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Fisher information for generalized Rayleigh distribution in ranked set sampling design with application to parameter estimation

SHEN Bing-liang CHEN Wang-xue*
WANG Shuo CHEN Meng

Abstract. In the current paper, we considered the Fisher information matrix from the generalized Rayleigh distribution (GR) distribution in ranked set sampling (RSS). The numerical results show that the ranked set sample carries more information about λ and α than a simple random sample of equivalent size. In order to give more insight into the performance of RSS with respect to (w.r.t.) simple random sampling (SRS), a modified unbiased estimator and a modified best linear unbiased estimator (BLUE) of scale and shape λ and α from GR distribution in SRS and RSS are studied. The numerical results show that the modified unbiased estimator and the modified BLUE of λ and α in RSS are significantly more efficient than the ones in SRS.

§1 Introduction

Ranked set sampling (RSS) was introduced by McIntyre (1952) for estimating the pasture yields. It is appropriate for situations where quantification of sampling units is either costly or difficult, but ranking the units in a small set is easy and inexpensive. For a further introduction of RSS, refer to Chen et al. (2017), Chen et al. (2003), Dong and Zhang (2020), Li et al. (2015), Qian et al. (2021), Qiu and Eftekharian (2021) and Yang et al. (2020).

The procedure of RSS involves randomly drawing m^2 units from the population and then randomly partitioning them into m sets of size m. The units are then ranked within each set. Here ranking could be judgement, visual perception, covariates, or any other method

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*Corresponding author.

that does not require actual measurement of the units. For each set, one unit is selected and measured. The basic version of RSS can be elucidated as follows. First, the experimenter draws m independent simple random samples, each of size m from the population, Then units within the $ith(i = 1, 2, \ldots, m)$ sample are subjected to judgement ordering, with negligible cost, and the unit possessing ith lowest rank is identified. Finally, the identified units are measured. Proceeding in this way, we attain a ranked set sample of size m. If needed, this process can be replicated r times (cycles) to yield a sample of desired size mr.

Generalized Rayleigh (GR) distribution is a useful distribution in modelling lifetime data with scale $\lambda > 0$ and shape $\alpha > 0$ parameters, and has the following distribution function

$$F(x; \lambda, \alpha) = \left(1 - e^{-(\lambda x)^2}\right)^{\alpha} I(x > 0) \tag{1}$$

and probability distribution function (pdf)

$$f(x; \lambda, \alpha) = 2\alpha\lambda^2 x e^{-(\lambda x)^2} \left(1 - e^{-(\lambda x)^2}\right)^{\alpha - 1}.$$

Among recent literature, Kundu and Raqab (2005) discussed Fisher information matrix from GR distribution in simple random sampling (SRS). Dey et al. (2017) considered the estimation of GR distribution using Bayesian methods in SRS and RSS. Melek and Selma (2018) considered the estimation of GR distribution using maximum likelihood methods in RSS.

In this article, we considered the Fisher information matrix from GR distribution in RSS. The numerical results show that the ranked set sample carries more information about λ and α than a simple random sample of equivalent size. In order to give more insight into the performance of RSS with respect to (w.r.t.) SRS, a modified unbiased estimator and a modified best linear unbiased estimator (BLUE) of λ and α from GR distribution in SRS and RSS are studied. The numerical results show that the modified unbiased estimator and the modified BLUE of λ and α in RSS are significantly more efficient than the ones in SRS. The current paper is organized as follows. In Sect. 2, the Fisher information matrix from GR distribution in SRS and RSS are given. In Sect. 3, the modified unbiased estimator and the modified BLUE of λ in SRS and RSS are given. In Sect. 4, the modified unbiased estimator and the modified BLUE of α in SRS and RSS are given. A comparison and conclusions will be respectively presented in Sects. 5 and 6.

§2 Fisher information matrix

In this section, the Fisher information matrix from GR distribution in SRS and RSS are respectively given.

2.1 Fisher information matrix in SRS

Let $\{X_1, X_2, X_3, \cdots, X_m\}$ be a simple random sample of size m from GR distribution, then the Fisher information number based on this sample is given by Kundu and Raqab (2005)

$$I_{SRS}(\lambda, \alpha) = \begin{bmatrix} I_{11,\ SRS} & I_{12,\ SRS} \\ I_{12,\ SRS} & I_{22,\ SRS} \end{bmatrix}, \tag{2}$$

where

$$I_{11,\ SRS} = \frac{2m}{\lambda^2} + \frac{2m\alpha}{\lambda^2}\sum_{j=0}^{\infty}\begin{pmatrix}\alpha-1\\ j\end{pmatrix}(-1)^j(j+1)^{-2}$$
$$-\frac{m\left(\alpha^2-\alpha\right)}{\lambda^2}\sum_{j=0}^{\infty}\begin{pmatrix}\alpha-3\\ j\end{pmatrix}(-1)^j\left(2(j+2)^{-2}-2(j+3)^{-2}-8(j+2)^{-3}\right),$$

$$I_{12,\ SRS} = -\frac{2m\alpha}{\lambda}\sum_{j=0}^{\infty}\begin{pmatrix}\alpha-2\\ j\end{pmatrix}(-1)^j(j+2)^{-2}$$

and

$$I_{22,\ SRS} = -E\left(\frac{\partial^2 L^*_{SRS}}{\partial\alpha^2}\right) = \frac{m}{\alpha^2}.$$

2.2 Fisher information matrix in RSS

In this section, we will study the Fisher information matrix from GR distribution in RSS.

Let $\left\{X_{1(1)}, X_{2(2)}, X_{3(3)}, \cdots, X_{m(m)}\right\}$ be a ranked set sample of size m from GR distribution, then the pdf of $X_{i(i)}$ is

$$f_i(x;\lambda,\alpha) = c(i,m)\,2\alpha\lambda^2 x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\alpha i-1}\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\alpha}\right]^{m-i}, \qquad (3)$$

where $c(i,m) = \dfrac{m!}{(m-i)!\,(i-1)!}$. Then the Fisher information matrix from GR distribution in RSS is given in the following result, the detailed proof is given in the Appendix.

Theorem 1. Suppose that the usual regularity assumptions hold (see Azzalini, 1996, p.71). Then the Fisher information matrix from GR distribution in RSS is given by

$$I_{RSS}(\lambda,\alpha) = \begin{bmatrix} I_{11,\ RSS} & I_{12,\ RSS}\\ I_{12,\ RSS} & I_{22,\ RSS}\end{bmatrix}, \qquad (4)$$

where

$$I_{11,\ RSS} = -\frac{2}{\lambda^2}\sum_{i=1}^{m}c(i,m)\left(\alpha^2 i-\alpha\right)\sum_{j=0}^{m-i}\begin{pmatrix}m-i\\ j\end{pmatrix}(-1)^j\sum_{k=0}^{\infty}\begin{pmatrix}\alpha i+\alpha j-3\\ k\end{pmatrix}(-1)^k$$
$$\left((k+2)^{-2}-(k+3)^{-2}\ -4(k+2)^{-3}\right)+\frac{\alpha^2}{\lambda^2}\sum_{i=1}^{m}c(i,m)(m-i)\left\{\sum_{j=0}^{m-i-1}\begin{pmatrix}m-i-1\\ j\end{pmatrix}(-1)^j\right.$$
$$\left[8(\alpha-1)\sum_{k=0}^{\infty}\begin{pmatrix}\alpha i+\alpha j+\alpha-3\\ k\end{pmatrix}(-1)^k(k+3)^{-3}\right.$$
$$\left.+\sum_{k=0}^{\infty}\begin{pmatrix}\alpha i+\alpha j+\alpha-2\\ k\end{pmatrix}(-1)^k\left(2(k+2)^{-2}-8(k+2)^{-3}\right)\right]$$
$$\left.+\,8\alpha\sum_{j=0}^{m-i-2}\begin{pmatrix}m-i-2\\ j\end{pmatrix}(-1)^j\sum_{k=0}^{\infty}\begin{pmatrix}\alpha i+\alpha j+2\alpha-3\\ k\end{pmatrix}(-1)^k(k+3)^{-3}\right\}$$
$$+\frac{2m}{\lambda^2}+\frac{2\alpha}{\lambda^2}\sum_{i=1}^{m}c(i,m)\sum_{j=0}^{m-i}\begin{pmatrix}m-i\\ j\end{pmatrix}(-1)^j\sum_{k=0}^{\infty}\begin{pmatrix}\alpha i+\alpha j-1\\ k\end{pmatrix}(-1)^k(k+1)^{-2},$$

$$I_{12,\ RSS}=-\frac{4\alpha}{\lambda}\sum_{i=1}^{m}c\left(i,m\right)i\sum_{j=0}^{m-i}\left(\begin{array}{c}m-i\\ j\end{array}\right)\left(-1\right)^{j}\sum_{k=0}^{\infty}\left(\begin{array}{c}\alpha i+\alpha j-2\\ k\end{array}\right)\left(-1\right)^{k}\left(k+2\right)^{-2}+\frac{2\alpha}{\lambda}$$
$$\sum_{i=1}^{m}c\left(i,m\right)\left(m-i\right)\left[\sum_{j=0}^{m-i-1}\left(\begin{array}{c}m-i-1\\ j\end{array}\right)\left(-1\right)^{j}\sum_{k=0}^{\infty}\left(\begin{array}{c}\alpha i+\alpha j+\alpha-2\\ k\end{array}\right)\left(-1\right)^{k}\left(k+2\right)^{-2}-\right.$$
$$\left.\alpha\sum_{j=0}^{m-i-2}\left(\begin{array}{c}m-i-2\\ j\end{array}\right)\left(-1\right)^{j}\left(\sum_{n=1}^{\infty}\frac{1}{n\left(\alpha i+\alpha j+\alpha-1+n\right)^{2}}-\sum_{n=1}^{\infty}\frac{1}{n\left(\alpha i+\alpha j+\alpha+n\right)^{2}}\right)\right]$$

and

$$I_{22,\ RSS}=\frac{m}{\alpha^{2}}+\frac{2}{\alpha^{2}}\sum_{i=1}^{m}c\left(i,m\right)\left(m-i\right)\sum_{j=0}^{m-i-2}\left(\begin{array}{c}m-i-2\\ j\end{array}\right)\left(-1\right)^{j}\left(i+j+1\right)^{-3}.$$

In Sect. 2, the Fisher information matrix from GR distribution in SRS and RSS were given. The numerical results from Tables 1 and 2 show that the ranked set sample carry more information about λ and α than a simple random sample of equivalent size. In order to give more insight into the performance of RSS w.r.t. SRS, a modified unbiased estimator and a modified BLUE of λ and α from GR distribution in SRS and RSS will be studied in Sects. 3 and 4.

§3 Modified unbiased estimator and modified BLUE of λ

In this section, a modified unbiased estimator and a modified BLUE of λ from GR distribution in SRS and RSS will be studied.

3.1 Modified BLUE of λ in SRS

Let $\{X_1, X_2, X_3, \cdots, X_m\}$ be a simple random sample of size m from GR distribution in which α is known, then

$$\begin{aligned}E\left(X\right)&=\int_{0}^{\infty}2\alpha\lambda^{2}t^{2}e^{-(\lambda t)^{2}}\left(1-e^{-(\lambda t)^{2}}\right)^{\alpha-1}dt\\&=2\alpha\lambda^{2}\sum_{i=0}^{\infty}\left(\begin{array}{c}\alpha-1\\ i\end{array}\right)\left(-1\right)^{i}\int_{0}^{\infty}t^{2}e^{-(1+i)(\lambda t)^{2}}dt\\&=\frac{\alpha}{\lambda}\sum_{i=0}^{\infty}\left(\begin{array}{c}\alpha-1\\ i\end{array}\right)\left(-1\right)^{i}\frac{1}{\left(1+i\right)^{\frac{3}{2}}}\int_{0}^{\infty}t^{\frac{1}{2}}e^{-t}dt\\&=\frac{\sqrt{\pi}\alpha}{2\lambda}\sum_{i=0}^{\infty}\left(\begin{array}{c}\alpha-1\\ i\end{array}\right)\left(-1\right)^{i}\left(1+i\right)^{-\frac{3}{2}}\\&=\frac{1}{\lambda}c_{\alpha},\end{aligned}\tag{5}$$

where $c_\alpha=\frac{\sqrt{\pi}\alpha}{2}\sum_{i=0}^{\infty}\left(\begin{array}{c}\alpha-1\\ i\end{array}\right)\left(-1\right)^{i}\left(1+i\right)^{-\frac{3}{2}}$. From (5), we can obtain that

$c_\alpha^{-1}\left(\frac{1}{m}\sum_{i=1}^{m}X_i\right)$ is a BLUE of λ^{-1}. Thus we suggest the following estimator of λ

$$\widehat{\lambda}_{SRS,\ MBLUE}=c_\alpha\left(\frac{1}{m}\sum_{i=1}^{m}X_i\right)^{-1}, \tag{6}$$

which is called a modified BLUE. This technique was used by Abu-Dayyeh et al. (2013), He et al. (2021a) and He et al. (2021b).

3.2 Modified unbiased estimator and modified BLUE of λ in RSS

Let $\left\{X_{1(1)},X_{2(2)},X_{3(3)},\cdots,X_{m(m)}\right\}$ be a ranked set sample of size m from GR distribution in which α is known, then we can obtain

Theorem 2. A modified unbiased estimator of λ

$$\widehat{\lambda}_{RSS,\ MUE}=c_\alpha\left(\frac{1}{m}\sum_{i=1}^{m}X_{i(i)}\right)^{-1} \tag{7}$$

and a modified BLUE of λ

$$\widehat{\lambda}_{RSS,\ MBLUE}=\left(\sum_{i=1}^{m}\frac{b_i^{-1}X_{i(i)}}{d_ib_i^{-2}-1}\right)^{-1}\sum_{i=1}^{m}\frac{1}{d_ib_i^{-2}-1}, \tag{8}$$

where $b_i=\frac{\sqrt{\pi}}{2}c\left(i,m\right)\alpha\sum_{j=0}^{m-i}\left(\begin{array}{c}m-i\\ j\end{array}\right)(-1)^j\sum_{k=0}^{\infty}\left(\begin{array}{c}\alpha i+\alpha j-1\\ k\end{array}\right)(-1)^k\left(1+k\right)^{-\frac{3}{2}}$ and

$d_i=c\left(i,m\right)\alpha\sum_{j=0}^{m-i}\left(\begin{array}{c}m-i\\ j\end{array}\right)(-1)^j\sum_{k=0}^{\infty}\left(\begin{array}{c}\alpha i+\alpha j-1\\ k\end{array}\right)(-1)^k\left(1+k\right)^{-2}$.

Proof. Use the basic identities

$$\sum_{i=1}^{m}E\left(X_{i(i)}\right)=mE\left(X\right) \tag{9}$$

of order statistics (see Stokes (1995)) and (5), we can obtain $c_\alpha^{-1}\left(\frac{1}{m}\sum_{i=1}^{m}X_{i(i)}\right)$ is an unbiased estimator of λ^{-1}. Thus we suggest the following estimator of λ

$$\widehat{\lambda}_{RSS,\ MUE}=c_\alpha\left(\frac{1}{m}\sum_{i=1}^{m}X_{i(i)}\right)^{-1}, \tag{10}$$

which is called a modified unbiased estimator. In order to obtain a modified BLUE of λ in RSS, we need compute

$$\begin{aligned}E\left(X_{i(i)}\right)&=\int_0^\infty c\left(i,m\right)2\alpha\lambda^2t^2e^{-(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i-1}\left[1-\left(1-e^{-(\lambda t)^2}\right)^{\alpha}\right]^{m-i}dt\\ &=c\left(i,m\right)2\alpha\lambda^2\sum_{j=0}^{m-i}\left(\begin{array}{c}m-i\\ j\end{array}\right)(-1)^j\int_0^\infty t^2e^{-(\lambda t)^2}\left[1-e^{-(\lambda t)^2}\right]^{\alpha i+\alpha j-1}dt\\ &=c\left(i,m\right)2\alpha\lambda^2\sum_{j=0}^{m-i}\left(\begin{array}{c}m-i\\ j\end{array}\right)(-1)^j\sum_{k=0}^{\infty}\left(\begin{array}{c}\alpha i+\alpha j-1\\ k\end{array}\right)(-1)^k\int_0^\infty t^2e^{-(1+k)(\lambda t)^2}dt\end{aligned} \tag{11}$$

$$
\begin{aligned}
&= \frac{1}{\lambda} c\left(i, m\right) \alpha \sum_{j=0}^{m-i} \binom{m-i}{j} (-1)^j \sum_{k=0}^{\infty} \binom{\alpha i + \alpha j - 1}{k} (-1)^k \left(1+k\right)^{-\frac{3}{2}} \int_0^{\infty} t^{\frac{1}{2}} e^{-t} dt \\
&= \frac{1}{\lambda} \frac{\sqrt{\pi}}{2} c\left(i, m\right) \alpha \sum_{j=0}^{m-i} \binom{m-i}{j} (-1)^j \sum_{k=0}^{\infty} \binom{\alpha i + \alpha j - 1}{k} (-1)^k \left(1+k\right)^{-\frac{3}{2}} \qquad (12) \\
&= \frac{1}{\lambda} b_i
\end{aligned}
$$

and

$$
\begin{aligned}
&Var\left(X_{i(i)}\right) = E\left(X_{i(i)}^2\right) - \left[E\left(X_{i(i)}\right)\right]^2 \\
&= \int_0^{\infty} c\left(i, m\right) 2\alpha\lambda^2 t^3 e^{-(\lambda t)^2} \left(1 - e^{-(\lambda t)^2}\right)^{\alpha i - 1} \left[1 - \left(1 - e^{-(\lambda t)^2}\right)^{\alpha}\right]^{m-i} dt - \left(\frac{1}{\lambda} b_i\right)^2 \\
&= c\left(i, m\right) 2\alpha\lambda^2 t \sum_{j=0}^{m-i} \binom{m-i}{j} (-1)^j \sum_{k=0}^{\infty} \binom{\alpha i + \alpha j - 1}{k} (-1)^k \int_0^{\infty} t^3 e^{-(1+k)(\lambda t)^2} dt - \left(\frac{1}{\lambda} b_i\right)^2 \\
&= \frac{1}{\lambda^2} c\left(i, m\right) \alpha \sum_{j=0}^{m-i} \binom{m-i}{j} (-1)^j \sum_{k=0}^{\infty} \binom{\alpha i + \alpha j - 1}{k} (-1)^k \left(1+k\right)^{-2} \int_0^{\infty} t e^{-t} dt - \left(\frac{1}{\lambda} b_i\right)^2 \\
&= \frac{1}{\lambda^2} c\left(i, m\right) \alpha \sum_{j=0}^{m-i} \binom{m-i}{j} (-1)^j \sum_{k=0}^{\infty} \binom{\alpha i + \alpha j - 1}{k} (-1)^k \left(1+k\right)^{-2} - \left(\frac{1}{\lambda} b_i\right)^2 \\
&= \frac{1}{\lambda^2} \left(d_i - b_i^2\right).
\end{aligned}
\qquad (13)
$$

Use (11) and (12), we can obtain that $\left(\sum_{i=1}^{m} \frac{1}{d_i b_i^{-2} - 1}\right)^{-1} \sum_{i=1}^{m} \frac{b_i^{-1} X_{i(i)}}{d_i b_i^{-2} - 1}$ is a BLUE of λ^{-1} from Casella and Berger (1990, p.338). Thus we suggest the following estimator of λ

$$
\widehat{\lambda}_{RSS,\ MBLUE} = \left(\sum_{i=1}^{m} \frac{b_i^{-1} X_{i(i)}}{d_i b_i^{-2} - 1}\right)^{-1} \sum_{i=1}^{m} \frac{1}{d_i b_i^{-2} - 1}, \qquad (14)
$$

which is called a modified BLUE. This completes the proof of Theorem 2.

§4 Modified unbiased estimator and modified BLUE of α

In this section, a modified unbiased estimator and a modified BLUE of α from GR distribution in SRS and RSS will be studied.

4.1 Modified BLUE of α in SRS

Let $\{X_1, X_2, X_3, \cdots, X_m\}$ be a simple random sample of size m from GR distribution in which λ is known, then $Y = -\ln\left(1 - e^{-(\lambda X)^2}\right)$ has exponential distribution $\exp(\alpha)$ with rate parameter and the pdf of Y is given by

$$
g\left(y\right) = \alpha e^{-\alpha y} I(y > 0).
$$

According to Lehmann (1983, p.133), we can obtain that $\frac{1}{m}\sum_{i=1}^{m} Y_i$ is a BLUE of α^{-1}, where

$Y_i = \ln\left(1 - e^{-(\lambda X_i)^2}\right)$. Therefore, we suggest the following estimator of α

$$\widehat{\alpha}_{SRS,\ MBLUE} = m\left(\sum_{i=1}^{m} Y_i\right)^{-1}. \tag{15}$$

which is called a modified BLUE.

4.2 Modified unbiased estimator and modified BLUE of α in RSS

Let $\left\{X_{1(1)}, X_{2(2)}, X_{3(3)}, \cdots, X_{m(m)}\right\}$ be a ranked set sample of size m from GR distribution in which λ is known, then we can obtain

Theorem 3. A modified unbiased estimator of α

$$\widehat{\alpha}_{RSS,\ MUE} = m\left(\sum_{i=1}^{m} Y_{i(i)}\right)^{-1} \tag{16}$$

and a modified BLUE of λ

$$\widehat{\alpha}_{RSS,\ MBLUE} = \left(\sum_{i=1}^{m} \frac{t_i Y_{i(i)}}{v_i}\right)^{-1} \sum_{i=1}^{m} \frac{t_i^2}{v_i}, \tag{17}$$

where $t_i = \sum_{j=1}^{i} \frac{1}{m-j+i}$ and $v_i = \sum_{j=1}^{i} (m-j+i)^{-2}$.

Proof. Let $Y_{i(i)} = -\ln\left(1 - e^{-\left(\lambda X_{i(i)}\right)^2}\right)$, then $Y_{1(1)}, Y_{2(2)}, Y_{3(3)}, \cdots, Y_{m(m)}$ be a ranked set sample of size m from $\exp(\alpha)$. Then

$$E\left(\frac{1}{m}\sum_{i=1}^{m} Y_{i(i)}\right) = mEY = \frac{1}{\alpha}. \tag{18}$$

Thus we suggest the following estimator of α

$$\widehat{\alpha}_{RSS,\ MUE} = m\left(\sum_{i=1}^{m} Y_{i(i)}\right)^{-1}, \tag{19}$$

which is called a modified unbiased estimator. Since

$$E\left(t_i^{-1} Y_{i(i)}\right) = \alpha^{-1} \tag{20}$$

with

$$Var\left(t_i^{-1} Y_{i(i)}\right) = \alpha^{-2} t_i^{-2} v_i. \tag{21}$$

Then we can obtain that $\left(\sum_{i=1}^{m} \frac{t_i^2}{v_i}\right)^{-1} \sum_{i=1}^{m} \frac{t_i Y_{i(i)}}{v_i}$ is a BLUE of α^{-1} from Casella and Berger (1990, p.338). Therefore, we suggest the following estimator of α

$$\widehat{\alpha}_{RSS,\ MBLUE} = \left(\sum_{i=1}^{m} \frac{t_i Y_{i(i)}}{v_i}\right)^{-1} \sum_{i=1}^{m} \frac{t_i^2}{v_i} \tag{22}$$

which is called a modified BLUE. This completes the proof of Theorem 3.

§5 Numerical comparison

5.1 Simulation studies

In this subsection, we will compare the relative efficiencies of the above estimators.

The relative efficiency of $I_{11,\ RSS}$ w.r.t. $I_{11,\ SRS}$ may be denoted as

$$RE^1 = \frac{I_{11,\ RSS}}{I_{11,\ SRS}} \tag{23}$$

from Barabesi and El-Sharaawi (2001). Similarly the relative efficiency of $I_{22,\ RSS}$ w.r.t. $I_{11,\ SRS}$ may be denoted as RE^2. It can be seen that RE^1 are free of λ, RE^2 is free of λ and α. The efficiency of $\widehat{\lambda}_{RSS,\ MUE}$ w.r.t. $\widehat{\lambda}_{SRS,\ MBLUE}$ may be denoted as

$$eff^1 = \frac{MSE\left(\hat{\lambda}_{SRS,\ BLUE}\right)}{MSE\left(\hat{\lambda}_{RSS,\ UE}\right)}, \tag{24}$$

where MSE is an abbreviation of the mean square error. Similarly, the efficiencies of $\widehat{\lambda}_{RSS,\ MBLUE}$ w.r.t. $\widehat{\lambda}_{SRS,\ MBLUE}$, $\widehat{\alpha}_{RSS,\ MUE}$ w.r.t. $\widehat{\alpha}_{SRS,\ MBLUE}$ and $\widehat{\alpha}_{RSS,\ MBLUE}$ w.r.t. $\widehat{\alpha}_{SRS,\ MBLUE}$ may be denoted as eff^2, eff^3, eff^4, respectively.

From Tables 1-2, we conclude the following:

(1) $RE^1 > 1$, which means the ranked set sample carry more information about the scale parameter λ than a simple random sample of equivalent size.

(2) $RE^2 > 1$, which means the ranked set sample carry more information about the shape parameter α than a simple random sample of equivalent size.

(3) In conclusion, the ranked set sample carry more information about the scale and shape parameters λ and α than a simple random sample of equivalent size.

From Tables 3-4, we conclude the following:

(4) $eff^1 > 1$, which means $\widehat{\lambda}_{RSS,\ MUE}$ is more efficient $\widehat{\lambda}_{SRS,\ MBLUE}$.

(5) $eff^2 > 1$, which means $\widehat{\lambda}_{RSS,\ MBLUE}$ is more efficient $\widehat{\lambda}_{SRS,\ MBLUE}$.

(6) $eff^3 > 1$, which means $\widehat{\alpha}_{RSS,\ MUE}$ is more efficient $\widehat{\alpha}_{SRS,\ MBLUE}$.

(7) $eff^4 > 1$, which means $\widehat{\alpha}_{RSS,\ MBLUE}$ is more efficient $\widehat{\alpha}_{SRS,\ MBLUE}$.

(8) In conclusion, the modified unbiased estimator and the modified BLUE of λ and α from GR distribution in RSS are significantly more efficient than the ones in SRS.

5.2 A real data application

In this subsection, we present a data analysis with a real data was originated by Bader and Priest (1982). Data set includes the 69 strength measured in GPA (giga-Pascals), for single carbon fibers and impregnated 1000 carbon fiber tows. Melek and Selma (2018) proved that data set fitted with GR distribution with $\lambda = 0.77510$ and $\alpha = 3.24615$ very well. The result of the analysis presented in Tables 5 and 6. It can be seen from the tables that there are the same conclusions as simulation results of the previous sections. This agrees with the simulation results of the previous sections.

§6 Conclusion

In this paper, we considered the Fisher information matrix from GR distribution in both SRS and RSS designs. We showed that the Fisher information in RSS is more than its counterpart in SRS for different parameters of GR distribution. In order to give more insight into the performance of RSS w.r.t. SRS, we proposed a modified unbiased estimator and a BLUE of λ

and α from GR distribution in SRS and RSS. The numerical results show that the modified unbiased estimator and the modified BLUE of λ and α in RSS are significantly more efficient than the ones in SRS. A further study would be to extend the use of moving extremes ranked set sampling methods to GR distribution.

Table 1. The Fisher information number and the relative efficiency of $I_{11,\ RSS}$ w.r.t. $I_{11,\ SRS}$.

α	m	$\lambda^2 I_{11,\ SRS}$	$\lambda^2 I_{11,\ RSS}$	RE^1
3	3	30	48.0750	1.60
	4	40	83.6763	2.09
	5	50	128.4771	2.57
	6	60	183.3105	3.06
	7	70	247.4842	3.54
4	3	37.3333	61.5707	1.65
	4	49.7778	106.0731	2.13
	5	62.2222	163.1008	2.62
	6	74.6667	232.0105	3.11
	7	87.1111	312.8632	3.59
5	3	43.9444	73.1658	1.66
	4	58.5926	126.6676	2.16
	5	73.2407	193.7451	2.65
	6	87.8889	275.8960	3.14
	7	102.5370	371.2359	3.62

Table 2. The Fisher information number and the relative efficiency of $I_{22,\ RSS}$ w.r.t. $I_{22,\ SRS}$.

m	$\alpha^2 I_{22,\ SRS}$	$\alpha^2 I_{22,\ RSS}$	RE^2
3	3	5.4233	1.81
4	4	8.8538	2.21
5	5	13.0731	2.61
6	6	18.1326	3.02
7	7	23.9857	3.43

Table 3. The efficiency of $\widehat{\lambda}_{RSS,\ MUE}$ w.r.t. $\widehat{\lambda}_{SRS,\ MBLUE}$ and $\widehat{\lambda}_{RSS,\ MBLUE}$ w.r.t. $\widehat{\lambda}_{SRS,\ MBLUE}$.

$(\alpha,\ \lambda)$	m	eff^1	eff^2
(3,1)	3	3.68	4.72
	4	5.86	6.34
	5	6.18	7.58
	6	11.93	13.67
	7	13.92	15.04

Table 3. Continued

(4,1)	3	4.29	4.48
	4	5.28	6.49
	5	8.49	9.06
	6	11.45	13.91
	7	13.35	15.17
(5,1)	3	4.33	4.80
	4	5.95	7.68
	5	7.07	9.57
	6	12.31	14.20
	7	15.94	17.83

Table 4. The efficiency of $\widehat{\alpha}_{RSS,\ MUE}$ w.r.t. $\widehat{\alpha}_{SRS,\ MBLUE}$ and $\widehat{\alpha}_{RSS,\ MBLUE}$ w.r.t. $\widehat{\alpha}_{SRS,\ MBLUE}$.

α	m	eff^3	eff^4
3	3	4.33	4.98
	4	4.40	5.30
	6	4.79	5.78
	7	4.82	5.58
4	3	4.39	4.60
	4	4.16	4.70
	5	4.19	5.31
	6	4.47	6.63
	7	5.10	5.94
5	3	4.54	5.31
	4	3.74	4.99
	5	4.14	4.18
	6	4.61	5.52
	7	4.79	5.85

Table 5. The efficiency of $\widehat{\lambda}_{RSS,\ MUE}$ w.r.t. $\widehat{\lambda}_{SRS,\ MBLUE}$ and $\widehat{\lambda}_{RSS,\ MBLUE}$ w.r.t. $\widehat{\lambda}_{SRS,\ MBLUE}$.

$(\alpha,\ \lambda)$	m	eff^1	eff^2
(3.24615,0.77510)	3	4.33	4.42
	4	4.40	6.50
	5	6.32	6.64

Table 6. The efficiency of $\widehat{\alpha}_{RSS,\ MUE}$ w.r.t. $\widehat{\alpha}_{SRS,\ MBLUE}$ and $\widehat{\alpha}_{RSS,\ MBLUE}$ w.r.t. $\widehat{\alpha}_{SRS,\ MBLUE}$.

$(\alpha,\ \lambda)$	m	eff^3	eff^4
(3.24615,0.77510)	3	4.99	5.30
	4	4.42	4.79
	5	4.52	5.34

Appendix. Proof of Theorem 1. The log-likelihood function for a single observation

$$L_{RSS}^{*}=d_0+(\alpha i-1)\ln\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)+(m-i)\ln\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\right)+\ln\alpha+\ln 2X_{i(i)} + 2\ln\lambda-\lambda^2X_{i(i)}^2,$$

where d_0 is a value which is free of α and λ. In order to compute the Fisher information matrix in RSS, the first derivative and the second derivative of L_{RSS}^{*} are respectively computed as

$$\frac{\partial L_{RSS}^{*}}{\partial\lambda}=(\alpha i-1)\frac{2\lambda X_{i(i)}^2e^{-\left(\lambda X_{i(i)}\right)^2}}{1-e^{-\left(\lambda X_{i(i)}\right)^2}}-(m-i)\frac{2\alpha\lambda X_{i(i)}^2e^{-\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-1}}{1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}} + \frac{2}{\lambda}-2\lambda X_{i(i)}^2,$$

$$\frac{\partial L_{RSS}^{*}}{\partial\alpha}=\frac{1}{\alpha}+i\ln\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)-(m-i)\frac{\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\ln\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)}{1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}},$$

$$\begin{aligned}\frac{\partial^2 L_{RSS}^{*}}{\partial\lambda^2}=&(\alpha i-1)\frac{2X_{i(i)}^2e^{-\left(\lambda X_{i(i)}\right)^2}-2X_{i(i)}^2e^{-2\left(\lambda X_{i(i)}\right)^2}-4\lambda^2X_{i(i)}^4e^{-\left(\lambda X_{i(i)}\right)^2}}{\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^2}-2X_{i(i)}^2\\
&-(m-i)\left[\frac{4\left(\alpha^2-\alpha\right)\lambda^2X_{i(i)}^4e^{-2\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-2}+2\alpha X_{i(i)}^2e^{-\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-1}}{1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}}\right.\\
&\left.-\frac{4\alpha\lambda^2X_{i(i)}^4e^{-\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-1}}{1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}}+\frac{4\alpha^2\lambda^2X_{i(i)}^4e^{-2\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{2\alpha-2}}{\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\right)^2}\right]-\frac{2}{\lambda^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 L_{RSS}^{*}}{\partial\lambda\partial\alpha}=&\,i\frac{2\lambda X_{i(i)}^2e^{-\left(\lambda X_{i(i)}\right)^2}}{1-e^{-\left(\lambda X_{i(i)}\right)^2}}-(m-i)\left[\frac{2\lambda X_{i(i)}^2e^{-\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-1}}{1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}}\right.\\
&\left.+\frac{2\alpha\lambda X_{i(i)}^2e^{-\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-1}\ln\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)}{\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\right)^2}\right]\end{aligned}$$

and

$$\frac{\partial^2 L_{RSS}^{*}}{\partial\alpha^2}=-\frac{1}{\alpha^2}-(m-i)\frac{\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\ln^2\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)}{\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\right)^2}.$$

Then under the assumed regularity conditions of Theorem 1

$$I_{11,\ RSS} = -\sum_{i=1}^{m} E\left(\frac{\partial^2 L^*_{RSS}}{\partial\lambda^2}\right)$$

$$E\left[\frac{4\left(\alpha^2-\alpha\right)\lambda^2 X^4_{i(i)} e^{-2\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-3} + 2\alpha X^2_{i(i)} e^{-\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-2}}{1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}}\right.$$

$$\left. - \frac{4\alpha\lambda^2 X^4_{i(i)} e^{-\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-2}}{1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}} + \frac{4\alpha^2\lambda^2 X^4_{i(i)} e^{-2\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{2\alpha-2}}{\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\right)^2}\right] + \frac{2m}{\lambda^2}$$

$$+ 2\sum_{i=1}^{m} E\left[X^2_{i(i)}\right]$$

$$= -2\lambda^2 \sum_{i=1}^{m} c\left(i,m\right)\left(\alpha^2 i-\alpha\right)\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^j \int_0^\infty \left(2t^3 e^{-2(\lambda t)^2} - 2t^3 e^{-3(\lambda t)^2}\right.$$

$$\left. - 4\lambda^2 t^5 e^{-2(\lambda X)^2}\right)\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha j-3} dt + 2\alpha^2\lambda^2 \sum_{i=1}^{m} c\left(i,m\right)\left(m-i\right)$$

$$\left[\int_0^\infty \left(4\left(\alpha-1\right)\lambda^2 t^5 e^{-3(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha-3} + 2t^3 e^{-2(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha-2}\right.\right.$$

$$\left. - 4\lambda^2 t^5 e^{-2(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha-2}\right)\left(1-\left(1-e^{-(\lambda t)^2}\right)^{\alpha}\right)^{m-i-1} dt$$

$$\left. + \int_0^\infty 4\alpha\lambda^2 t^5 e^{-3(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+2\alpha-3}\left(1-\left(1-e^{-(\lambda t)^2}\right)^{\alpha}\right)^{m-i-2} dt\right]$$

$$+ \frac{2m}{\lambda^2} + \sum_{i=1}^{m} c\,(i,m) 4\alpha\lambda^2 \int_0^\infty t^3 e^{-(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i-1}\left(1-\left(1-e^{-(\lambda t)^2}\right)^{\alpha}\right)^{m-i} dt$$

$$= -2\lambda^2 \sum_{i=1}^{m} c\left(i,m\right)\left(\alpha^2 i-\alpha\right)\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^j \sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-3}{k}(-1)^k \int_0^\infty \left(2t^3 e^{-(k+2)(\lambda t)^2}\right.$$

$$\left. - 2t^3 e^{-(k+3)(\lambda t)^2} - 4\lambda^2 t^5 e^{-(k+2)(\lambda t)^2}\right) dt + 2\alpha^2\lambda^2 \sum_{i=1}^{m} c\left(i,m\right)\left(m-i\right)$$

$$\left[\sum_{j=0}^{m-i-1}\binom{m-i-1}{j}(-1)^j \left(4\left(\alpha-1\right)\lambda^2 \int_0^\infty t^5 e^{-3(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha j+\alpha-3} dt\right.\right.$$

$$\left. + \int_0^\infty \left(2t^3 e^{-2(\lambda t)^2} - 4\lambda^2 t^5 e^{-2(\lambda t)^2}\right)\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha j+\alpha-2} dt\right)$$

$$\left. + 4\alpha\lambda^2 \sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-1)^j \int_0^\infty t^5 e^{-3(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha j+2\alpha-3} dt\right]$$

$$+ \frac{2m}{\lambda^2} + \sum_{i=1}^{m} c\,(i,m) 4\alpha\lambda^2 \sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^j \int_0^\infty t^3 e^{-(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha j+1} dt$$

$$= -\frac{2}{\lambda^2}\sum_{i=1}^{m} c\left(i,m\right)\left(\alpha^2 i-\alpha\right)\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^j \sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-3}{k}(-1)^k \left((k+2)^{-2}\int_0^\infty te^{-t}dt\right.$$

$$\left. - (k+3)^{-2}\int_0^\infty te^{-t}dt - (k+2)^{-3}\int_0^\infty te^{-t}dt\right) + 2\alpha^2\lambda^2 \sum_{i=1}^{m} c\left(i,m\right)\left(m-i\right)$$

$$\left\{\sum_{j=0}^{m-i-1}\binom{m-i-1}{j}(-1)^j\left[4(\alpha-1)\lambda^2\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+\alpha-3}{k}(-1)^k\int_0^{\infty}t^5e^{-(k+3)(\lambda t)^2}dt\right.\right.$$

$$+\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+\alpha-2}{k}(-1)^k\left(\int_0^{\infty}2t^3e^{-(k+2)(\lambda t)^2}dt-\int_0^{\infty}4\lambda^2t^5e^{-(k+2)(\lambda t)^2}dt\right)$$

$$\left.+\,4\alpha\lambda^2\sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-1)^j\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+2\alpha-3}{k}(-1)^k\int_0^{\infty}t^5e^{-(k+3)(\lambda t)^2}dt\right\}$$

$$+\frac{2m}{\lambda^2}+\sum_{i=1}^{m}c(i,m)4\alpha\lambda^2\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^j\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-1}{k}(-1)^k\int_0^{\infty}t^3e^{-(k+1)(\lambda t)^2}dt$$

$$=-\frac{2}{\lambda^2}\sum_{i=1}^{m}c(i,m)\left(\alpha^2 i-\alpha\right)\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^j\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-3}{k}(-1)^k\left((k+2)^{-2}\right.$$

$$\left.-(k+3)^{-2}-4(k+2)^{-3}\right)+\frac{\alpha^2}{\lambda^2}\sum_{i=1}^{m}c(i,m)(m-i)$$

$$\left\{\sum_{j=0}^{m-i-1}\binom{m-i-1}{j}(-1)^j\left[8(\alpha-1)\lambda^2\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+\alpha-3}{k}(-1)^k(k+3)^{-3}\right.\right.$$

$$\left.\int_0^{\infty}te^{-t}dt+\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+\alpha-2}{k}(-1)^k\left(2(k+2)^{-2}\int_0^{\infty}te^{-t}dt-4(k+2)^{-3}\int_0^{\infty}t^2e^{-t}dt\right)\right]$$

$$\left.+\,4\alpha\sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-1)^j\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+2\alpha-3}{k}(-1)^k(k+3)^{-3}\int_0^{\infty}t^2e^{-t}dt\right\}$$

$$+\frac{2m}{\lambda^2}+\frac{2\alpha}{\lambda^2}\sum_{i=1}^{m}c(i,m)\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^j\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-1}{k}(-1)^k(k+1)^{-2}\int_0^{\infty}te^{-t}dt$$

$$=-\frac{2}{\lambda^2}\sum_{i=1}^{m}c(i,m)\left(\alpha^2 i-\alpha\right)\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^j$$

$$\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-3}{k}(-1)^k\left((k+2)^{-2}-(k+3)^{-2}-4(k+2)^{-3}\right)$$

$$+\frac{\alpha^2}{\lambda^2}\sum_{i=1}^{m}c(i,m)(m-i)\left\{\sum_{j=0}^{m-i-1}\binom{m-i-1}{j}(-1)^j\left[8(\alpha-1)\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+\alpha-3}{k}\right.\right.$$

$$\left.(-1)^k(k+3)^{-3}+\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+\alpha-2}{k}(-1)^k\left(2(k+2)^{-2}-8(k+2)^{-3}\right)\right]$$

$$\left.+\,8\alpha\sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-1)^j\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+2\alpha-3}{k}(-1)^k(k+3)^{-3}\right\}$$

$$+\frac{2m}{\lambda^2}+\frac{2\alpha}{\lambda^2}\sum_{i=1}^{m}c(i,m)\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^j\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-1}{k}(-1)^k(k+1)^{-2},$$

$$I_{12,\ RSS}=-\sum_{i=1}^{m}E\left(\frac{\partial^2L^*_{RSS}}{\partial\lambda\partial\alpha}\right)$$

$$=-\sum_{i=1}^{m}iE\left[\frac{2\lambda X_{i(i)}^2e^{-\left(\lambda X_{i(i)}\right)^2}}{1-e^{-\left(\lambda X_{i(i)}\right)^2}}\right]+\sum_{i=1}^{m}(m-i)E\left[\frac{2\lambda X_{i(i)}^2e^{-\left(\lambda X_{i(i)}\right)^2}\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha-1}}{1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}}+\right.$$

$$+\left.\frac{2\alpha\lambda X_{i(i)}^{2}e^{-\left(\lambda X_{i(i)}\right)^{2}}\left(1-e^{-\left(\lambda X_{i(i)}\right)^{2}}\right)^{\alpha-1}\ln\left(1-e^{-\left(\lambda X_{i(i)}\right)^{2}}\right)}{\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^{2}}\right)^{\alpha}\right)^{2}}\right]$$

$$=-4\alpha\lambda^{3}\sum_{i=1}^{m}c(i,m)\,i\int_{0}^{\infty}t^{3}e^{-2(\lambda t)^{2}}\left(1-e^{-(\lambda t)^{2}}\right)^{\alpha i-2}\left(1-\left(1-e^{-(\lambda t)^{2}}\right)^{\alpha}\right)^{m-i}dt$$

$$+2\alpha\lambda^{2}\sum_{i=1}^{m}c(i,m)(m-i)\int_{0}^{\infty}2\lambda t^{3}e^{-2(\lambda t)^{2}}\left(1-e^{-(\lambda t)^{2}}\right)^{\alpha i+\alpha-2}\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^{2}}\right)^{\alpha}\right)^{m-i-1}$$

$$+2\alpha\lambda t^{3}e^{-2(\lambda t)^{2}}\left(1-e^{-(\lambda t)^{2}}\right)^{\alpha i+\alpha-2}\ln\left(1-e^{-(\lambda t)^{2}}\right)\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^{2}}\right)^{\alpha}\right)^{m-i-2}dt$$

$$=-4\alpha\lambda^{3}\sum_{i=1}^{m}c(i,m)\,i\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^{j}\int_{0}^{\infty}t^{3}e^{-2(\lambda t)^{2}}\left(1-e^{-(\lambda t)^{2}}\right)^{\alpha i+\alpha j-2}dt$$

$$+2\alpha\lambda^{3}\sum_{i=1}^{m}c(i,m)(m-i)\left[\sum_{j=0}^{m-i-1}\binom{m-i-1}{j}(-1)^{j}\int_{0}^{\infty}2t^{3}e^{-2(\lambda t)^{2}}\left(1-e^{-(\lambda t)^{2}}\right)^{\alpha i+\alpha j+\alpha-2}dt\right.$$

$$\left.+\sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-1)^{j}\int_{0}^{\infty}2\alpha t^{3}e^{-2(\lambda t)^{2}}\left(1-e^{-(\lambda t)^{2}}\right)^{\alpha i+\alpha j+\alpha-2}\ln\left(1-e^{-(\lambda t)^{2}}\right)dt\right]$$

$$=-4\alpha\lambda^{3}\sum_{i=1}^{m}c(i,m)\,i\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-2}{k}(-1)^{k}\int_{0}^{\infty}t^{3}e^{-(k+2)(\lambda t)^{2}}dt$$

$$+2\alpha\lambda^{3}\sum_{i=1}^{m}c(i,m)(m-i)\left[\sum_{j=0}^{m-i-1}\binom{m-i-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+\alpha-2}{k}(-1)^{k}\right.$$

$$\left.\int_{0}^{\infty}2t^{3}e^{-(k+2)(\lambda t)^{2}}dt-\alpha\sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-1)^{j}\int_{0}^{1}\ln t\cdot t(1-t)^{\alpha i+\alpha j+\alpha-2}\ln(1-t)dt\right]$$

$$=-\frac{4\alpha}{\lambda}\sum_{i=1}^{m}c(i,m)\,i\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-2}{k}(-1)^{k}(k+2)^{-2}\int_{0}^{\infty}te^{-t}dt$$

$$+\frac{2\alpha}{\lambda}\sum_{i=1}^{m}c(i,m)(m-i)\left[\sum_{j=0}^{m-i-1}\binom{m-i-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+\alpha-2}{k}(-1)^{k}(k+2)^{-2}\int_{0}^{\infty}\right.$$

$$\left.te^{-t}dt-\alpha\sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-1)^{j}\left(\int_{0}^{1}\ln t\cdot t^{\alpha i+\alpha j+\alpha-2}\ln(1-t)dt-\int_{0}^{1}\ln t\cdot t^{\alpha i+\alpha j+\alpha-1}\ln(1-t)dt\right)\right]$$

$$=-\frac{4\alpha}{\lambda}\sum_{i=1}^{m}c(i,m)\,i\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j-2}{k}(-1)^{k}(k+2)^{-2}$$

$$+\frac{2\alpha}{\lambda}\sum_{i=1}^{m}c(i,m)(m-i)\left[\sum_{j=0}^{m-i-1}\binom{m-i-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{\alpha i+\alpha j+\alpha-2}{k}(-1)^{k}(k+2)^{-2}\right.$$

$$\left.-\alpha\sum_{j=0}^{m-i-2}\binom{m-i-2}{j}(-1)^{j}\left(\sum_{n=1}^{\infty}\frac{1}{n(\alpha i+\alpha j+\alpha-1+n)^{2}}-\sum_{n=1}^{\infty}\frac{1}{n(\alpha i+\alpha j+\alpha+n)^{2}}\right)\right]$$

and

$$I_{22,\ RSS}=-\sum_{i=1}^{m}E\left(\frac{\partial^{2}L_{RSS}^{*}}{\partial\alpha^{2}}\right)$$

$$= \frac{m}{\alpha^2} + \sum_{i=1}^{m}(m-i)E\left[\frac{\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\ln^2\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)}{\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\right)^2}\right]$$

$$= \frac{m}{\alpha^2} + c\left(i,m\right)\alpha\sum_{i=1}^{m}(m-i)\int_0^{\infty}2\lambda^2 te^{-(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha-1}\ln^2\left(1-e^{-(\lambda t)^2}\right)$$
$$\left(1-\left(1-e^{-\left(\lambda X_{i(i)}\right)^2}\right)^{\alpha}\right)^{m-i-2}dt$$

$$= \frac{m}{\alpha^2} + \alpha\sum_{i=1}^{m}c\left(i,m\right)(m-i)\sum_{j=0}^{m-i-2}\left(\begin{array}{c} m-i-2 \\ j \end{array}\right)(-1)^j\int_0^{\infty}2\lambda^2 te^{-(\lambda t)^2}\left(1-e^{-(\lambda t)^2}\right)^{\alpha i+\alpha j+\alpha-1}$$
$$\ln^2\left(1-e^{-(\lambda t)^2}\right)dt$$

$$= \frac{m}{\alpha^2} + \alpha\sum_{i=1}^{m}c\left(i,m\right)(m-i)\sum_{j=0}^{m-i-2}\left(\begin{array}{c} m-i-2 \\ j \end{array}\right)(-1)^j\int_0^{1}t^{\alpha i+\alpha j+\alpha-1}\ln^2 tdt$$

$$= \frac{m}{\alpha^2} + \frac{2}{\alpha^2}\sum_{i=1}^{m}c\left(i,m\right)(m-i)\sum_{j=0}^{m-i-2}\left(\begin{array}{c} m-i-2 \\ j \end{array}\right)(-1)^j\left(i+j+1\right)^{-3}.$$

The Theorem is proved by combining $I_{11,\ RSS}$, $I_{12,\ RSS}$ with $I_{22,\ RSS}$.

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Department of Mathematics and Statistics, Jishou University, Jishou 416000, China.

Email: chenwangxue2017@163.com