{\Gamma^*}(\mathbb{M})$. Assume that $\overline{\Gamma^*}(\mathbb{M})$ is toroidal.

Claim 1. $k \leq 4$ and $q \leq 5$. Suppose not, we have $k \geq 5$ or $q \geq 6$.

If $q \geq 6$, let $x_1 = \alpha_1$, $x_2 = a_1\alpha_1$, $x_3 = a_2\alpha_1$, $x_4 = a_3\alpha_1$, $x_5 = a_4\alpha_1$, $x_6 = \alpha_2$, $x_7 = a_1\alpha_2$, $x_8 = a_2\alpha_2$ and $x_9 = a_3\alpha_2$. Then the subgraph of $\overline{\Gamma^*}(\mathbb{M})$ induced by $\Omega = \{x_i : 1 \leq i \leq 9\}$ contains $K_{5,4}$ as a subgraph. By Lemma 5.2, $g(\overline{\Gamma^*}(\mathbb{M})) \geq 2$, which is a contradiction to $\overline{\Gamma^*}(\mathbb{M})$ is toroidal. Hence $q \leq 5$.

If $k \geq 5$, let $x_1 = \alpha_1$, $x_2 = \alpha_2$, $x_3 = \alpha_3$, $x_4 = \alpha_4$, $x_5 = \alpha_5$, $x_6 = \alpha_2 + \alpha_3$, $x_7 = \alpha_1 + \alpha_2$, $x_8 = \alpha_1 + \alpha_3$, $x_9 = \alpha_4 + \alpha_5$, $x_{10} = \alpha_2 + \alpha_4$ and $x_{11} = \alpha_1 + \alpha_4$. Then the subgraph of $\overline{\Gamma^*}(\mathbb{M})$ induced by $\Omega = \{x_i : 1 \leq i \leq 11\}$ contains a subdivision of $K_{5,4}$. By Lemma 5.2, $g(\overline{\Gamma^*}(\mathbb{M})) \geq 2$, which is a contradiction. Hence $k \leq 4$.

Claim 2. When $k = 2$, q must be either 4 or 5. If $q \leq 3$, by Theorem 5.5, $g(\overline{\Gamma^*}(\mathbb{M})) = 0$ which is a contradiction. Since $q \leq 5$, q must be 4 or 5.

Claim 3. When k is 3, q must be 3. Suppose $q \geq 4$. Consider the elements $x_1 = \alpha_1$, $x_2 = a_1\alpha_1$, $x_3 = a_2\alpha_1$, $x_4 = \alpha_2$, $x_5 = a_1\alpha_2$, $x_6 = a_2\alpha_2$, $x_7 = \alpha_3$, $x_8 = a_1\alpha_3$, $x_9 = a_2\alpha_3$, $x_{10} = \alpha_1 + \alpha_3$ and $x_{11} = a_1\alpha_1 + a_1\alpha_3$ of $V(\overline{\Gamma^*}(\mathbb{M}))$. Then the subgraph induced by $\Omega = \{x_i : 1 \leq i \leq 11\}$ contains a subdivision of $K_{5,4}$. By Lemma 5.2, $g(\overline{\Gamma^*}(\mathbb{M})) \geq 2$, which is a contradiction. Hence $q \leq 3$.

If $q = 2$, by Theorem 5.5, $g(\overline{\Gamma^*}(\mathbb{M})) = 0$, which is a contradiction. Therefore $q = 3$.

Claim 4. When k is 4, q must be 2. Suppose $q \geq 3$. Consider the elements $x_1 = \alpha_1$, $x_2 = a_1\alpha_1$, $x_3 = \alpha_2$, $x_4 = a_1\alpha_2$, $x_5 = \alpha_1 + \alpha_2$, $x_6 = \alpha_3$, $x_7 = a_1\alpha_3$, $x_8 = \alpha_4$ and $x_9 = a_1\alpha_4$ of $V(\overline{\Gamma^*}(\mathbb{M}))$. Then the subgraph induced by $\Omega = \{x_i : 1 \leq i \leq 9\}$ contains $K_{5,4}$. By Lemma 5.2, $g(\overline{\Gamma^*}(\mathbb{M})) \geq 2$, which is a contradiction. Hence $q = 2$.

For converse part, let us consider the following cases.

Case 1. Assume that $k = 2$ and $q = 4$ or 5. By Theorem 2.5, $\overline{\Gamma^*}(\mathbb{M}) \cong K_{3,3}$ or $K_{4,4}$ and so by Lemma 5.2 $g(K_{3,3}) = 1 = g(K_{4,4})$.

Case 2. Assume that $k = 3$ and $q = 3$. Now the vertices of $\overline{\Gamma^*}(\mathbb{M})$ are $x_1 = \alpha_1$, $x_2 = \alpha_2$, $x_3 = \alpha_3$, $x_4 = a_1\alpha_1$, $x_5 = a_1\alpha_2$, $x_6 = a_1\alpha_3$, $x_7 = \alpha_1 + \alpha_2$, $x_8 = \alpha_1 + a_1\alpha_2$, $x_9 = a_1\alpha_1 + \alpha_2$, $x_{10} = a_1\alpha_1 + a_1\alpha_2$, $x_{11} = \alpha_1 + \alpha_3$, $x_{12} = \alpha_1 + a_1\alpha_3$, $x_{13} = a_1\alpha_1 + \alpha_3$, $x_{14} = a_1\alpha_1 + a_1\alpha_3$, $x_{15} = \alpha_2 + \alpha_3$, $x_{16} = a_1\alpha_2 + \alpha_3$, $x_{17} = \alpha_2 + a_1\alpha_3$ and $x_{18} = a_1\alpha_2 + a_1\alpha_3$. An embedding of $\overline{\Gamma^*}(\mathbb{M})$ in the torus is given in the Figure 2.

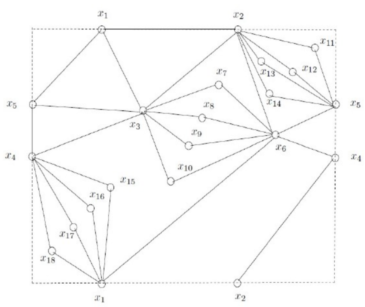

Figure 2. Embedding of $\overline{\Gamma^*}(\mathbb{M})$ in the torus.

Case 3. Assume that $k = 4$ and $q = 2$. Then the vertices of $\overline{\Gamma^*}(\mathbb{M})$ are $x_1 = \alpha_1$, $x_2 = \alpha_2$, $x_3 = \alpha_3$, $x_4 = \alpha_4$, $x_5 = \alpha_1 + \alpha_2$, $x_6 = \alpha_2 + \alpha_3$, $x_7 = \alpha_3 + \alpha_4$, $x_8 = \alpha_1 + \alpha_4$, $x_9 =$

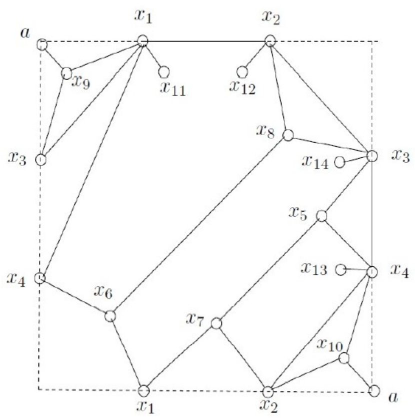

Figure 3. Embedding of $\overline{\Gamma^*}(\mathbb{M})$ in the torus.

$\alpha_2 + \alpha_4,\ x_{10} = \alpha_1 + \alpha_3,\ x_{11} = \alpha_2 + \alpha_3 + \alpha_4,\ x_{12} = \alpha_1 + \alpha_3 + \alpha_4,\ x_{13} = \alpha_1 + \alpha_2 + \alpha_3$ and $x_{14} = \alpha_1 + \alpha_2 + \alpha_4$. An embedding of $\overline{\Gamma^*}(\mathbb{M})$ in the torus is given in the Figure 3. □

Theorem 5.7. *Let $\mathbb{M}$ be a k-finitely generated free semimodule over a finite semiring $\mathbb{S}$ with q elements. Then*

$$g(\overline{\Gamma^*}(\mathbb{M})) = \begin{cases} 0 & \text{if } (k=2 \text{ and } q \leq 3) \text{ or } (k=3 \text{ and } q=2); \\ 1 & \text{if } (k=2 \text{ and } q=4 \text{ or } 5) \text{ or } (k=3 \text{ and } q=3) \\ & \qquad \text{or } (k=4 \text{ and } q=2); \\ \geq 3 & \text{otherwise.} \end{cases}$$

Proof. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be the basis of $\overline{\Gamma^*}(\mathbb{M})$.

Suppose $k \geq 5$. Let $x_1 = \alpha_1,\ x_2 = \alpha_2,\ x_3 = \alpha_3,\ x_4 = \alpha_1 + \alpha_2,\ x_5 = \alpha_1 + \alpha_3,\ x_6 = \alpha_4,\ x_7 = \alpha_5,\ x_8 = \alpha_4 + \alpha_5,\ x_9 = \alpha_2 + \alpha_3,\ x_{10} = \alpha_2 + \alpha_4 + \alpha_5,\ x_{11} = \alpha_1 + \alpha_4,\ x_{12} = \alpha_1 + \alpha_5,\ x_{13} = \alpha_2 + \alpha_3 + \alpha_4,\ x_{14} = \alpha_3 + \alpha_4$ and $x_{15} = \alpha_2 + \alpha_4$. Then the subgraph of $\overline{\Gamma^*}(\mathbb{M})$ induced by $\Omega = \{x_i\ :\ i = 1 \text{ to } 15\}$ contains a subdivision of $K_{5,5}$. Hence, by Lemma 5.2, $g(\overline{\Gamma^*}(\mathbb{M})) \geq 3$.

If $q \geq 6$, let $x_1 = \alpha_1,\ x_2 = a_1\alpha_1,\ x_3 = a_2\alpha_1,\ x_4 = a_3\alpha_1,\ x_5 = a_4\alpha_1$. Since $k \geq 2$, $x_6 = \alpha_2,\ x_7 = a_1\alpha_2,\ x_8 = a_2\alpha_2,\ x_9 = a_3\alpha_2$ and $x_{10} = a_4\alpha_2$. Then the subgraph of $\overline{\Gamma^*}(\mathbb{M})$ induced by $\Omega = \{x_i\ :\ i = 1 \text{ to } 10\}$ contains $K_{5,5}$. Therefore, by Lemma 5.2, $g(\overline{\Gamma^*}(\mathbb{M})) \geq 3$.

Hence we have $k \leq 4$ and $q \leq 5$.

Case 1.1. If $k = 2$ and $q \leq 3$, then by Theorem 5.5, $g(\overline{\Gamma^*}(\mathbb{M})) = 0$.

Case 1.2. If $k = 3$ and $q = 2$, then by Theorem 5.5, $g(\overline{\Gamma^*}(\mathbb{M})) = 0$.

Case 2.1. If $k = 2$ and $q = 4$ or 5, then by Theorem 5.6, $g(\overline{\Gamma^*}(\mathbb{M})) = 1$.

Case 2.2. If $k = 3$ and $q = 3$, then by Theorem 5.6, $g(\overline{\Gamma^*}(\mathbb{M})) = 1$.

Case 2.3. Assume that $k = 4$. If $q = 2$, then by Theorem 5.6, $g(\overline{\Gamma^*}(\mathbb{M})) = 1$.

Case 3.1. Assume that $k = 3$ and $q = 4$. By Theorem 3.3, $n = 27$ and $m = 108$. Now, by Theorem 5.4, $g(\overline{\Gamma^*}(\mathbb{M})) \geq \frac{m}{6} - \frac{n}{2} + 1 \geq 2.5$ and so $g(\overline{\Gamma^*}(\mathbb{M})) \geq 3$.

Case 3.2. Assume that $k = 3$ and $q \geq 5$. Let $x_1 = \alpha_1,\ x_2 = a_1\alpha_1,\ x_3 = a_2\alpha_1,\ x_4 = a_3\alpha_1,\ x_5 = \alpha_2,\ x_6 = a_1\alpha_2,\ x_7 = \alpha_3,\ x_8 = a_1\alpha_3,\ x_9 = a_2\alpha_3,\ x_{10} = a_3\alpha_3$ and $x_{11} = \alpha_1 + \alpha_3$. Then the subgraph of $\overline{\Gamma^*}(\mathbb{M})$ induced by $\Omega = \{x_i\ :\ i = 1 \text{ to } 11\}$ contains a subdivision of $K_{5,5}$ as a subgraph. Therefore by Lemma 5.2, $g(\overline{\Gamma^*}(\mathbb{M})) \geq 3$.

Case 3.3 Assume that $k = 4$ and $q \geq 3$. Let $x_1 = \alpha_1,\ x_2 = a_1\alpha_1,\ x_3 = \alpha_2,\ x_4 = a_1\alpha_2,\ x_5 = \alpha_1 + \alpha_2,\ x_6 = \alpha_3,\ x_7 = a_1\alpha_3,\ x_8 = \alpha_4,\ x_9 = a_1\alpha_4$ and $x_{10} = \alpha_3 + \alpha_4$. Then the subgraph of $\overline{\Gamma^*}(\mathbb{M})$ induced by $\Omega = \{x_i\ :\ i = 1 \text{ to } 10\} \subseteq V(\overline{\Gamma^*}(\mathbb{M}))$ contains $K_{5,5}$ as a subgraph. Therefore, by Lemma 5.2, $g(\overline{\Gamma^*}(\mathbb{M})) \geq 3$. □

§6 Conclusion

In this paper, we study various properties of the complement graph $\overline{\Gamma^*}(\mathbb{M})$ of the reduced non-zero component graph $\Gamma^*(\mathbb{M})$ of a finitely generated free semimodule over a semiring $\mathbb{S}$. In fact, we have studied about the connectedness, diameter, girth, domination number, clique number and chromatic number of the graph $\overline{\Gamma^*}(\mathbb{M})$. Using these parameters, it is proved that $\overline{\Gamma^*}(\mathbb{M})$ is a perfect graph. Further, we obtain a characterization for $\overline{\Gamma^*}(\mathbb{M})$ to be complete or complete bipartite. When $\mathbb{S}$ is finite, we could find the degree of all vertices, maximum degree and minimum degree in $\overline{\Gamma^*}(\mathbb{M})$. In the last part of the paper, we have characterized all finite dimensional free semimodules $\mathbb{M}$ over a semiring $\mathbb{S}$ for $\overline{\Gamma^*}(\mathbb{M})$ is planar and torodial. At last it is proved that there exists no free semimodules $\mathbb{M}$ for which the graph $\overline{\Gamma^*}(\mathbb{M})$ is of genus 2. As a further study, one can study about various domination parameters of $\overline{\Gamma^*}(\mathbb{M})$.

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