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Symmetries and conservation laws associated with a hyperbolic mean curvature flow

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Abstract. Under investigation in this paper is a hyperbolic mean curvature flow for convex evolving curves. Firstly, in view of Lie group analysis, infinitesimal generators, symmetry groups and an optimal system of symmetries of the considered hyperbolic mean curvature flow are presented. At the same time, some group invariant solutions are computed through reduced equations. In particular, we construct explicit solutions by applying the power series method. Furthermore, the convergence of the solutions of power series is certificated. Finally, conservation laws of the hyperbolic mean curvature flow are established via Ibragimov's approach.

§1 Introduction

As the mean curvature of the hypersurface is the main driving factor, LeFloch established a model and referred to this model as the hyperbolic mean curvature flow (HMCF) [17]. The hyperbolic mean curvature flow for convex evolving curves was introduced in [18],

$$\frac{\partial^2\gamma}{\partial t^2}(u,t)=\frac{1}{2}(1+|\frac{\partial\gamma}{\partial t}|^2)k(u,t)\vec{N}(u,t)-\nabla\rho(u,t),\quad \forall(u,t)\in I\times[0,T),\tag{1}$$

where $I\subset(-\infty,+\infty)$, k stands for the mean curvature of curve $\gamma(u,t)$, $\vec{N}$ represents the inner unit normal of curve $\gamma(u,t)$, $\nabla\rho=(\frac{\partial^2\gamma}{\partial s\partial t},\frac{\partial\gamma}{\partial t})\vec{T}$. Hyperbolic mean curvature flow has been widely studied both in mathematics and physics. The lifespan of the classical solution to the Cauchy problem was derived in [18], the uniqueness of the short-time smooth solution and nonlinear wave equations for curvatures were obtained in [9], self-similar solutions were provided in [8], development of singularity was discussed in [15], local solvability of the hyperbolic Gauss curvature flow and several results on finite time blow-up were given in [4], contraction of the hyperbolic curve was analyzed in [14], applications in Einstein equations were studied in [16]. Two forced hyperbolic mean curvature flows were investigated in [19], the asymptotical behavior of a strictly convex closed planar curve driven by a hyperbolic normal flow was shown in [26].

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As everyone knows, Lie group method plays an important role in studying the geometric properties of the geometric flow [23,24,25]. The main thoughts of the symmtery method are to construct invariance condition and obtain reductions to differential equations [3,5,7,20]. Once the reduced equations are given, a large number of corresponding exact solutions can be obtained. In order to obtain the classification of all reduction equations, we require an optimal system of one-dimensional subalgebra constructed by the symmetry method [20]. Utilizing Lie group analysis, we are going to get an optimal system of (1), in which some fascinating special solutions are presented. Another significant field is the conservation laws of PDEs which have a major effect on constructing solutions of PDEs [7,22]. We will obtain conservation laws of Eq.(1) by using Ibragimov's method [10].

The remaining of the paper is arranged as follows. Symmetries and related optimal system of the hyperbolic mean curvature flow are analyzed in Section 2; Section 3 considers the symmetry reductions by means of similar variables; in Section 4, some new explicit solutions are provided with help of the power series method, and the convergence of the solutions of power series is presented; in Section 5, nonlinearly self-adjointness of Eq.(1) is proved and its conservation laws are established by using Ibragimov's technique. These conservation laws can be used to interpret some physical phenomena.

§2 Lie point symmetry

Through the support function of convex evolving curves, an associated hyperbolic equation of Eq.(1) is obtained as follows(see [18] for more details)

$$hh_{\tau\tau} + h_{\tau\tau}h_{\theta\theta} - h_{\theta\tau}^2 + \frac{1}{2}(1 + h_\tau^2) = 0, \quad \forall(\theta,\tau) \in I \times [0,T). \tag{2}$$

Next, we perform Lie symmetry technique for Eq.(2). First of all, let's think about a vector field of infinitesimal transformations of Eq.(2) with the form

$$X = \xi^1(\theta,\tau,h)\partial_\theta + \xi^2(\theta,\tau,h)\partial_\tau + \eta(\theta,\tau,h)\partial_h. \tag{3}$$

Based on the transformation (3), applying the invariance conditions to Eq.(2), we get [2,20]

$$\mathrm{pr}^{(2)}X(\Delta)|_{\Delta=0} = 0,$$

where $\mathrm{pr}^{(2)}X$ is the 2th-order prolongation of X [2,20] and $\Delta = hh_{\tau\tau} + h_{\tau\tau}h_{\theta\theta} - h_{\theta\tau}^2 + \frac{1}{2}(1 + h_\tau^2)$. For Eq.(2), $\mathrm{pr}^{(2)}X$ is

$$\mathrm{pr}^{(2)}X = X + \eta_\tau^{(1)}\frac{\partial}{\partial h_\tau} + \eta_{\theta\theta}^{(2)}\frac{\partial}{\partial h_{\theta\theta}} + \eta_{\theta\tau}^{(2)}\frac{\partial}{\partial h_{\theta\tau}} + \eta_{\tau\tau}^{(2)}\frac{\partial}{\partial h_{\tau\tau}},$$

where

$$\begin{aligned}
\eta_\tau^{(1)} &= D_\tau(\eta) - h_\theta D_\tau(\xi^1) - h_\tau D_\tau(\xi^2),\\
\eta_{\theta\theta}^{(2)} &= D_\theta^2(\eta - \xi^1 h_\theta - \xi^2 h_\tau) + \xi^1 h_{\theta\theta\theta} + \xi^2 h_{\theta\theta\tau},\\
\eta_{\theta\tau}^{(2)} &= D_\theta D_\tau(\eta - \xi^1 h_\theta - \xi^2 h_\tau) + \xi^1 h_{\theta\theta\tau} + \xi^2 h_{\theta\tau\tau},\\
\eta_{\tau\tau}^{(2)} &= D_\tau^2(\eta - \xi^1 h_\theta - \xi^2 h_\tau) + \xi^1 h_{\theta\tau\tau} + \xi^2 h_{\tau\tau\tau},
\end{aligned}$$

and D_θ, D_τ represent the total differential operators, for example,

$$D_\tau = \frac{\partial}{\partial \tau} + h_\tau \frac{\partial}{\partial h} + h_{\theta\tau} \frac{\partial}{\partial h_\theta} + h_{\tau\tau} \frac{\partial}{\partial h_\tau} + \cdots .$$

Next, we get an over determined system of equations of ξ^1, ξ^2 and η

$$\begin{aligned} &\xi^1_\theta = \xi^1_\tau = \xi^1_h = 0, \\ &\xi^2_{\theta\theta} = \xi^2_{\tau\theta} = \xi^2_{\tau\tau} = 0, \ \xi^2_h = 0, \\ &\eta_\tau = 0, \ \eta_h = \xi^2_\tau, \ \eta_{\theta\theta} = \xi^2_\tau h - \eta. \end{aligned}$$

Solving above equations, one get

$$\xi^1 = c_1, \ \xi^2 = c_2\tau + c_3\theta + c_4, \ \eta = c_2 h + c_5 \mathrm{sin}\theta + c_6 \mathrm{cos}\theta,$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants. Therefore, Lie algebra L_6 of the transformations of Eq.(2) is spanned by the following vector fields

$$X_1 = \partial_\theta, \ X_2 = \tau\partial_\tau + h\partial_h, \ X_3 = \theta\partial_\tau, \ X_4 = \partial_\tau, \ X_5 = \mathrm{sin}\theta\partial_h, \ X_6 = \mathrm{cos}\theta\partial_h.$$

To obtain the symmetry groups, we solve the initial problems of the following ordinary differential equations

$$\begin{aligned} &\frac{d\tilde\theta}{d\epsilon} = \xi^1(\tilde\theta, \tilde\tau, \tilde h), \ \tilde\theta|_{\epsilon=0} = \theta, \\ &\frac{d\tilde\tau}{d\epsilon} = \xi^2(\tilde\theta, \tilde\tau, \tilde h), \ \tilde\tau|_{\epsilon=0} = \tau, \\ &\frac{d\tilde h}{d\epsilon} = \eta(\tilde\theta, \tilde\tau, \tilde h), \ \tilde h|_{\epsilon=0} = h, \end{aligned}$$

then we get the one-parameter symmetry groups $G_i : (\theta, \tau, h) \to (\tilde\theta, \tilde\tau, \tilde h)$ of the infinitesimal generators $X_i (i = 1, 2, \cdots, 6)$ as follows,

$$\begin{aligned} &G_1 : \ (\theta, \tau, h) \to (\theta + \epsilon, \tau, h), \\ &G_2 : \ (\theta, \tau, h) \to (\theta, \tau e^\epsilon, h e^\epsilon), \\ &G_3 : \ (\theta, \tau, h) \to (\theta, \tau + \epsilon\theta, h), \\ &G_4 : \ (\theta, \tau, h) \to (\theta, \tau + \epsilon, h), \\ &G_5 : \ (\theta, \tau, h) \to (\theta, \tau, h + \epsilon\mathrm{sin}\theta), \\ &G_6 : \ (\theta, \tau, h) \to (\theta, \tau, h + \epsilon\mathrm{cos}\theta). \end{aligned}$$

Based on the above discussion, we obtain the following theorem.

Theorem 2.1. *If $h = f(\theta, \tau)$ is a solution of Eq.(2), then by applying the above-mentioned groups $G_i (i = 1, 2, \cdots, 6)$, the corresponding new solutions $h_i (i = 1, 2, \cdots, 6)$ can be presented respectively as follows:*

$$\begin{aligned} &h_1 = f(\theta - \epsilon, \tau), \\ &h_2 = e^\epsilon f(\theta, \tau e^{-\epsilon}), \\ &h_3 = f(\theta, \tau - \epsilon\theta), \\ &h_4 = f(\theta, \tau - \epsilon), \end{aligned}$$

$$h_5 = f(\theta, \tau) + \epsilon sin\theta,$$
$$h_6 = f(\theta, \tau) + \epsilon cos\theta.$$

In order to get the classification of all the invariant solutions, we require an optimal system of subgroups. Next, we only use the commutator table to construct the optimal system of one-dimensional subalgebras for Eq.(2) [6]. By applying the commutator $[X_m, X_n] = X_m X_n - X_n X_m$, we determine the commutation relations of $X_1, X_2, X_3, X_4, X_5, X_6$ shown in Table 1.

Table 1. Table of Lie brackets.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	X_4	0	X_6	$-X_5$
X_2	0	0	$-X_3$	$-X_4$	$-X_5$	$-X_6$
X_3	$-X_4$	X_3	0	0	0	0
X_4	0	X_4	0	0	0	0
X_5	$-X_6$	X_5	0	0	0	0
X_6	X_5	X_6	0	0	0	0

An arbitrary operator $X \in L_6$ is shown as

$$X = l_1 X_1 + l_2 X_2 + l_3 X_3 + l_4 X_4 + l_5 X_5 + l_6 X_6.$$

To discover the linear transformations about the vector $l = (l_1, l_2, l_3, l_4, l_5, l_6)$, we define

$$E_i = c_{ij}^k l_j \partial_{l_k}, \ i = 1, 2, 3, 4, 5, 6, \tag{4}$$

where c_{ij}^k is given as the formula $[X_i, X_j] = c_{ij}^k X_k$. Based on Eq.(4) and Table 1, $E_1, E_2, E_3, E_4, E_5, E_6$ are shown as

$$E_1 = l_3 \partial_{l_4} + l_5 \partial_{l_6} - l_6 \partial_{l_5},$$
$$E_2 = -l_3 \partial_{l_3} - l_4 \partial_{l_4} - l_5 \partial_{l_5} - l_6 \partial_{l_6},$$
$$E_3 = -l_1 \partial_{l_4} + l_2 \partial_{l_3},$$
$$E_4 = l_2 \partial_{l_4},$$
$$E_5 = -l_1 \partial_{l_6} + l_2 \partial_{l_5},$$
$$E_6 = l_1 \partial_{l_5} + l_2 \partial_{l_6}.$$

For $E_1, E_2, E_3, E_4, E_5, E_6$, the Lie equations which have parameters $a_1, a_2, a_3, a_4, a_5, a_6$ with the initial condition $\tilde{l}|_{a_i=0} = l, \ i = 1, 2, 3, 4, 5, 6$ can be given as

$$\frac{d\tilde{l_1}}{da_1} = 0, \ \frac{d\tilde{l_2}}{da_1} = 0, \ \frac{d\tilde{l_3}}{da_1} = 0, \ \frac{d\tilde{l_4}}{da_1} = \tilde{l_3}, \ \frac{d\tilde{l_5}}{da_1} = -\tilde{l_6}, \ \frac{d\tilde{l_6}}{da_1} = \tilde{l_5},$$
$$\frac{d\tilde{l_1}}{da_2} = 0, \ \frac{d\tilde{l_2}}{da_2} = 0, \ \frac{d\tilde{l_3}}{da_2} = -\tilde{l_3}, \ \frac{d\tilde{l_4}}{da_2} = -\tilde{l_4}, \ \frac{d\tilde{l_5}}{da_2} = -\tilde{l_5}, \ \frac{d\tilde{l_6}}{da_2} = -\tilde{l_6},$$
$$\frac{d\tilde{l_1}}{da_3} = 0, \ \frac{d\tilde{l_2}}{da_3} = 0, \ \frac{d\tilde{l_3}}{da_3} = \tilde{l_2}, \ \frac{d\tilde{l_4}}{da_3} = -\tilde{l_1}, \ \frac{d\tilde{l_5}}{da_3} = 0, \ \frac{d\tilde{l_6}}{da_3} = 0,$$

$$\frac{d\tilde{l_1}}{da_4}=0,\ \frac{d\tilde{l_2}}{da_4}=0,\ \frac{d\tilde{l_3}}{da_4}=0,\ \frac{d\tilde{l_4}}{da_4}=\tilde{l_2},\ \frac{d\tilde{l_5}}{da_4}=0,\ \frac{d\tilde{l_6}}{da_4}=0,$$
$$\frac{d\tilde{l_1}}{da_5}=0,\ \frac{d\tilde{l_2}}{da_5}=0,\ \frac{d\tilde{l_3}}{da_5}=0,\ \frac{d\tilde{l_4}}{da_5}=0,\ \frac{d\tilde{l_5}}{da_5}=\tilde{l_2},\ \frac{d\tilde{l_6}}{da_5}=-\tilde{l_1},$$
$$\frac{d\tilde{l_1}}{da_6}=0,\ \frac{d\tilde{l_2}}{da_6}=0,\ \frac{d\tilde{l_3}}{da_6}=0,\ \frac{d\tilde{l_4}}{da_6}=0,\ \frac{d\tilde{l_5}}{da_6}=\tilde{l_1},\ \frac{d\tilde{l_6}}{da_6}=\tilde{l_2}.$$

Using the solutions of the above equations, the transformations are constructed as follows

$$T_1:\ \tilde{l_1}=l_1,\ \tilde{l_2}=l_2,\ \tilde{l_3}=l_3,\ \tilde{l_4}=a_1l_3+l_4,\ \tilde{l_5}=-l_6\mathrm{sin}a_1+l_5\mathrm{cos}a_1,\ \tilde{l_6}=l_6\mathrm{cos}a_1+l_5\mathrm{sin}a_1,$$
$$T_2:\ \tilde{l_1}=l_1,\ \tilde{l_2}=l_2,\ \tilde{l_3}=e^{-a_2}l_3,\ \tilde{l_4}=e^{-a_2}l_4,\ \tilde{l_5}=e^{-a_2}l_5,\ \tilde{l_6}=e^{-a_2}l_6,$$
$$T_3:\ \tilde{l_1}=l_1,\ \tilde{l_2}=l_2,\ \tilde{l_3}=a_3l_2+l_3,\ \tilde{l_4}=-a_3l_1+l_4,\ \tilde{l_5}=l_5,\ \tilde{l_6}=l_6,$$
$$T_4:\ \tilde{l_1}=l_1,\ \tilde{l_2}=l_2,\ \tilde{l_3}=l_3,\ \tilde{l_4}=a_4l_2+l_4,\ \tilde{l_5}=l_5,\ \tilde{l_6}=l_6,$$
$$T_5:\ \tilde{l_1}=l_1,\ \tilde{l_2}=l_2,\ \tilde{l_3}=l_3,\ \tilde{l_4}=l_4,\ \tilde{l_5}=a_5l_2+l_5,\ \tilde{l_6}=-a_5l_1+l_6,$$
$$T_6:\ \tilde{l_1}=l_1,\ \tilde{l_2}=l_2,\ \tilde{l_3}=l_3,\ \tilde{l_4}=l_4,\ \tilde{l_5}=a_6l_1+l_5,\ \tilde{l_6}=a_6l_2+l_6.$$

The structure of the optimal system demands a reduction of the vector

$$l=(l_1,l_2,l_3,l_4,l_5,l_6), \tag{5}$$

via the transformations T_1-T_6. Our work is to seek the simplest representative of each type of similar vectors (5). The construction is divided into two situations.

Case 2.1 $l_2\neq 0$

By letting $a_3=-\frac{l_3}{l_2}$, $a_4=-\frac{l_4}{l_2}$, $a_5=-\frac{l_5}{l_2}$, and $a_6=-\frac{l_6}{l_2}$ in T_3,T_4,T_5 and T_6 respectively, we can enable $\tilde{l_3}=\tilde{l_4}=\tilde{l_5}=\tilde{l_6}=0$. The vector (5) is simplified as follows

$$(l_1,l_2,0,0,0,0).$$

We derive the representatives

$$X_2,\ X_1\pm X_2. \tag{6}$$

Case 2.2 $l_2=0$

The vector (5) becomes to

$$(l_1,0,l_3,l_4,l_5,l_6). \tag{7}$$

2.2.1 $l_1\neq 0$

By letting $a_3=\frac{l_4}{l_1}$, $a_5=\frac{l_6}{l_1}$ and $a_6=-\frac{l_5}{l_1}$ in T_3,T_5 and T_6 respectively, we can enable $\tilde{l_4}=\tilde{l_5}=\tilde{l_6}=0$. We simplify the vector (7) to

$$(l_1,0,l_3,0,0,0).$$

Considering all the combinations, we get the representatives as follows

$$X_1,\ X_1\pm X_3. \tag{8}$$

2.2.2 $l_1=0$

The vector (7) is simplified as follows

$$(0,0,l_3,l_4,l_5,l_6). \tag{9}$$

2.2.2.1 $l_3\neq 0$

By letting $a_1 = -\frac{l_4}{l_3}$ in T_1, we can make $\tilde{l_4} = 0$. The vector (9) is simplified as follows

$$(0, 0, l_3, 0, l_5, l_6). \tag{10}$$

2.2.2.1.1 $l_5 \neq 0$

By letting $a_1 = -\arctan\frac{l_6}{l_5}$ in T_1, we can make $\tilde{l_6} = 0$. The vector (10) is equivalent to

$$(0, 0, l_3, 0, l_5, 0).$$

We get the representatives as follows

$$X_3 \pm X_5. \tag{11}$$

2.2.2.1.2 $l_5 = 0$

The vector (10) becomes to

$$(0, 0, l_3, 0, 0, l_6).$$

We get the representatives as follows

$$X_3,\ X_3 \pm X_6. \tag{12}$$

2.2.2.2 $l_3 = 0$

The vector (9) is simplified as follows

$$(0, 0, 0, l_4, l_5, l_6). \tag{13}$$

2.2.2.2.1 $l_5 \neq 0$

By letting $a_1 = -\arctan\frac{l_6}{l_5}$ in T_1, we can make $\tilde{l_6} = 0$. The vector (13) is equivalent to

$$(0, 0, 0, l_4, l_5, 0).$$

We get the representatives as follows

$$X_5, X_4 \pm X_5. \tag{14}$$

2.2.2.2.2 $l_5 = 0$

The vector (13) becomes to

$$(0, 0, 0, l_4, 0, l_6).$$

Thinking over all the combinations, we get the representatives as follows

$$X_4,\ X_6,\ X_4 \pm X_6. \tag{15}$$

Ultimately, by collecting the operators(6,8,11,12,14 and 15), we obtain the theorem as follows.

Theorem 2.2. *An optimal system of* $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ *is created by:*

$$X_2,\ X_1 \pm X_2,\ X_1,\ X_1 \pm X_3,\ X_3 \pm X_5,\ X_3,\ X_3 \pm X_6,\ X_5,\ X_4 \pm X_5,\ X_4,\ X_6,\ X_4 \pm X_6.$$

§3 Similarity reductions

In this section, with the help of Theorem 2.2, we are going to cope with similarity reductions and find several exact solutions of Eq.(2).

Case 3.1

For generator X_2, the invariants are $z=\theta,\ f(z)=\frac{h}{\tau}$, Eq.(2) becomes to

$$-f'^2+\frac{1}{2}f^2+\frac{1}{2}=0, \tag{16}$$

where $f'=\frac{df}{dz}$. By solving Eq.(16), one can obtain $h(\theta,\tau)=\tau\sinh(\frac{\sqrt{2}}{2}(c_1-\theta))$ which is a solution of separation variables of Eq.(2), where c_1 is an arbitrary constant. Furthermore, we have $h(\theta,\tau)\to\infty$, as $\tau\to+\infty$, which shows that the metric dilates infinitely. By using G_1,G_3,G_4,G_5,G_6, we get more solutions to Eq.(2) by means of Theorem 2.1,

$$h(\theta,\tau)=(\tau-\epsilon_4-\epsilon_3\theta)\sinh(\frac{\sqrt{2}}{2}(c_1-\theta+\epsilon_1))+\epsilon_5\sin\theta+\epsilon_6\cos\theta,$$

where $\epsilon_i,\ i=1,3,4,5,6$, are parameters.

Case 3.2

For generator X_1, analogously, we have $z=\tau, h=f(z)$. The reduction of Eq.(2) is

$$ff''+\frac{1}{2}f'^2+\frac{1}{2}=0, \tag{17}$$

where $f'=\frac{df}{dz}$. The invariant solution of Eq.(2) is $h(\theta,\tau)=f(\tau)$. Obviously, in this case, variable θ has no effect on the solution of Eq.(2).

Case 3.3

For generator X_1+X_2, the invariants are $z=\tau e^{-\theta},\ f(z)=he^{-\theta}$, Eq.(2) can be reduced to

$$-zf'f''+2ff''+\frac{1}{2}f'^2+\frac{1}{2}=0, \tag{18}$$

where $f'=\frac{df}{dz}$. The group invariant solution to Eq.(2) is $h(\theta,\tau)=e^{\theta}f(\tau e^{-\theta})$.

Case 3.4

For generator X_1+X_3, we have $z=-\frac{1}{2}\theta^2+\tau,\ f(z)=h$. The reduction of Eq.(2) is

$$-f'f''+ff''+\frac{1}{2}f'^2+\frac{1}{2}=0, \tag{19}$$

where $f'=\frac{df}{dz}$. The group invariant solution to Eq.(2) is $h(\theta,\tau)=f(-\frac{1}{2}\theta^2+\tau)$.

Remark 3.1 Unfortunately, the group invariant solutions can not be obtained for other generators in the optimal system of Theorem 2.2.

§4 Power series solutions

Next, by means of power series method which is an very useful technique for treating PDEs [1], we will discuss cases 3.2, 3.3 and 3.4.

For Case 3.2, assuming that the power series solution to Eq.(17) is as follows

$$f(z)=\sum_{n=0}^{\infty}p_nz^n, \tag{20}$$

where the coefficients p_n are constants to be resolved.

Putting (20) into Eq.(17), we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n-k+1)(n-k+2)p_{n-k+2}p_k z^n$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n-k+1)(k+1)p_{n-k+1}p_{k+1}z^n+\frac{1}{2}=0. \tag{21}$$

Comparing coefficients for (21), we get

$$p_2=-\frac{1+p_1^2}{4p_0}. \tag{22}$$

Generally, for $n\geq 1$, we have

$$p_{n+2}=-\frac{1}{2(n+1)(n+2)p_0}\{(n+1)p_1p_{n+1}$$
$$+\sum_{k=1}^{n}(n-k+1)[2(n-k+2)p_{n-k+2}p_k+(k+1)p_{n-k+1}p_{k+1}]\}. \tag{23}$$

In view of Eq.(23), the coefficients $p_i,(i\geq 3)$ of (20) can be obtained, e.g.,

$$p_3=-\frac{2p_1p_2}{3p_0},$$
$$p_4=-\frac{9p_1p_3+4p_2^2}{12p_0}.$$

Therefore, for arbitrary constants $p_0\neq 0$ and p_1, the other terms of the sequences $\{p_n\}_{n=0}^{\infty}$, according to (22) and (23), can be determined. This implies that there is a power series solution (20) whose coefficients are composed of (22) and (23). Furthermore, for Eq.(17), we certificate the convergence of (20). In fact, from (23), we get

$$|p_{n+2}|\leq M[|p_{n+1}|+\sum_{k=1}^{n}(|p_{n-k+2}||p_k|+|p_{n-k+1}||p_{k+1}|)],$$

where $M=\max\{|\frac{p_1}{2p_0}|,|\frac{1}{p_0}|\}$.

Next, we construct another power series $R=R(z)=\sum_{n=0}^{\infty}r_nz^n$, by

$$r_i=|p_i|,\quad i=0,1,2,$$

and

$$r_{n+2}=M[r_{n+1}+\sum_{k=1}^{n}(r_{n-k+2}r_k+r_{n-k+1}r_{k+1})],$$

where $n=1,2,\cdots$. It is easily seen that

$$|p_n|\leq r_n,\quad n=0,1,2,\cdots.$$

Therefore, $R=R(z)=\sum_{n=0}^{\infty}r_nz^n$ is a majorant series of (20). Next, we prove that $R=R(z)$

has a positive radius of convergence.

$$\begin{aligned}R(z) &= r_0 + r_1 z + r_2 z^2 + \sum_{n=1}^{\infty} r_{n+2} z^{n+2} = r_0 + r_1 z + r_2 z^2 + M[\sum_{n=1}^{\infty} r_{n+1} z^{n+2} \\ &\quad + \sum_{n=1}^{\infty}\sum_{k=1}^{n} r_{n-k+2} r_k z^{n+2} + \sum_{n=1}^{\infty}\sum_{k=1}^{n} r_{n-k+1} r_{k+1} z^{n+2}] \\ &= r_0 + r_1 z + r_2 z^2 + M[z(R - r_0 - r_1 z) + (R - r_0)(R - r_0 - r_1 z) \\ &\quad + (R - r_0)(R - r_0 - r_1 z)].\end{aligned}$$

Then, we discuss the implicit functional equation about the independent variable z,

$$\begin{aligned}F(z,R) &= R - r_0 - r_1 z - r_2 z^2 - M[z(R - r_0 - r_1 z) + (R - r_0)(R - r_0 - r_1 z) \\ &\quad + (R - r_0)(R - r_0 - r_1 z)].\end{aligned}$$

Based on the implicit function theorem [21], because F is analytic in the $(z,R)-$plane and $F(0,r_0) = 0, F'_R(0,r_0) = 1 \neq 0$, we reach that $R = R(z)$ is analytic in a neighborhood of the point $(0,r_0)$ and has the positive radius. This shows that (20) converges in a neighborhood of the point $(0,r_0)$ of the plane. The proof is completed. Thus the power series solution (20) for Eq.(17) is analytic and can be described as

$$\begin{aligned}f(z) &= p_0 + p_1 z + p_2 z^2 + \sum_{n=1}^{\infty} p_{n+2} z^{n+2} \\ &= p_0 + p_1 z - \frac{1+p_1^2}{4p_0} z^2 - \sum_{n=1}^{\infty} \frac{1}{2(n+1)(n+2)p_0}\{(n+1)p_1 p_{n+1} \\ &\quad + \sum_{k=1}^{n}(n-k+1)[2(n-k+2)p_{n-k+2}p_k + (k+1)p_{n-k+1}p_{k+1}]\} z^{n+2}.\end{aligned}$$

Moreover, the power series solution of Eq.(2) is

$$\begin{aligned}h(\theta,\tau) &= p_0 + p_1 \tau + p_2 \tau^2 + \sum_{n=1}^{\infty} p_{n+2} \tau^{n+2} \\ &= p_0 + p_1 \tau - \frac{1+p_1^2}{4p_0} \tau^2 - \sum_{n=1}^{\infty} \frac{1}{2(n+1)(n+2)p_0}\{(n+1)p_1 p_{n+1} \\ &\quad + \sum_{k=1}^{n}(n-k+1)[2(n-k+2)p_{n-k+2}p_k + (k+1)p_{n-k+1}p_{k+1}]\} \tau^{n+2}, \qquad (24)\end{aligned}$$

where $p_0 \neq 0$ and p_1 are arbitrary constants, the other terms $p_n (n \geq 2)$ can be provided according to (22) and (23). Obviously, when $\tau \to +\infty$, we have $h(\theta,\tau) \to \infty$. We take the first five terms of Eq.(24) as approximate to $h(\theta,\tau)$. Then the approximations of h are depicted in Fig.1.

By acting G_2, G_3, G_4, G_5, G_6, we obtain more solutions to Eq.(2) by means of Theorem 2.1,

$$h(\theta,\tau) = p_0 e^{\epsilon_2} + p_1(\tau - \epsilon_4 - \epsilon_3\theta) - \frac{1+p_1^2}{4p_0}(\tau - \epsilon_4 - \epsilon_3\theta)^2 e^{-\epsilon_2}$$

$$-\sum_{n=1}^{\infty}\frac{1}{2(n+1)(n+2)p_0}\{(n+1)p_1p_{n+1}+\sum_{k=1}^{n}(n-k+1)[2(n-k+2)p_{n-k+2}p_k$$
$$+(k+1)p_{n-k+1}p_{k+1}](\tau-\epsilon_4-\epsilon_3\theta)^{n+2}e^{-(n+1)\epsilon_2}+\epsilon_5\sin\theta+\epsilon_6\cos\theta\},$$

where $\epsilon_i,\ i=2,3,4,5,6$, are parameters.

Similarly, we can also get the power series solutions to (18) and (19) which are listed in Table 2. Proofs of convergence of the power series solutions of (18) and (19) are similar to the one of Eq.(17). The details are omitted here.

Table 2. Table of power series solutions.

Cases	Power series solutions
X_1+X_2	$h(\theta,\tau)=p_0e^{\theta}+p_1\tau-\frac{1+p_1^2}{8p_0}\tau^2e^{-\theta}-\sum_{n=1}^{\infty}\frac{1}{2(n+1)(n+2)p_0}\{4p_2p_n+(1-n^2)p_1p_{n+1}$ $+\sum_{k=1}^{n-1}(n-k+1)[2(n-k+2)p_{n-k+2}p_k+(k+1)(k-n+\frac{1}{2})p_{n-k+1}p_{k+1}]\}$ $\tau^{n+2}e^{-(n+1)\theta}$, where $p_0\neq 0$ and p_1 are arbitrary constants.
X_1+X_3	$h(\theta,\tau)=p_0+p_1(-\frac{1}{2}\theta^2+\tau)+\frac{1+p_1^2}{4(p_1-p_0)}(-\frac{1}{2}\theta^2+\tau)^2+\sum_{n=1}^{\infty}\frac{1}{2(n+1)(n+2)(p_1-p_0)}$ $\{(n+1)p_1p_{n+1}+\sum_{k=1}^{n}2(n-k+1)[(n-k+2)p_{n-k+2}(p_k-(k+1)p_{k+1})$ $+(k+1)p_{n-k+1}p_{k+1}]\}(-\frac{1}{2}\theta^2+\tau)^{n+2}$, where p_0 and $p_1(p_0\neq p_1)$ are arbitrary constants.

We take the first five terms of power series solutions of (18) and (19) as approximate to $h(\theta,\tau)$ respectively, then the approximations of h are depicted in Figs.2 and 3.

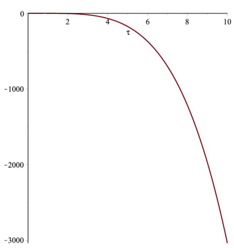

Fig 1. Power series soluti on of Eq.(17) for $p_0=1, p_1=1$.

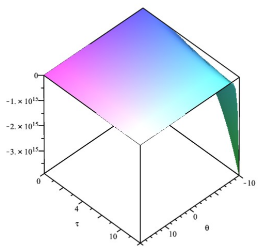

Fig 2. Power series soluti on of Eq.(18) for $p_0=1, p_1=1$.

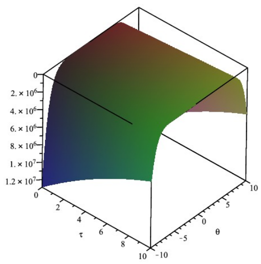

Fig 3. Power series soluti on of Eq.(19) for $p_0=1, p_1=2$.

In the same way, based on Theorem 2.1, by using G_3, G_4, G_5, G_6 to power series solution of (18), we obtain more solutions to Eq.(2) as follows,

$$h(\theta,\tau)=p_0e^{\theta}+p_1(\tau-\epsilon_4-\epsilon_3\theta)-\frac{1+p_1^2}{8p_0}(\tau-\epsilon_4-\epsilon_3\theta)^2e^{-\theta}$$
$$-\sum_{n=1}^{\infty}\frac{(\tau-\epsilon_4-\epsilon_3\theta)^{n+2}e^{-(n+1)\theta}+\epsilon_5\sin\theta+\epsilon_6\cos\theta}{2(n+1)(n+2)p_0}\{4p_2p_n+(1-n^2)p_1p_{n+1}$$
$$+\sum_{k=1}^{n-1}(n-k+1)[2(n-k+2)p_{n-k+2}p_k+(k+1)(k-n+\frac{1}{2})p_{n-k+1}p_{k+1}]\},$$

where $\epsilon_i,\ i=3,4,5,6$, are parameters.

By acting G_4, G_5, G_6 to power series solution of (19), we obtain more solutions to Eq.(2),

$$h(\theta,\tau)=p_0+p_1(-\frac{1}{2}\theta^2+\tau-\epsilon_4)+\frac{1+p_1^2}{4(p_1-p_0)}(-\frac{1}{2}\theta^2+\tau-\epsilon_4)^2+\epsilon_5\sin\theta+\epsilon_6\cos\theta$$
$$+\sum_{n=1}^{\infty}\frac{1}{2(n+1)(n+2)(p_1-p_0)}\{(n+1)p_1p_{n+1}+\sum_{k=1}^{n}2(n-k+1)[(n-k+2)$$
$$p_{n-k+2}(p_k-(k+1)p_{k+1})+(k+1)p_{n-k+1}p_{k+1}]\}(-\frac{1}{2}\theta^2+\tau-\epsilon_4)^{n+2},$$

where $\epsilon_i,\ i=4,5,6$, are parameters.

§5 Conservation laws

In this section, via Ibragimov's approach [10,11], we will establish conservation laws of Eq.(2). Next, nonlinear self-adjointness of Eq.(2) is proved.

5.1 Nonlinear self-adjointness

First, conservation laws multiplier of Eq.(2) is defined as

$$\Lambda=\Lambda(\theta,\tau,h).$$

Then,

$$E_h[\Lambda(hh_{\tau\tau}+h_{\tau\tau}h_{\theta\theta}-h_{\theta\tau}^2+\frac{1}{2}(1+h_\tau^2))]=0, \tag{25}$$

where the Euler operator E_h is shown as

$$E_h=\frac{\partial}{\partial h}-D_\theta\frac{\partial}{\partial h_\theta}-D_\tau\frac{\partial}{\partial h_\tau}+D_\theta^2\frac{\partial}{\partial h_{\theta\theta}}+D_\tau^2\frac{\partial}{\partial h_{\tau\tau}}+D_\theta D_\tau\frac{\partial}{\partial h_{\theta\tau}}\cdots. \tag{26}$$

Substituting (26) into (25), we obtain the following system which has only one unknown variable Λ,

$$\Lambda_h=0,\ \Lambda_\tau=0,\ \Lambda_{\theta\theta}+\Lambda=0.$$

Solving above equations, one get $\Lambda=c_1\sin\theta+c_2\cos\theta$, where c_1 and c_2 are arbitrary constants.

For a mth-order PDE system

$$\mathcal{R}^\alpha(x,u,\cdots,u_{(k)})=0,\ \alpha=1,\cdots,m, \tag{27}$$

where $x=(x^1,x^2,\cdots,x^n)$, $u=(u^1,u^2,\cdots,u^m)$ and $u_{(1)},u_{(2)},\cdots,u_{(k)}$ express the set of all first, second,..., kth-order derivatives of u associated with x.

The adjoint equations of Eq.(27) are defined as

$$(\mathcal{R}^\alpha)^*(x,u,v,\cdots,u_{(k)},v_{(k)})=0,\ \alpha=1,\cdots,m,\ v=v(x).$$

Besides,

$$(\mathcal{R}^\alpha)^*(x,u,v,\cdots,u_{(k)},v_{(k)})=\frac{\delta\mathcal{L}}{\delta u^\alpha},$$

where $\mathcal{L}$ is the following form of Lagrangian

$$\mathcal{L}=v^\beta\mathcal{R}^\beta(x,u,\cdots,u_{(k)}),\ \beta=1,2,\cdots,m,$$

and the Euler-Lagrange operator is defined as

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}+\sum_{j=1}^{\infty}(-1)^j D_{i_1}\cdots D_{i_j}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_j}},\ \alpha=1,2,\cdots,m.$$

Definition 5.1[12]. The system (27) is nonlinearly self-adjoint if the adjoint system satisfies all the solutions u of (27) upon a substitution $v=\varphi(x,u)$ such that $\varphi(x,u)\neq 0$. Particularly, the system

$$(\mathcal{R}^\alpha)^*(x,u,\varphi,\cdots,u_{(k)},\varphi_{(k)})=0,\ \alpha=1,\cdots,m,$$

can be replaced by the following system

$$\lambda^\beta_\alpha\mathcal{R}^\beta(x,u,u,\cdots,u_{(k)},u_{(k)})=0,\ \beta=1,\cdots,m,$$

namely,

$$(\mathcal{R}^\alpha)^*|_{v=\varphi(x,u)}=\lambda^\beta_\alpha\mathcal{R}^\beta,\beta=1,\cdots,m,$$

where λ^β_α is a function.

Theorem 5.1 (13). . *The overdetermined system of $\Lambda(x,u)$ of system (27) can be replaced by the system of nonlinearly self-adjoint substitution.*

If the Lagrangian of Eq.(2) is written as

$$\mathcal{L}=\varphi(\theta,\tau,h)(hh_{\tau\tau}+h_{\tau\tau}h_{\theta\theta}-h^2_{\theta\tau}+\frac{1}{2}(1+h^2_\tau)),$$

Applying the Theorem 5.1, we get

$$\varphi(\theta,\tau,h)=\Lambda(\theta,\tau,h)=c_1\sin\theta+c_2\cos\theta. \tag{28}$$

Therefore, Eq.(2) is nonlinearly self-adjoint with substitution (28).

5.2 Structure of conservation laws

Theorem 5.2 (12). . *The system of Eq.(27) is nonlinearly self-adjoint. Then each Lie point, Lie-Bäcklund, nonlocal symmetry*

$$X=\xi^i(x,u,u_{(1)},\cdots)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u,u_{(1)},\cdots)\frac{\partial}{\partial u^\alpha},$$

recognized by the system of Eq.(27) leads to a conservation law, where the constituents $\mathcal{C}^i$ of the conserved vector $\mathcal{C}=(\mathcal{C}^1,\cdots,\mathcal{C}^n)$ are computed by

$$\mathcal{C}^i=W^\alpha[\frac{\partial\mathcal{L}}{\partial u^\alpha_i}-D_j(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij}})$$
$$+D_jD_k(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}})-\cdots]+D_j(W^\alpha)[\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij}}-D_k(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}})+\cdots]+D_jD_k(W^\alpha)[\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}}-\cdots],$$

and $W^\alpha=\eta^\alpha-\xi^j u^\alpha_j$. The Lagrangian $\mathcal{L}$ can be expressed in the symmetric form about mixed derivatives $u^\alpha_{ij},u^\alpha_{ijk},\cdots$.

The Lagrangian $\mathcal{L}$ of Eq.(2) is given as follows

$$\mathcal{L}=(c_1\sin\theta+c_2\cos\theta)(hh_{\tau\tau}+h_{\tau\tau}h_{\theta\theta}-h^2_{\theta\tau}+\frac{1}{2}(1+h^2_\tau)).$$

For the generator $X=\xi^1\partial_\theta+\xi^2\partial_\tau+\phi\partial_h$, based on the Theorem 5.2, we get $W=\phi-\xi^1h_\theta-$

$\xi^2 h_\tau$, so the constituents of the conservation vector are

$$\mathcal{C}^\theta = W[\frac{\partial \mathcal{L}}{\partial h_\theta} - D_\theta(\frac{\partial \mathcal{L}}{\partial h_{\theta\theta}}) - D_\tau(\frac{\partial \mathcal{L}}{\partial h_{\theta\tau}})] + D_\theta(W)\frac{\partial \mathcal{L}}{\partial h_{\theta\theta}} + D_\tau(W)\frac{\partial \mathcal{L}}{\partial h_{\theta\tau}},$$
$$\mathcal{C}^\tau = W[\frac{\partial \mathcal{L}}{\partial h_\tau} - D_\tau(\frac{\partial \mathcal{L}}{\partial h_{\tau\tau}}) - D_\theta(\frac{\partial \mathcal{L}}{\partial h_{\theta\tau}})] + D_\tau(W)\frac{\partial \mathcal{L}}{\partial h_{\tau\tau}} + D_\theta(W)\frac{\partial \mathcal{L}}{\partial h_{\theta\tau}}.$$

By putting $\mathcal{L}$ into above constituents of the conservation vector, $\mathcal{C}^\theta, \mathcal{C}^\tau$ are simplified as

$$\begin{aligned}\mathcal{C}^\theta =& -W[(c_1\cos\theta - c_2\sin\theta)h_{\tau\tau} - (c_1\sin\theta + c_2\cos\theta)h_{\theta\tau\tau}]\\ &+ D_\theta(W)(c_1\sin\theta + c_2\cos\theta)h_{\tau\tau} - 2D_\tau(W)(c_1\sin\theta + c_2\cos\theta)h_{\theta\tau},\end{aligned} \tag{29}$$

$$\begin{aligned}\mathcal{C}^\tau =& W[2(c_1\cos\theta - c_2\sin\theta)h_{\theta\tau} + (c_1\sin\theta + c_2\cos\theta)h_{\theta\theta\tau}]\\ &+ D_\tau(W)(c_1\sin\theta + c_2\cos\theta)(h + h_{\theta\theta}) - 2D_\theta(W)(c_1\sin\theta + c_2\cos\theta)h_{\theta\tau}.\end{aligned} \tag{30}$$

For generator $X_1 = \partial_\theta$, we have $W = -h_\theta$. According to the formulas (29) and (30), the constituents of the conserved vector of X_1 are

$$\mathcal{C}_1^\theta = h_\theta[(c_1\cos\theta - c_2\sin\theta)h_{\tau\tau} - (c_1\sin\theta + c_2\cos\theta)h_{\theta\tau\tau}] + (2h_{\theta\tau}^2 - h_{\theta\theta}h_{\tau\tau})(c_1\sin\theta + c_2\cos\theta),$$
$$\mathcal{C}_1^\tau = -h_\theta[2(c_1\cos\theta - c_2\sin\theta)h_{\theta\tau} + (c_1\sin\theta + c_2\cos\theta)h_{\theta\theta\tau}] + h_{\theta\tau}(h_{\theta\theta} - h)(c_1\sin\theta + c_2\cos\theta).$$

For generator $X_2 = \tau\partial_\tau + h\partial_h$, we have $W = h - \tau h_\tau$. According to the formulas (29) and (30), the constituents of the conserved vector of X_2 are

$$\begin{aligned}\mathcal{C}_2^\theta =& (\tau h_\tau - h)[(c_1\cos\theta - c_2\sin\theta)h_{\tau\tau} - (c_1\sin\theta + c_2\cos\theta)h_{\theta\tau\tau}]\\ &+ h_{\tau\tau}(h_\theta + \tau h_{\theta\tau})(c_1\sin\theta + c_2\cos\theta),\\ \mathcal{C}_2^\tau =& (h - \tau h_\tau)[2(c_1\cos\theta - c_2\sin\theta)h_{\theta\tau} + (c_1\sin\theta + c_2\cos\theta)h_{\theta\theta\tau}]\\ &- [\tau h_{\tau\tau}(h + h_{\theta\theta}) + 2h_{\theta\tau}(h_\theta - \tau h_{\theta\tau})](c_1\sin\theta + c_2\cos\theta).\end{aligned}$$

For generator $X_3 = \theta\partial_\tau$, we have $W = -\theta h_\tau$. According to the formulas (29) and (30), the constituents of the conserved vector of X_3 are

$$\begin{aligned}\mathcal{C}_3^\theta =& \theta h_\tau[(c_1\cos\theta - c_2\sin\theta)h_{\tau\tau} - (c_1\sin\theta + c_2\cos\theta)h_{\theta\tau\tau}] + h_{\tau\tau}(\theta h_{\theta\tau} - h_\tau)(c_1\sin\theta + c_2\cos\theta),\\ \mathcal{C}_3^\tau =& -\theta h_\tau[2(c_1\cos\theta - c_2\sin\theta)h_{\theta\tau} + (c_1\sin\theta + c_2\cos\theta)h_{\theta\theta\tau}]\\ &- [\theta h_{\tau\tau}(h + h_{\theta\theta}) - 2h_{\theta\tau}(h_\tau + \theta h_{\theta\tau})](c_1\sin\theta + c_2\cos\theta).\end{aligned}$$

For generator $X_4 = \partial_\tau$, we have $W = -h_\tau$. According to the formulas (29) and (30), the constituents of the conserved vector of X_4 are

$$\begin{aligned}\mathcal{C}_4^\theta =& h_\tau[(c_1\cos\theta - c_2\sin\theta)h_{\tau\tau} - (c_1\sin\theta + c_2\cos\theta)h_{\theta\tau\tau}] + h_{\theta\tau}h_{\tau\tau}(c_1\sin\theta + c_2\cos\theta),\\ \mathcal{C}_4^\tau =& -h_\tau[2(c_1\cos\theta - c_2\sin\theta)h_{\theta\tau} + (c_1\sin\theta + c_2\cos\theta)h_{\theta\theta\tau}]\\ &- [h_{\tau\tau}(h + h_{\theta\theta}) - 2h_{\theta\tau}^2](c_1\sin\theta + c_2\cos\theta).\end{aligned}$$

For generator $X_5 = \sin\theta\partial_h$, we have $W = \sin\theta$. According to the formulas (29) and (30), the constituents of the conserved vector of X_5 are

$$\mathcal{C}_5^\theta = -\sin\theta[(c_1\cos\theta - c_2\sin\theta)h_{\tau\tau} - (c_1\sin\theta + c_2\cos\theta)h_{\theta\tau\tau}] + h_{\tau\tau}\cos\theta(c_1\sin\theta + c_2\cos\theta),$$
$$\mathcal{C}_5^\tau = \sin\theta[2(c_1\cos\theta - c_2\sin\theta)h_{\theta\tau} + (c_1\sin\theta + c_2\cos\theta)h_{\theta\theta\tau}] - 2h_{\theta\tau}\cos\theta(c_1\sin\theta + c_2\cos\theta).$$

For generator $X_6 = \cos\theta\partial_h$, we have $W = \cos\theta$. According to the formulas (29) and (30), the constituents of the conserved vector of X_6 are

$$\mathcal{C}_6^\theta = -\cos\theta[(c_1\cos\theta - c_2\sin\theta)h_{\tau\tau} - (c_1\sin\theta + c_2\cos\theta)h_{\theta\tau\tau}] - h_{\tau\tau}\sin\theta(c_1\sin\theta + c_2\cos\theta),$$

$$\mathcal{C}_6^\tau = \cos\theta[2(c_1\cos\theta - c_2\sin\theta)h_{\theta\tau} + (c_1\sin\theta + c_2\cos\theta)h_{\theta\theta\tau}] + 2h_{\theta\tau}\sin\theta(c_1\sin\theta + c_2\cos\theta).$$

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