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Fuzzy rough sets in Šostak sense

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Abstract. In this paper, we defined the fuzzy operator Φ_λ in a fuzzy ideal approximation space $(X, R, \mathcal{I})$ associated with a fuzzy rough set λ in Šostak sense. Associated with Φ_λ, there are fuzzy ideal interior and closure operators int_Φ^λ and cl_Φ^λ, respectively. r-fuzzy separation axioms, r-fuzzy connectedness and r-fuzzy compactness in fuzzy ideal approximation spaces are defined and compared with the relative notions in r-fuzzy approximation spaces. There are many differences when studying these notions related with a fuzzy ideal different from studying these notions in usual fuzzy approximation spaces. Lastly, using a fuzzy grill, we will get the same results given during the context.

§1 Introduction

In 1982, Pawlak ([15]) introduced the notion of rough sets dealing with the uncertainty of intelligent systems. By an equivalence relation R on a set X, we have the notion of approximation space (X, R). For a subset $A \subseteq X$, the boundary region set $A^b = A^u \setminus A^l$ describes the roughness of the set A whenever the upper approximation set A^u is greater than the lower approximation set A^l of A. in the approximation space (X, R). If $A^u = A^l$, then the set A is not a rough set. The notions of ideal and fuzzy ideal in sense of Chang [4] were given in [9] and [18], respectively. Fuzzy ideals in Šostak sense [19] were given in [16]. The local closed set A^* of a set A was given in [20] associated with an ideal defined on the usual approximation space(X, R). Many research papers were based on studying a topological space joined with an ideal as in [5, 10, 11]. Studying the roughness of a fuzzy set was given in many articles like [3, 12, 13].

Fuzzy separation axioms with grades were given in [7], and studying separation axioms with respect to an ideal was given in [2]. In [14], the authors studied fuzzy soft separation axioms and fuzzy soft connectedness in fuzzy topological spaces in sense of Chang. The definition of fuzzy grills was given in [1], and a research paper studying the fuzzy topological spaces via ideals and grills was given in [8].

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The motivation of this paper is to define the fuzzy approximation lower and upper sets, and moreover to define the fuzzy approximation interior and closure operators on a fuzzy approximation space in sense of Šostak. During these fuzzy operators associated with some $r \in I_0$, we defied fuzzy approximation separation axioms, fuzzy approximation connectedness and fuzzy approximation com- pactness. A generalization of these definitions is illustrated using a fuzzy ideal constructed on the fuzzy approximation space.

In the paper, we define the local r-fuzzy closed set $\Phi_\lambda(\mu, r)$ of a fuzzy set μ associated with the rough fuzzy set λ and $r \in (0, 1]$ as a new generalization of the notion of a local fuzzy closed set. Also, we define the r-fuzzy cosets for each element $x \in X$, and introduce the definitions of r-fuzzy lower and r-fuzzy upper approximation sets according to these r-fuzzy cosets in the fuzzy approximation space (X, R). The fuzzy boundary region set is defined according to the fuzzy difference given in [8]. The definitions of r-fuzzy lower, r-fuzzy upper and r-fuzzy boundary region sets are satisfying most of the properties of the corresponding ones in the general case. r-fuzzy approximation connectedness and r-fuzzy ideal approximation connectedness are defined and studied. Also, r-fuzzy approximation compactness and r-fuzzy ideal approximation compactness are defined and studied.

Throughout the paper, let X be a finite set of objects and I the closed unit interval $[0, 1]$, $I_0 = (0, 1]$. Assume that an order-reversing involution $\alpha \mapsto \alpha^c$ of I is fixed. I^X denotes all fuzzy subsets of X, and $\lambda^c(x) = 1 - \lambda(x)\ \forall x \in X$ for all $\lambda \in I^X$. A constant fuzzy set $\overline{t}$ for all $t \in I$ is defined by $\overline{t}(x) = t\ \forall x \in X$. Infimum and supremum of a fuzzy set $\lambda \in I^X$ are given as: $\inf \lambda = \bigwedge_{x \in X} \lambda(x)$ and $\sup \lambda = \bigvee_{x \in X} \lambda(x)$.
If $f : X \to Y$ is a mapping, $\mu \in I^X, \nu \in I^Y$, then

$$(f(\mu))(y) = \bigvee_{x \in f^{-1}(y)} \mu(x)\ \forall y \in Y \quad \text{and} \quad f^{-1}(\nu) = \nu \circ f.$$

Assume a fuzzy relation $R : X \times X \to I$ is defined so that $R(x, x) = 1\ \forall x \in X$, $R(x, y) = R(y, x)\ \forall x, y \in X$ and $R(x, y) \geq (R(x, z) \wedge R(z, y))\ \forall x, y, z \in X$. That is, R is a fuzzy equivalence relation on X. (X, R) is called a fuzzy approximation space based on the fuzzy equivalence relation R on X.

Definition 1.1. For each $x \in X$ and $r \in I_0$, define a fuzzy coset $[x]_r : X \to I$ as follows:

$$[x]_r(y) = r \wedge R(x, y) \quad \forall y \in X. \tag{1}$$

Recall that $[x]_1(y) = R(x, y)\ \forall y \in X$, which means that $[x]_1(y) = 1$ iff $R(x, y) = 1$. All elements $y \in X$ with fuzzy relation value $R(x, y) > 0$ are elements in the fuzzy coset $[x]_r$ with membership value $(r \wedge R(x, y))$, and any element $y \in X$ with $R(x, y) = 0$ does not belong to the fuzzy coset $[x]_r$. Any fuzzy coset $[x]_r$ includes at least the element $x \in X$, and consequently $\bigvee_{z \in X} [x]_r(z) = r$ for all $x \in X$. Also, $\bigvee_{z \in X} [z]_r(y) = r\ \forall y \in X$ (i.e. $\bigvee_{z \in X} [z]_r = \overline{r}$). If $R(x, y) > 0$ and $r \in I_0$, then the fuzzy cosets $[x]_r, [y]_r$ (as fuzzy sets) are containing the same elements of X with some non-zero membership values, and moreover if $[y]_r(z) = 0$, then it must be that $[x]_r(z) = 0$ whenever $R(x, y) > 0$. That is, any two fuzzy cosets are either two fuzzy sets containing the same elements of X with some non-zero membership values or containing completely different elements of X with some non-zero membership values.

Note that: $[x]_r \neq \overline{0}\ \forall x \in X$, while may be all elements $z \in X$ are given such that $[x]_r(z) > 0$ in case of $R(x,z) > 0\ \forall z \in X$, and thus the fuzzy approximation space (X, R) looks like fuzzy partitioned into only one fuzzy coset. The fuzzy cosets could be such that $[x]_r(x) = r$ and $[x]_r(z) = 0\ \forall z \neq x$, which means (X, R) looks like fuzzy partitioned into completely disjoint fuzzy cosets. Putting $I = \{0, 1\}$ as a crisp case, we get exactly the usual meaning of partitioning of a set X based on an equivalence relation R on X.

Recall that the fuzzy difference between two fuzzy sets was defined as follows ([8]):

$$(\lambda \bar{\wedge} \mu) \ = \begin{cases} \overline{0} & \text{if } \ \lambda \leq \mu, \\ \lambda \wedge \mu^c & \text{otherwise.} \end{cases} \tag{2}$$

§2 r-fuzzy lower, r-fuzzy upper and r-fuzzy boundary region sets

Definition 2.1. Let $\lambda \in I^X$, $r \in I_0$ and R a fuzzy equivalence relation on X and the fuzzy cosets are defined as in (1). Then, the r-fuzzy lower set λ_R^r, the r-fuzzy upper set λ_r^R and the r-fuzzy boundary region set λ_r^B are defined as follow:

$$\lambda_R^r(x) \ = \ \lambda(x) \ \wedge \ (\bigvee_{\lambda^c(z)>0,\ z\neq x} [x]_r(z))^c \ \ \forall x \in X, \tag{3}$$

$$\lambda_r^R(x) \ = \ \lambda(x) \ \vee \bigvee_{\lambda(z)>0, z\neq x} [x]_r(z) \ \ \forall x \in X, \tag{4}$$

$$\lambda_r^B \ = \ \lambda_r^R \ \bar{\wedge}\ \lambda_R^r \ = \begin{cases} \overline{0} & \text{if } \ \lambda_r^R \leq \lambda_R^r \\ \lambda_r^R \wedge (\lambda_R^r)^c & \text{otherwise.} \end{cases} \tag{5}$$

λ_R^r, λ_r^R and λ_r^B are then called r-fuzzy lower, r-fuzzy upper and r-fuzzy boundary region sets associated with the fuzzy set λ in I^X, $r \in I_0$ and based on the fuzzy equivalence relation R in a fuzzy approximation space (X, R).

From Equations (3) and (4), we get that $\lambda_R^r \leq \lambda \leq \lambda_r^R \ \ \forall \lambda \in I^X, \forall r \in I_0$. Whenever λ_r^R be so that $\lambda_r^R \leq \lambda_R^r$, we get that $\lambda = \lambda_R^r = \lambda_r^R$ and then from Equation (5), we have $\lambda_r^B = \overline{0}$. Otherwise, $\lambda_r^B = \lambda_r^R \wedge (\lambda_R^r)^c$. The fuzzy accuracy $\alpha_R^r(\lambda)$ of approximation of the fuzzy set λ could be characterized numerically by $\alpha_R^r(\lambda) = \frac{\inf \lambda_R^r}{\sup \lambda_r^R}$, where $0 \leq \alpha_R^r(\lambda) \leq 1$. If $\alpha_R^r(\lambda) = 1$, then λ is crisp with respect to R ($\lambda_R^r = \lambda_r^R$ and λ is precise with respect to R). Otherwise, if $\alpha_R^r(\lambda) < 1$, then λ is rough with respect to R.

Lemma 2.1. *For any fuzzy set $\lambda \in I^X$ and $r \in I_0$ we get that:*

(1) $\lambda_R^r \ \leq \ \lambda \ \leq \ \lambda_r^R$,

(2) $\overline{0}_R^r = \overline{0}_r^R = \overline{0}$ and $\overline{1}_R^r = \overline{1}_r^R = \overline{1}$,

(3) $(\lambda \vee \mu)_R^r \ \geq \ \lambda_R^r \vee \mu_R^r$,

(4) $(\lambda \wedge \mu)_r^R \ \leq \ \lambda_r^R \wedge \mu_r^R$,

(5) $\lambda \leq \mu$ implies that $\lambda_R^r \leq \mu_R^r$ and $\lambda_r^R \leq \mu_r^R$,

(6) $(\lambda \vee \mu)_r^R = \lambda_r^R \vee \mu_r^R$,

(7) $(\lambda \wedge \mu)_R^r = \lambda_R^r \wedge \mu_R^r$,

(8) $(\lambda_r^R)^c = (\lambda^c)_R^r$ and $(\lambda_R^r)^c = (\lambda^c)_r^R$

(9) $(\lambda_R^r)_r^R \geq (\lambda_R^r)_R^r = \lambda_R^r$

(10) $(\lambda_r^R)_R^r \leq (\lambda_r^R)_r^R = \lambda_r^R$

Proof. From Equations (3), (4), we get that (1), (2), (5), (8) are proved directly.

For (3), we have

$$\begin{aligned}(\lambda \vee \mu)_R^r(x) &= (\lambda \vee \mu)(x) \wedge \Big(\bigvee_{(\lambda\vee\mu)^c(z)>0,\ z\neq x} [x]_r(z)\Big)^c \\ &\geq \Big(\lambda(x) \wedge \Big(\bigvee_{\lambda^c(z)>0,\ z\neq x} [x]_r(z)\Big)^c\Big) \vee \Big(\mu(x) \wedge \Big(\bigvee_{\mu^c(z)>0,\ z\neq x} [x]_r(z)\Big)^c\Big) \\ &= (\lambda_R^r \vee \mu_R^r)(x) \quad \forall x \in X.\end{aligned}$$

For (4), we have

$$\begin{aligned}(\lambda \wedge \mu)_r^R(x) &= (\lambda \wedge \mu)(x) \vee \bigvee_{(\lambda\wedge\mu)(z)>0,\ z\neq x} [x]_r(z) \\ &\leq \Big(\lambda(x) \vee \bigvee_{\lambda(z)>0, z\neq x} [x]_r(z)\Big) \wedge \Big(\mu(x) \vee \bigvee_{\mu(z)>0, z\neq x} [x]_r(z)\Big) \\ &= (\lambda_r^R \wedge \mu_r^R)(x) \quad \forall x \in X.\end{aligned}$$

For (6), we have

$$\begin{aligned}(\lambda \vee \mu)_r^R(x) &= (\lambda \vee \mu)(x) \vee \bigvee_{(\lambda\vee\mu)(z)>0,\ z\neq x} [x]_r(z) \\ &= \Big(\lambda(x) \vee \bigvee_{\lambda(z)>0, z\neq x} [x]_r(z)\Big) \vee \Big(\mu(x) \vee \bigvee_{\mu(z)>0, z\neq x} [x]_r(z)\Big) \\ &= (\lambda_r^R \vee \mu_r^R)(x) \quad \forall x \in X.\end{aligned}$$

For (7), we have

$$\begin{aligned}(\lambda \wedge \mu)_R^r(x) &= (\lambda \wedge \mu)(x) \wedge \Big(\bigvee_{(\lambda\wedge\mu)^c(z)>0,\ z\neq x} [x]_r(z)\Big)^c \\ &= \Big(\lambda(x) \wedge \Big(\bigvee_{\lambda^c(z)>0, z\neq x} [x]_r(z)\Big)^c\Big) \wedge \Big(\mu(x) \wedge \Big(\bigvee_{\mu^c(z)>0, z\neq x} [x]_r(z)\Big)^c\Big) \\ &= (\lambda_R^r \wedge \mu_R^r)(x) \quad \forall x \in X.\end{aligned}$$

For (9), we have

$$\begin{aligned}(\lambda_R^r)_r^R(x) &= \lambda_R^r(x) \vee \bigvee_{\lambda_R^r(z)>0, z\neq x} [x]_r(z) \\ &\geq \lambda_R^r(x) \\ &= \Big[\lambda(x) \wedge \Big(\bigvee_{\lambda^c(z)>0, z\neq x} [x]_r(z)\Big)^c\Big] \wedge \Big(\bigvee_{\lambda^c(z)>0, z\neq x} [x]_r(z)\Big)^c \\ &= (\lambda_R^r)_R^r\,(x) \quad \forall x \in X.\end{aligned}$$

For (10), we have

$$\begin{aligned}(\lambda_r^R)_R^r(x) &= \lambda_r^R(x) \ \wedge \ (\bigvee_{(\lambda_r^R)^c(z)>0,\ z\neq x} [x]_r(z))^c \\ &\leq \lambda_r^R(x) \\ &= [\lambda(x) \ \vee \bigvee_{\lambda(z)>0, z\neq x} [x]_r(z)] \ \vee \bigvee_{\lambda(z)>0, z\neq x} [x]_r(z) \\ &= (\lambda_r^R)_r^R\ (x) \ \ \forall x \in X. \ \Box\end{aligned}$$

Remark 2.1. According to the above two cases of the r-fuzzy boundary regions, we can say that there are only two possible cases, one representing the non rough case and the other one representing the rough case:

(B1) A crisp (an exact) (no roughness) fuzzy set λ with respect to the fuzzy equivalence relation R when

$$\lambda_r^B = \overline{0} \ \text{ whenever } \ \lambda_R^r = \lambda_r^R = \lambda.$$

(B2) A roughly fuzzy R-definable set λ, that is, some elements of X have membership values of both of λ and λ^c when

$$\lambda_r^B = \lambda_r^R \wedge (\lambda_R^r)^c \ \text{ whenever } \ \lambda_r^R \nleq \lambda_R^r.$$

The following example for the case in which all elements are in fuzzy relation together with a nonzero membership value.

Example 2.1. Let R be a fuzzy relation on a set $X = \{a, b, c, d\}$ given as:

R	a	b	c	d
a	1	0.8	0.3	0.1
b	0.8	1	0.3	0.1
c	0.3	0.3	1	0.1
d	0.1	0.1	0.1	1

Assume that $\lambda = \{0.3, 0.4, 1, 0.2\}$ and let $r = 0.4$. Then,
$\lambda_R^r = \{0.3, 0.4, 0.7, 0.2\}$, $\lambda_r^R = \{0.4, 0.4, 1, 0.2\}$ and $\lambda_r^B = \{0.4, 0.4, 0.3, 0.2\}$.

If $r = 0.6$, then $\lambda_R^r = \{0.3, 0.4, 0.7, 0.2\}$, $\lambda_r^R = \{0.6, 0.6, 1, 0.2\}$ and $\lambda_r^B = \{0.6, 0.6, 0.3, 0.2\}$.
For $\mu = \{1, 0.2, 0, 0.9\}$ and $r = 0.2$, we get that
$\mu_R^r = \{0.8, 0.2, 0, 0.9\}$, $\mu_r^R = \{1, 0.2, 0.3, 0.9\}$ and $\mu_r^B = \{0.2, 0.2, 0.3, 0.1\}$.

The following example for the case in which some elements have a fuzzy relation in between with zero membership value.

Example 2.2. Let R be a fuzzy relation on a set $X = \{a, b, c, d\}$ as shown below.

R	a	b	c	d
a	1	0.2	0.7	0
b	0.2	1	0.2	0
c	0.7	0.2	1	0
d	0	0	0	1

Assume that $\lambda = \{0.2, 0.8, 0.6, 0.1\}$ and $r = 0.8$. Then,
$\lambda_R^r = \{0.2, 0.8, 0.3, 0.1\}$, $\lambda_r^R = \{0.7, 0.8, 0.7, 0.1\}$ and $\lambda_r^B = \{0.7, 0.2, 0.7, 0.1\}$.

If $r = 0.5$, then $\lambda_R^r = \{0.2, 0.8, 0.5, 0.1\}$, $\lambda_r^R = \{0.5, 0.8, 0.6, 0.1\}$ and $\lambda_r^B = \{0.5, 0.2, 0.5, 0.1\}$.

For $\mu = \{1, 0.7, 0, 0\}$ and $r = 0.5$, we get that
$\mu_R^r = \{0.5, 0.7, 0, 0\}$, $\mu_r^R = \{1, 0.7, 0.5, 0\}$ and $\mu_r^B = \{0.5, 0.3, 0.5, 0\}$.

Associated with a fuzzy set λ and a value $r \in I_0$ in a fuzzy approximation space (X, R), we can define a fuzzy interior operator $\text{int}_R^\lambda : I^X \times I_0 \to I^X$ as follows:

$$\text{int}_R^\lambda(\nu, r) = \lambda_R^r \wedge \nu_R^r \quad \forall\, \nu \neq \overline{1} \quad \text{and} \quad \text{int}_R^\lambda(\overline{1}, r) = \overline{1}. \tag{6}$$

This is called a fuzzy interior associated with λ and $r \in I_0$ in the fuzzy approximation space (X, R).

Also, we can define a fuzzy closure operator $\text{cl}_R^\lambda : I^X \times I_0 \to I^X$ as follows:

$$\text{cl}_R^\lambda(\nu, r) = (\lambda_R^r)^c \vee \nu_r^R \quad \forall\, \nu \neq \overline{0} \quad \text{and} \quad \text{cl}_R^\lambda(\overline{0}, r) = \overline{0}. \tag{7}$$

It is called a fuzzy closure associated with λ and $r \in I_0$ in the fuzzy approximation space (X, R).

Note that: $\text{cl}_R^\lambda(\nu^R, r) = \text{cl}_R^\lambda(\nu, r)\ \forall \nu \in I^X$, $\text{int}_R^\lambda(\nu_R, r) = \text{int}_R^\lambda(\nu, r)\ \forall \nu \in I^X$, and moreover $\text{int}_R^\lambda(\nu^c, r) = (\text{cl}_R^\lambda(\nu, r))^c$ and $\text{cl}_R^\lambda(\nu^c, r) = (\text{int}_R^\lambda(\nu, r))^c\ \forall \nu \in I^X$.

Example 2.3. Let R be a fuzzy relation on a set $X = \{a, b, c\}$ as shown below.

R	a	b	c
a	1	0.8	0
b	0.8	1	0
c	0	0	1

Assume that $\lambda = \{0, 0, 0.5\}$ and $r = 0.6$. Then,
$\lambda_R^r = \lambda$, $\lambda_r^R = \lambda$, $\lambda_r^B = \overline{0}$ and $(\lambda_R^r)^c = \{1, 1, 0.5\}$.

For $\mu = \{0.8, 0.8, 0\}$, we get that $\mu_R^r = \{0.4, 0.4, 0\}$, $\mu_r^R = \{0.8, 0.8, 0\} = \mu$. Hence, $\text{int}_R^\lambda(\mu, r) = \lambda_R^r \wedge \mu_R^r = \overline{0}$ and $\text{cl}_R^\lambda(\mu, r) = (\lambda_R^r)^c \vee \mu_r^R = \{1, 1, 0.5\}$.

Now, if we changed λ to be $\lambda = \{0.2, 0.7, 1\}$ and changed r to be $r = 0.9$. Then,
$\lambda_R^r = \{0.2, 0.2, 1\}$, $\lambda_r^R = \{0.8, 0.8, 1\}$, $\lambda_r^B = \{0.8, 0.8, 0\}$ and $(\lambda_R^r)^c = \{0.8, 0.8, 0\}$.

Put the same $\mu = \{0.8, 0.8, 0\}$, we get that $\mu_R^r = \{0.2, 0.2, 0\}$, $\mu_r^R = \{0.8, 0.8, 0\} = \mu$. Hence, $\text{int}_R^\lambda(\mu, r) = \lambda_R^r \wedge \mu_R^r = \{0.2, 0.2, 0\}$ and $\text{cl}_R^\lambda(\mu, r) = (\lambda_R^r)^c \vee \mu_r^R = \{0.8, 0.8, 0\}$. Also, if we have $\nu = \{0.3, 1, 1\}, r = 0.9$, we get $\nu_R^r = \{0.3, 0.2, 1\}$, $\nu_r^R = \{0.8, 1, 1\}$.
Thus, $\text{int}_R^\lambda(\nu, r) = \{0.2, 0.2, 1\}$ and $\text{cl}_R^\lambda(\nu, r) = \{0.8, 1, 1\}$.

From these examples, we see that the computations depend on the fuzzy relation R, the value of, r and the fuzzy set λ associated with the fuzzy approximation space (X, R).

Definition 2.2. Let (X, R) be a fuzzy approximation space associated with $\lambda \in I^X$, $r \in I_0$. Then,

(1) μ is r-fuzzy preopen (resp. preclosed) set iff
$\mu \le \text{int}_R^\lambda(\text{cl}_R^\lambda(\mu, r), r)$ (resp. $\mu \ge \text{cl}_R^\lambda(\text{int}_R^\lambda(\mu, r), r)$).

(2) The r-fuzzy preinterior of μ, denoted by $\mathrm{p\,int}_R^\lambda(\mu,r)$ is defined by
$\mathrm{p\,int}_R^\lambda(\mu,r)=\bigvee\{\nu\in I^X:\mu\ge\nu,\ \nu \text{ is } r\text{−fuzzy preopen}\}$.

(3) The r-fuzzy preclosure of μ, denoted by $\mathrm{p\,cl}_R^\lambda(\mu,r)$ is defined by
$\mathrm{p\,cl}_R^\lambda(\mu,r)=\bigwedge\{\nu\in I^X:\mu\le\nu,\ \nu \text{ is } r\text{−fuzzy preclosed}\}$.

§3 Fuzzy ideal approximation spaces

A map $\mathcal{I}:I^X\to I$ is called a fuzzy ideal ([16]) on X if it satisfies:

(1) $\mathcal{I}(\bar{0})=1$,

(2) $\lambda\le\mu\ \Rightarrow\ \mathcal{I}(\lambda)\ge\mathcal{I}(\mu)$ for all $\lambda,\mu\in I^X$,

(3) $\mathcal{I}(\lambda\vee\mu)\ge\mathcal{I}(\lambda)\wedge\mathcal{I}(\mu)$ for all $\lambda,\mu\in I^X$.

If $\mathcal{I}_1$ and $\mathcal{I}_2$ are fuzzy ideals on X, we have $\mathcal{I}_1\le\mathcal{I}_2$ iff $\mathcal{I}_1(\mu)\le\mathcal{I}_2(\mu)\ \forall\mu\in I^X$. The triple $(X,R,\mathcal{I})$ is called a fuzzy ideal approximation space. Define the fuzzy ideal $\mathcal{I}^\circ$ as a fuzzy ideal $\mathcal{I}$ so that $\mathcal{I}(\mu)=0\ \forall\mu\ne\bar{0}$.

Definition 3.1. Let $(X,R,\mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda\in I^X$, $r\in I_0$. Then,

(1) The local r-fuzzy closed set $\Phi_\lambda(\mu,r)(R,\mathcal{I})$ of a set $\mu\in I^X$ is defined by:
$$\Phi_\lambda(\mu,r)(R,\mathcal{I})=\bigwedge\{\nu\in I^X:\mathcal{I}(\mu\bar{\wedge}\nu)\ge r,\ \mathrm{cl}_R^\lambda(\nu,r)=\nu\}. \tag{8}$$

(2) The local r-fuzzy preclosed set $\Phi_\lambda^p(\mu,r)$ of a set $\mu\in I^X$ is defined by:
$$\Phi_\lambda^p(\mu,r)=\bigwedge\{\nu\in I^X:\mathcal{I}(\mu\bar{\wedge}\nu)\ge r,\ \mathrm{p\,cl}_R^\lambda(\nu,r)=\nu\}. \tag{9}$$
We wrote $\Phi_\lambda(\mu,r)$ and $\Phi_\lambda^p(\mu,r)$ instead of $\Phi_\lambda(\mu,r)(R,\mathcal{I})$ and $\Phi_\lambda^p(\mu,r)(R,\mathcal{I})$, respectively.

Corollary 3.1. Let $(X,R,\mathcal{I}^\circ)$ be a fuzzy ideal approximation space, $\lambda\in I^X,r\in I_0$. Then, for each $\mu\in I^X$, we have $\Phi_\lambda(\mu,r)=\mathrm{cl}_R^\lambda(\mu,r)$, $\Phi_\lambda^p(\mu,r)=\mathrm{p\,cl}_R^\lambda(\mu,r)$.

Proposition 3.1. *Let $(X,R,\mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda\in I^X,r\in I_0$. Then,*

(1) $\mu\le\nu$ *implies* $\Phi_\lambda(\mu,r)\le\Phi_\lambda(\nu,r)$ *and* $\Phi_\lambda^p(\mu,r)\le\Phi_\lambda^p(\nu,r)$.

(2) *If* $\mathcal{I}_1,\mathcal{I}_2$ *are fuzzy ideals on* X *and* $\mathcal{I}_1\le\mathcal{I}_2$, *then*
$\Phi_\lambda(\mu,r)(\mathcal{I}_1)\ \ge\ \Phi_\lambda(\mu,r)(\mathcal{I}_2)$ *and* $\Phi_\lambda^p(\mu,r)(\mathcal{I}_1)\ \ge\ \Phi_\lambda^p(\mu,r)(\mathcal{I}_2)$.

(3) $\Phi_\lambda^p(\mu,r)\ \le\ \Phi_\lambda(\mu,r)\ =\ \mathrm{cl}_R^\lambda(\Phi_\lambda(\mu,r),r)\ \le\ \mathrm{cl}_R^\lambda(\mu,r)$, *and*
$\Phi_\lambda^p(\mu,r)\ =\ \mathrm{p\,cl}_R^\lambda(\Phi_\lambda^p(\mu,r),r)\ \le\ \mathrm{p\,cl}_R^\lambda(\mu,r)\ \le\ \mathrm{cl}_R^\lambda(\mu,r)$.

(4) $(\Phi_\lambda(\Phi_\lambda(\mu,r),r)\ \le\ \mathrm{cl}_R^\lambda(\Phi_\lambda(\mu,r),r)=\Phi_\lambda(\mu,r)$,

(5) $\Phi_\lambda^p(\Phi_\lambda^p(\mu,r),r)\le\mathrm{pcl}_R^\lambda(\Phi_\lambda^p(\mu,r),r)=\Phi_\lambda^p(\mu,r)$.

(6) $\Phi_\lambda(\mu, r) \ \vee \ \Phi_\lambda(\nu, r) \ \leq \ \Phi_\lambda((\mu \vee \nu), r)$ *and* $\Phi_\lambda(\mu, r) \ \wedge \ \Phi_\lambda(\nu, r) \ \geq \ \Phi_\lambda((\mu \wedge \nu), r)$.

Proof. Obvious. □

Definition 3.2. Let $(X, R, \mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda \in I^X, r \in I_0$. Then, for any $\mu \in I^X$, define the fuzzy operators cl^λ_Φ, $\mathrm{p\,cl}^\lambda_\Phi$, $\mathrm{int}^\lambda_\Phi$, $\mathrm{p\,int}^\lambda_\Phi$: $I^X \times I_0 \to I^X$ as follow:

$$\mathrm{cl}^\lambda_\Phi(\mu, r) \ = \ \mu \ \vee \ \Phi_\lambda(\mu, r)\ , \qquad \mathrm{p\,cl}^\lambda_\Phi(\mu, r) \ = \ \mu \ \vee \ \Phi^p_\lambda(\mu, r) \quad \forall \mu \in I^X. \tag{10}$$

$$\mathrm{int}^\lambda_\Phi(\mu, r) \ = \ \mu \ \wedge \ (\Phi_\lambda(\mu^c, r))^c\ , \quad \mathrm{p\,int}^\lambda_\Phi(\mu, r) \ = \ \mu \ \wedge \ (\Phi^p_\lambda(\mu^c, r))^c \quad \forall \mu \in I^X. \tag{11}$$

Now, if $\mathcal{I} = \mathcal{I}^\circ$, then from Corollary 3.1,

(1) $\mathrm{cl}^\lambda_\Phi(\mu, r) = \mathrm{cl}^\lambda_R(\mu, r) = \Phi_\lambda(\mu, r)$ and $\mathrm{int}^\lambda_\Phi(\mu, r) = \mathrm{int}^\lambda_R(\mu, r) = (\Phi_\lambda(\mu^c, r))^c \ \ \forall \mu \in I^X$.

(2) $\mathrm{p\,cl}^\lambda_\Phi(\mu, r) \ = \ \mathrm{p\,cl}^\lambda_R(\mu, r) \ = \ \Phi^p_\lambda(\mu, r)$ and $\mathrm{p\,int}^\lambda_\Phi(\mu, r) \ = \ \mathrm{p\,int}^\lambda_R(\mu, r) \ = \ (\Phi^p_\lambda(\mu^c, r))^c$ $\forall \mu \in I^X$.

Proposition 3.2. *Let $(X, R, \mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda \in I^X, r \in I_0$. Then, for any $\mu, \nu \in I^X$, we have:*

(1) $\mathrm{int}^\lambda_R(\mu, r) \leq \mathrm{p\,int}^\lambda_\Phi(\mu, r) \leq \mathrm{int}^\lambda_\Phi(\mu, r) \leq \mu \leq \ \mathrm{p\,cl}^\lambda_\Phi(\mu, r) \leq \mathrm{cl}^\lambda_\Phi(\mu, r) \leq \mathrm{cl}^\lambda_R(\mu, r)$.

(2) $\mathrm{cl}^\lambda_\Phi(\mu^c, r) \ = \ (\mathrm{int}^\lambda_\Phi(\mu, r))^c$ and $\mathrm{int}^\lambda_\Phi(\mu^c, r) \ = \ (\mathrm{cl}^\lambda_\Phi(\mu, r))^c$.

(3) $\mathrm{cl}^\lambda_\Phi((\mu \vee \nu), r) \ \geq \ \mathrm{cl}^\lambda_\Phi(\mu, r) \ \vee \ \mathrm{cl}^\lambda_\Phi(\nu, r), \ \ \mathrm{cl}^\lambda_\Phi((\mu \wedge \nu), r) \ \leq \ \mathrm{cl}^\lambda_\Phi(\mu, r) \ \wedge \ \mathrm{cl}^\lambda_\Phi(\nu, r)$.

(4) $\mathrm{int}^\lambda_\Phi((\mu \vee \nu), r) \geq \mathrm{int}^\lambda_\Phi(\mu, r) \vee \mathrm{int}^\lambda_\Phi(\nu, r), \ \ \mathrm{int}^\lambda_\Phi((\mu \wedge \nu), r) \leq \mathrm{int}^\lambda_\Phi(\mu, r) \wedge \mathrm{int}^\lambda_\Phi(\nu, r)$.

(5) $\mathrm{cl}^\lambda_\Phi(\mathrm{cl}^\lambda_\Phi(\mu, r), r) \ \geq \ \mathrm{cl}^\lambda_\Phi(\mu, r)$ and $\mathrm{int}^\lambda_\Phi(\mathrm{int}^\lambda_\Phi(\mu, r), r) \ \leq \ \mathrm{int}^\lambda_\Phi(\mu, r)$.

(6) If $\mu \leq \nu$, then $\mathrm{cl}^\lambda_\Phi(\mu, r) \leq \mathrm{cl}^\lambda_\Phi(\nu, r)$, $\mathrm{int}^\lambda_\Phi(\mu, r) \leq \mathrm{int}^\lambda_\Phi(\nu, r)$.

(7) $\mathrm{pcl}^\lambda_\Phi(\mu, r) \leq \mathrm{pcl}^\lambda_R(\mu, r)$.

Proof. $(1) - (6)$: Clear.

For (7): Suppose that $\mathrm{pcl}^\lambda_\Phi(\mu, r) \not\leq \ \mathrm{pcl}^\lambda_R(\mu, r)$, and if $\mathrm{pcl}^\lambda_R(\mu, r) = \nu$, then $\mu \leq \nu$ and ν is r-fuzzy preclosed set with $\mathrm{pcl}^\lambda_\Phi(\mu, r) \not\leq \nu$. But $\mu \leq \nu$ implies that $\mathcal{I}(\mu \bar{\wedge} \nu) \geq r$, and thus $\Phi^p_\lambda(\mu, r) \leq \nu$ which means that $\mathrm{pcl}^\lambda_\Phi(\mu, r) = \mu \vee \Phi^p_\lambda(\mu, r) \leq \mu \wedge \nu \leq \nu$, which is a contradiction. Hence, $\mathrm{pcl}^\lambda_\Phi(\mu, r) \leq \mathrm{pcl}^\lambda_R(\mu, r)$. □

Definition 3.3. $(X, R, \mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda \in I^X$, $r \in I_0$. Then,

(1) $\mu \in I^X$ is said to be r-fuzzy Φ-open if $\mu \leq \ \mathrm{int}^\lambda_R(\Phi_\lambda(\mu, r), r)$. The complement of r-fuzzy Φ-open is said to be r-fuzzy Φ-closed.

(2) $\mu \in I^X$ is called r-fuzzy dense in itself if $\mu \leq \ \Phi_\lambda(\mu, r)$.

(3) $\mu \in I^X$ is said to be r-fuzzy ideal preopen if $\mu \leq \ \mathrm{int}^\lambda_R(\mathrm{cl}^\lambda_\Phi(\mu, r), r)$. The complement of r-fuzzy ideal preopen is said to be r-fuzzy ideal preclosed.

Lemma 3.1. *Let $(X, R, \mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda \in I^X$, $r \in I_0$. Then,*

(1) *If* $\mu \in I^X$ *is r-fuzzy Φ-closed, then* $\mu \geq \Phi_\lambda(\mathrm{int}_R^\lambda(\mu, r), r)$.

(2) *If* $\mu \in I^X$ *is r-fuzzy ideal preclosed, then* $\mu \geq \mathrm{cl}_R^\lambda(\mathrm{int}_\Phi^\lambda(\mu, r), r)$.

Proof. For (1): Let μ be r-fuzzy Φ-closed. Then,
$\mu^c \leq \mathrm{int}_R^\lambda(\Phi_\lambda(\mu^c, r), r) \leq \mathrm{int}_R^\lambda(\mathrm{cl}_R^\lambda(\mu^c, r), r) = \mathrm{int}_R^\lambda((\mathrm{int}_R^\lambda(\mu, r))^c, r)$
$= (\mathrm{cl}_R^\lambda(\mathrm{int}_R^\lambda(\mu, r)), r)^c \leq (\Phi_\lambda(\mathrm{int}_R^\lambda(\mu, r)), r)^c$. Therefore, $\Phi_\lambda(\mathrm{int}_R^\lambda(\mu, r), r) \leq \mu$.
For (2), it is obvious. □

It is clear that:
r-fuzzy Φ-open (r-fuzzy Φ-closed) ⇒ r-fuzzy ideal preopen (r-fuzzy ideal preclosed)
⇒ r-fuzzy preopen (r-fuzzy preclosed).

Example 3.1. Let $X = \{a, b, c, d\}$,

R	a	b	c	d
a	1	1	0	0
b	1	1	0	0
c	0	0	1	0.6
d	0	0	0.6	1

Assume that $\lambda = \{0, 0, 0.5, 0.5\}$, $r = 0.8$ and a fuzzy ideal $\mathcal{I}$ is defined on X so that $\mathcal{I}(\nu) \geq 0.8\ \forall \nu \leq \{0.5, 0.5, 1, 1\}$. Then, $\lambda_R^r = \{0, 0, 0.4, 0.4\}$ and $(\lambda_R^r)^c = \{1, 1, 0.6, 0.6\}$.

For $\mu = \{0.3, 0, 3, 1, 1\}$, we get that μ is a r-fuzzy preopen set but neither r-fuzzy ideal preopen nor r-fuzzy Φ-open.

Example 3.2. Let R be a fuzzy relation on a set $X = \{a, b, c, d, e\}$ as shown down.

R	a	b	c	d	e
a	1	1	1	0	0
b	1	1	1	0	0
c	1	1	1	0	0
d	0	0	0	1	0.2
e	0	0	0	0.2	1

Assume that $\lambda = \{1, 1, 1, 0.8, 0.6\}$, $r = 0.6$ and a fuzzy ideal $\mathcal{I}$ is defined on X so that $\mathcal{I}(\nu) \geq 0.6\ \forall \nu \leq \{1, 1, 1, 0.8, 0.8\}$. Then, $\lambda_R^r = \{1, 1, 1, 0.8, 0.6\}$ and $(\lambda_R^r)^c = \{0, 0, 0, 0.2, 0.4\}$.

For $\mu = \{1, 1, 1, 0, 0\}$, we get that μ is a r-fuzzy ideal preopen set but it is not r-fuzzy Φ-open.

Theorem 3.1. *Let $(X, R, \mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda \in I^X, r \in I_0$. Then, the following are equivalent.*

(1) $\mu \in I^X$ *is r-fuzzy Φ-open.*

(2) $\mu \in I^X$ *is r-fuzzy ideal preopen and r-fuzzy ideal dense in itself.*

Proof. (1) $\Rightarrow$ (2): It is clear that every r-fuzzy Φ-open set is r-fuzzy ideal preopen. On the other hand, $\mu \le \text{int}_R^\lambda(\Phi_\lambda(\mu, r), r) \le \Phi_\lambda(\mu, r)$, which means μ is r-fuzzy ideal dense in itself.

(2) $\Rightarrow$ (1): By assumption, $\mu \le \text{int}_R^\lambda(\text{cl}_\Phi^\lambda(\mu, r), r) = \text{int}_R^\lambda((\mu \vee \Phi_\lambda(\mu, r)), r) = \text{int}_R^\lambda(\Phi_\lambda(\mu, r), r)$, and hence μ is r-fuzzy Φ-open. □

The following example shows that r-fuzzy ideal preopen and r-fuzzy ideal dense in itself are independent concepts.

Example 3.3. (1) In Example 3.2, we get that: For $\mu = \{1, 1, 1, 0, 0\}$, we have μ is r-fuzzy ideal preopen set but not r-fuzzy ideal dense in itself.

(2) Let $X = \{a, b, c, d\}$,

R	a	b	c	d
a	1	1	0	0
b	1	1	0	0
c	0	0	1	0.8
d	0	0	0.8	1

Assume that $\lambda = \{1, 1, 0.2, 0\}$, $r = 0.3$ and a fuzzy ideal $\mathcal{I}$ is defined on X so that $\mathcal{I}(\nu) \ge \nu \le \overline{0.3}$. Then, $\lambda_R^r = \{1, 1, 0.2, 0\}$ and $(\lambda_R^r)^c = \{0, 0, 0.8, 0\}$.

For $\mu = \{0, 0, 0.4, 0.4\}$, since $\mathcal{I}(\mu) \ngeq 0.3$, then $\Phi_\lambda(\mu, r) \ne \overline{0}$ and the smallest fuzzy set for which $\text{cl}_R^\lambda(\eta, r) = \eta$ is $\eta = \{0, 0, 0.8, 1\}$. That is, $\Phi_\lambda(\mu, r) = \{0, 0, 0.8, 1\}$ and $\mu \nleq \Phi_\lambda(\mu, r)$, that is, μ is r-fuzzy ideal dense in itself while μ is not r-fuzzy ideal preopen.

§4 Separation axioms in fuzzy ideal approximation spaces

Definition 4.1. Let $(X, R, \mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda \in I^X, r \in I_0$, $r \in I_0$. Then,

(1) A fuzzy ideal approximation space $(X, R, \mathcal{I})$ (resp. a fuzzy approximation space (X, R)) is called r-fuzzy ideal-$(t, s)T_0$ (resp. r-fuzzy $(t, s)T_0$) if for every $x \ne y \in X$, there exists $\mu \in I^X, t \in I_0$ with $\text{int}_\Phi^\lambda(\mu, r)(x) \ge t$ (resp. $\text{int}_R^\lambda(\mu, r)(x) \ge t$) such that $\mu(y) < t$ or there exists $\nu \in I^X, s \in I_0$ with $\text{int}_\Phi^\lambda(\nu, r)(y) \ge s$ (resp. $\text{int}_R^\lambda(\nu, r)(y) \ge s$) such that $\nu(x) < s$.

(2) A fuzzy ideal approximation space $(X, R, \mathcal{I})$ (resp. a fuzzy approximation space (X, R)) is called r-fuzzy ideal-$(t, s)T_1$ (resp. r-fuzzy $(t, s)T_1$) if for every $x \ne y \in X$, there exist $\mu, \nu \in I^X$; $t, s \in I_0$ with $\text{int}_\Phi^\lambda(\mu, r)(x) \ge t$ and $\text{int}_\Phi^\lambda(\nu, r)(y) \ge s$ (resp. $\text{int}_R^\lambda(\mu, r)(x) \ge t$ and $\text{int}_R^\lambda(\nu, r)(y) \ge s$) such that $\mu(y) < t$ and $\nu(x) < s$.

(3) A fuzzy ideal approximation space $(X, R, \mathcal{I})$ (resp. a fuzzy approximation space (X, R)) is called r-fuzzy ideal-$(t, s)T_2$ (resp. r-fuzzy $(t, s)T_2$) if for every $x \ne y \in X$, there exist $\mu, \nu \in I^X$; $t, s \in I_0$ with $\text{int}_\Phi^\lambda(\mu, r)(x) \ge t$ and $\text{int}_\Phi^\lambda(\nu, r)(y) \ge s$ (resp. $\text{int}_R^\lambda(\mu, r)(x) \ge t$ and $\text{int}_R^\lambda(\nu, r)(y) \ge s$) such that $\sup(\mu \wedge \nu) < (t \wedge s)$.

Remark 4.1. From (1) in Proposition 3.2, we have $\mathrm{int}^{\lambda}_{\Phi}(\mu, r) \geq \mathrm{int}^{\lambda}_{R}(\mu, r)\ \forall \mu \in I^X$. Denote for r-fuzzy ideal approximation $(t, s)T_i$ separation axioms by r- $(t, s)FI - T_i,\ i = 0, 1, 2,$ that is,

$$
\begin{array}{ccccc}
r-(t,s)T_2 & \longrightarrow & r-(t,s)T_1 & \longrightarrow & r-(t,s)T_0 \\
\downarrow & & \downarrow & & \downarrow \\
r-(t,s)FI-T_2 & \longrightarrow & r-(t,s)FI-T_1 & \longrightarrow & r-(t,s)FI-T_0
\end{array}
$$

Consider a fuzzy ideal approximation space $(X, R, \mathcal{I})$ associated with $\lambda \in I^X, r \in I_0$ and $\mathcal{I} = \mathcal{I}^{\circ}$. Then, the fuzzy ideal separation axioms r- $(t, s)FI - T_i$ are identical to the fuzzy separation axioms r- $(t, s)T_i$ of the fuzzy approximation space $(X, R),\ i = 0, 1, 2$.

Example 4.1. Let $\lambda = \{1, 0.8, 0\},\ r = 0.8,\ t = s = 0.5$ and R be a fuzzy relation on a set $X = \{a, b, c\}$ as shown in the matrix:

R	a	b	c
a	1	0.3	0
b	0.3	1	0
c	0	0	1

Then, we get that: $\lambda^r_R = \{0.7, 0.8, 0\},\ \lambda^R_r = \{1, 0.8, 0\},\ (\lambda^r_R)^c = \{0.3, 0.2, 1\}$.

Now, for the case $a \neq b$, there exists $\mu = \{0.8, 0, 0.4\}$, and then $\mu^r_R = \{0.7, 0, 0.4\}$, which means $\mathrm{int}^{\lambda}_{R}(\mu, r) = \{0.7, 0, 0\}$, and thus $\mathrm{int}^{\lambda}_{R}(\mu, r)(a) \geq t,\ \mu(b) < t$. Also, we can find $\nu = \{0, 0.6, 0.1\}$, and then $\nu^r_R = \{0, 0.6, 0.1\}$, which means $\mathrm{int}^{\lambda}_{R}(\nu, r) = \{0, 0.6, 0\}$, and thus $\mathrm{int}^{\lambda}_{R}(\nu, r)(b) \geq s,\ \nu(a) < s$.

For the cases $a \neq c$ and $b \neq c$, we can find $\eta = \mu \in I^X$ with $\mathrm{int}^{\lambda}_{R}(\eta, r)(a) \geq t$ such that $\eta(c) < t$ and $\eta = \nu \in I^X$ with $\mathrm{int}^{\lambda}_{R}(\eta, r)(b) \geq s$ such that $\eta(c) < s$, while we can not find $\eta \in I^X$ with $\mathrm{int}^{\lambda}_{R}(\eta, r)(c) \geq 0.5$. Hence, (X, R) is 0.8-fuzzy approximation $(0.5, 0.5)T_0$-space associated with λ. (X, R) could not be 0.8-fuzzy approximation $(0.5, 0.5)T_1$-space or 0.8-fuzzy approximation $(0.5, 0.5)T_2$-space.

Define a fuzzy ideal $\mathcal{I}$ on X so that $\mathcal{I}(\eta) \geq r\ \forall \eta \leq \overline{0.7}$. Then, we can find three fuzzy sets $\eta = \{0.8, 0, 0\},\ \xi = \{0, 0.8, 0\}$ and $\zeta = \{0, 0, 0.8\}$ by which $(X, R, \mathcal{I})$ is 0.8-fuzzy ideal approximation $(0.5, 0.5)T_i$-space, $i = 0, 1, 2$ while (X, R) is neither 0.8-fuzzy approximation $(0.5, 0.5)T_1$-space nor 0.8-fuzzy approximation $(0.5, 0.5)T_2$-space.

The following example is given to show that there is a r-fuzzy ideal approximation $(t, s)T_0$-space but not r-fuzzy approximation $(t, s)T_0$-space.

Example 4.2. Let $\lambda = \{0.6, 0, 0\},\ r = 0.9,\ t = s = 0.4$ and R be a fuzzy relation on a set $X = \{a, b, c\}$ as shown in the matrix:

R	a	b	c
a	1	0	0
b	0	1	0
c	0	0	1

Then, we get that: $\lambda_R^r = \{0.6, 0, 0\}$, $(\lambda_R^r)^c = \{0.4, 1, 1\}$.

Now, for the case $b \neq c$, it fails to find $\eta \in I^X$ with $\mathrm{int}_R^\lambda(\eta, r)(b) \geq t$ or $\mathrm{int}_R^\lambda(\eta, r)(c) \geq s$. Hence, (X, R) is not 0.9-fuzzy approximation $(0.4, 0.4)T_0$-space associated with $\lambda, r = 0.9$. Consequently, (X, R) could not be a 0.9-fuzzy approximation $(0.4, 0.4)T_1$-space or 0.9-fuzzy approximation $(0.4, 0.4)T_2$-space.

Define a fuzzy ideal $\mathcal{I}$ on X so that $\mathcal{I}(\eta) \geq r$ $\forall \eta \leq \{0.6, 1, 1\}$. Then, there exist $\mu = \{0.4, 0.4, 0\}$ and $\nu = \{0.4, 0, 0.4\}$ for which $\Phi_\lambda(\mu^c, r) = \overline{0}$ and $\Phi_\lambda(\nu^c, r) = \overline{0}$, which implies that $\mathrm{int}_\Phi^\lambda(\mu, r) = \mu = \{0.4, 0.4, 0\}$ and $\mathrm{int}_\Phi^\lambda(\nu, r) = \nu = \{0.4, 0, 0.4\}$, and thus $\mathrm{int}_\Phi^\lambda(\mu, r)(a) \geq 0.4$, $\mu(c) < 0.4$, $\mathrm{int}_\Phi^\lambda(\mu, r)(b) \geq 0.4$, $\mu(c) < 0.4$ and $\mathrm{int}_\Phi^\lambda(\nu, r)(a) \geq 0.4$,
$\nu(b) < 0.4$. That is, $(X, R, \mathcal{I})$ is 0.9-fuzzy ideal approximation $(0.4, 0.4)T_0$-space but (X, R) is not 0.9-fuzzy approximation $(0.4, 0.4)T_0$-space. Moreover, we can find $\eta = \{0.4, 0, 0\}$, $\xi = \{0, 0.4, 0\}$ and $\zeta = \{0, 0, 0.4\}$ concluding that $(X, R, \mathcal{I})$ is 0.9-fuzzy ideal approximation $(0.4, 0.4)T_1$-space and 0.9-fuzzy ideal approximation $(0.4, 0.4)T_2$-space.

If (X, R) and (Y, R^*) are r-fuzzy approximation spaces associated with $\lambda \in I^X$, $r \in I_0$ and $\mu \in I^Y, r \in I_0$, respectively, then a mapping $f : (X, R) \to (Y, R^*)$ is said to be r-fuzzy approximation continuous (FAC) if $\mathrm{int}_R^\lambda(f^{-1}(\eta), r) \geq f^{-1}(\mathrm{int}_{R^*}^\mu(\eta, r))$ $\forall \eta \in I^Y$. It is equivalent to $\mathrm{cl}_R^\lambda(f^{-1}(\eta), r) \leq f^{-1}(\mathrm{cl}_{R^*}^\mu(\eta, r))$ $\forall \eta \in I^Y$.

Now, with respect to $\lambda \in I^X$ and $\mu \in I^Y$, If $\mathcal{I}$, $\mathcal{I}^*$ are fuzzy ideals on X, Y, respectively, then a mapping $f : (X, R, \mathcal{I}) \to (Y, R^*)$ is called r-fuzzy ideal approximation continuous (FIAC) provided that $\mathrm{int}_\Phi^\lambda(f^{-1}(\eta), r) \geq f^{-1}(\mathrm{int}_{R^*}^\mu(\eta, r))$ $\forall \eta \in I^Y$.
It is easily shown that it is equivalent to $\mathrm{cl}_\Phi^\lambda(f^{-1}(\eta), r) \leq f^{-1}(\mathrm{cl}_{R^*}^\mu(\eta, r))$ $\forall \eta \in I^Y$.
Also, let us call $f : (X, R) \to (Y, R^*)$ an r-fuzzy approximation open (FAO) provided that $\mathrm{int}_{R^*}^\mu(f(\xi), r) \geq f(\mathrm{int}_R^\lambda(\xi, r))$ $\forall \xi \in I^X$.

$f : (X, R) \to (Y, R^*, \mathcal{I}^*)$ is r-fuzzy ideal approximation open (FIAO) provided that $\mathrm{int}_\Phi^\mu(f(\xi), r) \geq f(\mathrm{int}_R^\lambda(\xi, r))$ $\forall \xi \in I^X$.

Clearly, every (FAC) (resp. (FAO)) mapping will be (FIAC) (resp. (FIAO)) mapping as well (from (1) in Proposition 3.2).

Theorem 4.1. *Let (X, R), (Y, R^*) be r-fuzzy approximation spaces associated with $\lambda \in I^X$, $\mu \in I^Y, r \in I_0$, respectively, $\mathcal{I}$ a fuzzy ideal on X and $f : (X, R) \to (Y, R^*)$ is an injective (FAC) mapping with $f(\lambda) = \mu$. Then, $(X, R, \mathcal{I})$ is a r-fuzzy ideal approximation $(t, s)T_i$-space if (Y, R^*) is r-fuzzy approximation $(t, s)T_i$-space, $i = 0, 1, 2$.*

Proof. Since $x \neq y$ in X implies that $f(x) \neq f(y)$ in Y, and from Y is r-fuzzy approximation $(t, s)T_2$-space, then there exist $\eta, \zeta \in I^Y$ with $t \leq \mathrm{int}_{R^*}^\mu(\eta, r)(f(x))$,
$s \leq \mathrm{int}_{R^*}^\mu(\zeta, r)(f(y))$ such that $\sup(\eta \wedge \zeta) < (t \wedge s)$, that is, $t \leq f^{-1}(\mathrm{int}_{R^*}^\mu(\eta, r))(x)$, $s \leq f^{-1}(\mathrm{int}_{R^*}^\mu(\zeta, r))(y)$, and then $t \leq f^{-1}(\mathrm{int}_\Phi^\mu(\eta, r))(x)$, $s \leq f^{-1}(\mathrm{int}_\Phi^\mu(\zeta, r))(y)$. Since f is

(FAC), then $t \leq \mathrm{int}^{\lambda}_{R}(f^{-1}(\eta, r))(x)$, $s \leq \mathrm{int}^{\lambda}_{R}(f^{-1}(\zeta, r))(y)$, and then $t \leq \mathrm{int}^{\lambda}_{\Phi}(f^{-1}(\eta, r))(x)$, $s \leq \mathrm{int}^{\lambda}_{\Phi}(f^{-1}(\zeta, r))(y)$. That is, there exist $\rho = f^{-1}(\eta)$, $\omega = f^{-1}(\zeta)$ with $t \leq \mathrm{int}^{\lambda}_{\Phi}(\rho, r)(x)$, $s \leq \mathrm{int}^{\lambda}_{\Phi}(\omega, r)(y)$ and $\sup(\rho \wedge \omega) < (t \wedge s)$. Hence, $(X, R, \mathcal{I})$ is r-fuzzy ideal approximation $(t, s)T_2$-space. Other cases are similar. □

Theorem 4.2. *Let (X, R), (Y, R^*) be r-fuzzy approximation spaces associated with $\lambda \in I^X$, $\mu \in I^Y, r \in I_0$, respectively, $\mathcal{I}^*$ a fuzzy ideal on Y and $f : (X, R) \to (Y, R^*)$ is a surjective (FAO) mapping with $f^{-1}(\mu) = \lambda$. Then, $(Y, R^*, \mathcal{I}^*)$ is r-fuzzy ideal $(t, s)T_i$-space if (X, R) is an r-fuzzy approximation $(t, s)T_i$-space, $i = 0, 1, 2$.*

Proof. Since $p \neq q$ in Y implies that $f^{-1}(p) \neq f^{-1}(q)$ in X, and from (X, R) is r-fuzzy approximation $(t, s)T_2$-space, then there exist $\rho, \omega \in I^X$ with $t \leq \mathrm{int}^{\lambda}_{R}(\rho, r)(f^{-1}(p))$, $s \leq \mathrm{int}^{\lambda}_{R}(\omega, r)(f^{-1}(q))$ such that $\sup(\rho \wedge \omega) < (t \wedge s)$, that is, $t \leq f(\mathrm{int}^{\lambda}_{R}(\rho, r))(p)$, $s \leq f(\mathrm{int}^{\lambda}_{R}(\omega, r))(q)$. From f is (FAO), then, $t \leq \mathrm{int}^{\mu}_{R^*}(f(\rho), r)(p)$, $s \leq \mathrm{int}^{\mu}_{R^*}(f(\omega), r)(q)$, and thus $t \leq \mathrm{int}^{\mu}_{\Phi}(f(\rho), r)(p)$, $s \leq \mathrm{int}^{\mu}_{\Phi}(f(\omega), r)(q)$. That is, there exist $\eta = f(\rho)$, $\zeta = f(\omega)$ with $t \leq \mathrm{int}^{\mu}_{\Phi}(\eta, r)(p)$, $s \leq \mathrm{int}^{\mu}_{\Phi}(\zeta, r)(q)$ and $\sup(\eta \wedge \zeta) < (t \wedge s)$. Hence, $(Y, R^*, \mathcal{I}^*)$ is a r-fuzzy ideal approximation $(t, s)T_2$-space. Other cases are similar. □

§5 Connected fuzzy ideal approximation spaces

Definition 5.1. Let (X, R) be a fuzzy approximation space associated with $\lambda \in I^X, r \in I_0$, $r \in I_0$. Then,

(1) the fuzzy sets $\mu, \nu \in I^X$ are called r-fuzzy approximation preseparated (resp. separated) sets if
$\mathrm{p\,cl}^{\lambda}_{R}(\mu, r) \wedge \nu = \mu \wedge \mathrm{p\,cl}^{\lambda}_{R}(\nu, r) = \overline{0}$ (resp. $\mathrm{cl}^{\lambda}_{R}(\mu, r) \wedge \nu = \mu \wedge \mathrm{cl}^{\lambda}_{R}(\nu, r) = \overline{0}$).

(2) A fuzzy set $\eta \in I^X$ is called r-fuzzy approximation predisconnected (resp. disconnected) set if there exist r-fuzzy approximation preseparated (resp. separated) sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \eta$. A fuzzy set η is called r-fuzzy approximation preconnected (resp. connected) if it is not r-fuzzy approximation predisconnected (resp. disconnected).

(3) (X, R) is called r-fuzzy approximation predisconnected (resp. disconnected) space if there exist r-fuzzy approximation preseparated (resp. separated) sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \overline{1}$. A fuzzy approximation space(X, R) is called r-fuzzy approximation preconnected (resp. connected) space if it is not r-fuzzy approximation predisconnected (resp. disconnected) space.

Definition 5.2. Let $(X, R, \mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda \in I^X, r \in I_0$. Then,

(1) the fuzzy sets $\mu, \nu \in I^X$ are called r-fuzzy ideal approximation preseparated (resp. separated) sets if
$\mathrm{p\,cl}^{\lambda}_{\Phi}(\mu, r) \wedge \nu = \mu \wedge \mathrm{p\,cl}^{\lambda}_{\Phi}(\nu, r) = \overline{0}$ (resp. $\mathrm{cl}^{\lambda}_{\Phi}(\mu, r) \wedge \nu = \mu \wedge \mathrm{cl}^{\lambda}_{\Phi}(\nu, r) = \overline{0}$).

(2) A fuzzy set $\eta \in I^X$ is called r-fuzzy ideal approximation predisconnected (resp. disconnected) set if there exist r-fuzzy ideal approximation preseparated (resp. separated) sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \eta$. A fuzzy set η is called r-fuzzy ideal approximation preconnected (resp. connected) if it is not r-fuzzy ideal approximation predisconnected (resp. disconnected).

(3) $(X, R, \mathcal{I})$ is called r-fuzzy ideal approximation predisconnected (resp. disconnected) space if there exist r-fuzzy ideal approximation preseparated (resp. separated) sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \overline{1}$. A fuzzy ideal approximation space$(X, R, \mathcal{I})$ is called r-fuzzy ideal approximation preconnected (resp. connected) space if it is not r-fuzzy ideal approximation predisconnected (resp. disconnected) space.

Remark 5.1. We have the following implications.

$$\begin{array}{ccc} r-\textit{fuzzy separated} & \longrightarrow & r-\textit{fuzzy ideal separated} \\ \downarrow & & \downarrow \\ r-\textit{fuzzy preseparated} & \longrightarrow & r-\textit{fuzzy ideal preseparated} \end{array}$$

and hence,

$$\begin{array}{ccc} r-\textit{fuzzy ideal preconnected} & \longrightarrow & r-\textit{fuzzy preconnected} \\ \downarrow & & \downarrow \\ r-\textit{fuzzy ideal connected} & \longrightarrow & r-\textit{fuzzy connected} \end{array}$$

Example 5.1. Let $X = \{a, b, c, d, e\}$, R a fuzzy relation on X defined by

R	a	b	c	d	e
a	1	1	0.2	0	0
b	1	1	0.2	0	0
c	0.2	0.2	1	0	0
d	0	0	0	1	0
e	0	0	0	0	1

Suppose that $\lambda = \{0, 0, 0.4, 0.8, 0\}$, $r = 0.6$. Then, $\lambda_R^r = \{0, 0, 0.4, 0.8, 0\}$, and $(\lambda_R^r)^c = \{1, 1, 0.6, 0.2, 1\}$. Now, for $\mu = \{0.6, 0, 0, 0, 0\}$, $\nu = \{0, 0.6, 0, 0, 0\}$. Then, $\mu_r^R = \{0.6, 0.6, 0.2, 0, 0\}$, $\nu_r^R = \{0.6, 0.6, 0.2, 0, 0\}$, and thus $\text{cl}_R^\lambda(\mu, r) = \{1, 1, 0.6, 0.2, 1\}$ and $\text{cl}_R^\lambda(\nu, r) = \{1, 1, 0.6, 0.2, 1\}$. Moreover, $\mu_R^r = \{0.4, 0, 0, 0, 0\}$, $\nu_R^r = \{0, 0.4, 0, 0, 0\}$, and thus $\text{int}_R^\lambda(\mu, r) = \overline{0}$ and $\text{int}_R^\lambda(\nu, r) = \overline{0}$. Hence,

(1) μ, ν are 0.6-fuzzy approximation preseparated sets but not 0.6-fuzzy approximation separated sets.

(2) Consider a fuzzy ideal $\mathcal{I}$ defined on X so that $\mathcal{I}(\eta) \geq r\ \forall \eta \leq \overline{0.6}$. Then,
μ, ν are 0.6-fuzzy ideal approximation separated sets but not 0.6-fuzzy approximation separated sets.

(3) Consider a fuzzy ideal $\mathcal{I}$ defined on X so that $\mathcal{I}(\eta) \geq r\ \forall \eta \leq \overline{0.3}$. Then,
$\mathcal{I}(\mu) \not\geq r,\ \mathcal{I}(\nu) \not\geq r$, which implies that μ, ν are not 0.6-fuzzy ideal approximation separated sets.

But, μ, ν are 0.6-fuzzy approximation preclosed sets, and hence μ, ν are 0.6-fuzzy ideal approximation preseparated sets but not 0.6-fuzzy ideal approximation separated sets.

(4) Here, $\eta = \{0.6, 0, 0.6, 0, 0\}, \xi = \{0, 0.6, 0, 0.6, 0\}$ are not 0.6-fuzzy approximation preseparated. While, η, ξ are 0.6-fuzzy ideal approximation preseparated sets whenever $\mathcal{I}$ is a fuzzy ideal defined on X so that $\mathcal{I}(\zeta) \geq r\ \forall \zeta \leq \overline{0.6}$.

Proposition 5.1. *Let (X, R) be a fuzzy approximation space associated with $\lambda \in I^X, r \in I_0$. Then, the following are equivalent.*

(1) *(X, R) is r-fuzzy approximation preconnected.*

(2) $\mu \wedge \nu = \overline{0},\ \mathrm{p\,int}_R^\lambda(\mu, r) = \mu,\ \mathrm{p\,int}_R^\lambda(\nu, r) = \nu$ and $\mu \vee \nu = \overline{1}$ imply $\mu = \overline{0}$ or $\nu = \overline{0}$.

(3) $\mu \wedge \nu = \overline{0},\ \mathrm{p\,cl}_R^\lambda(\mu, r) = \mu,\ \mathrm{p\,cl}_R^\lambda(\nu, r) = \nu$ and $\mu \vee \nu = \overline{1}$ imply $\mu = \overline{0}$ or $\nu = \overline{0}$.

Proof. (1) $\Rightarrow$ (2): Let $\mu, \nu \in I^X$ with $\mathrm{p\,int}_R^\lambda(\mu, r) = \mu$, $\mathrm{p\,int}_R^\lambda(\nu, r) = \nu$ such that $\mu \wedge \nu = \overline{0}$ and $\mu \vee \nu = \overline{1}$. Then,

$$\mathrm{p\,cl}_R^\lambda(\mu, r) = \mathrm{p\,cl}_R^\lambda(\nu^c, r) = (\mathrm{p\,int}_R^\lambda(\nu, r))^c = \nu^c = \mu,$$
$$\mathrm{p\,cl}_R^\lambda(\nu, r) = \mathrm{p\,cl}_R^\lambda(\mu^c, r) = (\mathrm{p\,int}_R^\lambda(\mu, r))^c = \mu^c = \nu.$$

Hence, $\mathrm{p\,cl}_R^\lambda(\mu, r) \wedge \nu = \mu \wedge \mathrm{p\,cl}_R^\lambda(\nu, r) = \mu \wedge \nu = \overline{0}$. That is, μ, ν are r-fuzzy approximation preseparated sets so that $\mu \vee \nu = \overline{1}$. But (X, R) is r-fuzzy approximation connected implies that $\mu = \overline{0}$ or $\nu = \overline{0}$.

(2) $\Rightarrow$ (3): , (3) $\Rightarrow$ (1): Clear. □

Proposition 5.2. *Let (X, R) be a fuzzy approximation space associated with $\lambda \in I^X, r \in I_0$. Then, for $\mu \in I^X$, the following are equivalent.*

(1) *μ is r-fuzzy preconnected set.*

(2) *If ν, ρ are r-fuzzy approximation preseparated sets with $\mu \leq (\nu \vee \rho)$, then*
$\mu \wedge \nu = \overline{0}$ or $\mu \wedge \rho = \overline{0}$.

(3) *If ν, ρ are r-fuzzy approximation preseparated sets with $\mu \leq (\nu \vee \rho)$, then*
$\mu \leq \nu$ or $\mu \leq \rho$.

Proof. Direct □

Theorem 5.1. *Let* $(X,R),(Y,R^*)$ *be fuzzy approximation spaces associated with* $\lambda \in I^X$, $\mu \in I^Y$, *respectively,* $r \in I_0$, $\mathcal{I}$ *a fuzzy ideal on* X, *and* $f:(X,R,\mathcal{I}) \to (Y,R^*)$ *a fuzzy mapping such that* $\mathrm{p\,cl}^{\lambda}_{\Phi}(f^{-1}(\nu)) \leq f^{-1}(\mathrm{p\,cl}^{\mu}_{R^*}(\nu))$ $\forall \nu \in I^Y$. *Then,* $f(\eta) \in I^Y$ *is* r*-fuzzy approximation preconnected set if* η *is* r*-fuzzy ideal approximation preconnected in* X.

Proof. Let $\nu, \rho \in I^Y$ be r-fuzzy approximation preseparated sets with $f(\eta) = \nu \vee \rho$. That is, $\mathrm{p\,cl}^{\mu}_{R^*}(\nu, r) \wedge \rho = \mathrm{p\,cl}^{\mu}_{R^*}(\rho, r) \wedge \nu = \overline{0}$. Then, $\eta \leq (f^{-1}(\nu) \vee f^{-1}(\rho))$, and from the condition of f, we get that

$$\begin{aligned} \mathrm{p\,cl}^{\lambda}_{\Phi}(f^{-1}(\nu), r) \wedge f^{-1}(\rho) &\leq f^{-1}(\mathrm{p\,cl}^{\mu}_{R^*}(\nu, r)) \wedge f^{-1}(\rho) \\ &= f^{-1}(\mathrm{p\,cl}^{\mu}_{R^*}(\nu, r) \wedge \rho) = f^{-1}(\overline{0}) = \overline{0}, \end{aligned}$$

and in a similar way, we have

$$\begin{aligned} \mathrm{p\,cl}^{\lambda}_{\Phi}(f^{-1}(\rho), r) \wedge f^{-1}(\nu) &\leq f^{-1}(\mathrm{p\,cl}^{\mu}_{R^*}(\rho, r)) \wedge f^{-1}(\nu) \\ &= f^{-1}(\mathrm{cl}^{\mu}_{R^*}(\rho, r) \wedge \nu) = f^{-1}(\overline{0}) = \overline{0}. \end{aligned}$$

Hence, $f^{-1}(\nu)$ and $f^{-1}(\rho)$ are r-fuzzy ideal approximation preseparated sets in X so that $\eta \leq (f^{-1}(\nu) \vee f^{-1}(\rho))$. But from (3) in Proposition 5.2, we get that $\eta \leq f^{-1}(\nu)$ or $\eta \leq f^{-1}(\rho)$, which means that $f(\eta) \leq \nu$ or $f(\eta) \leq \rho$. Thus, from that η is r-fuzzy ideal approximation preconnected, and consequently η is an r-fuzzy approximation preconnected set in X, and again from (3) in Proposition 5.2, we get that $f(\eta)$ is r-fuzzy approximation preconnected in Y. □

§6 Compactness in fuzzy ideal approximation spaces

This section is devoted to introduce the notion of fuzzy ideal approximation compact spaces.

Definition 6.1. Let $(X, R, \mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda \in I^X, r \in I_0$. Then,

(1) μ is said to be r-fuzzy approximation compact (resp. r-fuzzy ideal approximation compact) if for any family $\{\mu_j \in I^X : \mathrm{int}^{\lambda}_{R}(\mu_j, r) = \mu_j, j \in J\}$ with $\mu \leq \bigvee_{j \in J} \mu_j$, there exists a finite subset J_0 of J so that $\mu \leq \bigvee_{j \in J_0} \mu_j$ (resp. $\mathcal{I}(\mu \,\overline{\wedge}\, (\bigvee_{j \in J_0} \mu_j)) \geq r$).

(2) μ is said to be r-fuzzy almost approximation compact (resp. r-fuzzy almost ideal approximation compact) if for any family $\{\mu_j \in I^X : \mathrm{int}^{\lambda}_{R}(\mu_j, r) = \mu_j, j \in J\}$ with $\mu \leq \bigvee_{j \in J} \mu_j$, there exists a finite subset J_0 of J such that $\mu \leq \bigvee_{j \in J_0} \mathrm{cl}^{\lambda}_{R}(\mu_j, r)$ (resp. $\mathcal{I}(\mu \,\overline{\wedge}\, (\bigvee_{j \in J_0} \mathrm{cl}^{\lambda}_{\Phi}(\mu_j, r))) \geq r$).

(3) μ is said to be r-fuzzy nearly approximation compact (resp. r-fuzzy nearly ideal approximation compact) if for any family $\{\mu_j \in I^X : \mathrm{int}^{\lambda}_{R}(\mu_j, r) = \mu_j, j \in J\}$ with $\mu \leq \bigvee_{j \in J} \mu_j$, there exists a finite subset J_0 of J such that $\mu \leq \bigvee_{j \in J_0} \mathrm{int}^{\lambda}_{R}(\mathrm{cl}^{\lambda}_{R}(\mu_j, r), r)$ (resp. $\mathcal{I}(\mu \,\overline{\wedge}\, (\bigvee_{j \in J_0} \mathrm{int}^{\lambda}_{R}(\mathrm{cl}^{\lambda}_{\Phi}(\mu_j, r), r))) \geq r$).

The fuzzy approximation space (X, R) (resp. The fuzzy ideal approximation space $(X, R, \mathcal{I})$ will be called r-fuzzy approximation compact, r-fuzzy almost approximation compact, r-fuzzy nearly approximation compact (resp. r-fuzzy ideal approximation compact, r-fuzzy almost ideal approximation compact, r-fuzzy nearly ideal approximation compact) if we replaced μ with $\overline{1}$.

It is clear that:
r-fuzzy approximation compact $\Longrightarrow$ r-fuzzy almost approximation compact $\Longrightarrow$ r-fuzzy nearly approximation compact.
(resp. r-fuzzy ideal approximation compact $\Longrightarrow$ r-fuzzy almost ideal approximation compact $\Longrightarrow$ r-fuzzy nearly ideal approximation compact).

If $\mathcal{I} = \mathcal{I}^\circ$, then

(1) r-fuzzy approximation compact and r-fuzzy ideal approximation compact are equivalent.

(2) r-fuzzy almost approximation compact and r-fuzzy almost ideal approximation compact are equivalent.

(3) r-fuzzy nearly approximation compact and r-fuzzy nearly ideal approximation compact are equivalent.

Definition 6.2. Let $(X, R, \mathcal{I})$ be a fuzzy ideal approximation space associated with $\lambda \in I^X, r \in I_0$. Then, X is said to be r-fuzzy regular (resp. r-fuzzy ideal regular) space if for each $\eta \in I^X$ with $\mathrm{int}^\lambda_R(\eta, r) = \eta$,

$$\eta = \bigvee_{j \in J} \{\eta_j \ : \ \mathrm{int}^\lambda_R(\eta_j, r) = \eta_j, \ \mathrm{cl}^\lambda_R(\eta_j, r) \leq \eta\}.$$

$$(\text{ resp. } \ \eta = \bigvee_{j \in J} \{\eta_j \ : \ \mathrm{int}^\lambda_R(\eta_j, r) = \eta_j, \ \mathrm{cl}^\lambda_\Phi(\eta_j, r) \leq \eta\}).$$

It is clear that every r-fuzzy regular space is r-fuzzy ideal regular space. If $\mathcal{I} = \mathcal{I}^\circ$, then the concepts of r-fuzzy regular and r-fuzzy ideal regular are identical.

Theorem 6.1. *Let $(X, R, \mathcal{I})$ be r-fuzzy almost ideal approximation compact and r-fuzzy ideal regular. Then, X is r-fuzzy ideal approximation compact space.*

Proof. Assume a family $\{\mu_j \in I^X : \mathrm{int}^\lambda_R(\mu_j, r) = \mu_j, \ j \in J\}$ with $\overline{1} = \bigvee_{j \in J} \mu_j$.
By r-fuzzy ideal regularity of X, then for each $\mathrm{int}^\lambda_R(\mu_j, r) = \mu_j$, we have

$$\mu_j = \bigvee_{j_k \in J_K} \{\mu_{j_k} : \ \mathrm{int}^\lambda_R(\mu_{j_k}, r) = \mu_{j_k}, \ \mathrm{cl}^\lambda_\Phi(\mu_{j_k}, r) \leq \mu_j\}.$$

Hence,$\overline{1} = \bigvee_{j \in J} (\bigvee_{j_k \in J_K} \mu_{j_k})$. Since X is r-fuzzy almost ideal approximation compact, then there exists a finite index subset $J_0 \times J_K$ of $J \times J$ such that

$$\mathcal{I}(\overline{1} \,\overline{\wedge}\, (\bigvee_{j \in J_0} (\bigvee_{j_k \in J_K} \mathrm{cl}^\lambda_\Phi(\mu_{j_k}, r)))) \geq r.$$

Since for each $j \in J_0$, we have $\bigvee_{j_k \in J_K} \mathrm{cl}^{\lambda}_{\Phi}(\mu_{j_k}, r) \le \mu_j$, then we get that

$$\overline{1} \,\overline{\wedge}\, (\bigvee_{j\in J_0} (\bigvee_{j_k\in J_K} \mathrm{cl}^{\lambda}_{\Phi}(\mu_{j_k}, r))) \ \ge\ \overline{1} \,\overline{\wedge}\, (\bigvee_{j\in J_0} \mu_j).$$ Therefore,

$\mathcal{I}(\overline{1} \,\overline{\wedge}\, (\bigvee_{j\in J_0} \mu_j)) \ge r$, and thus $(X, R, \mathcal{I})$ is r-fuzzy ideal approximation compact. □

Theorem 6.2. *Let $(X, R, \mathcal{I})$ be r-fuzzy nearly ideal approximation compact and r-fuzzy ideal regular. Then, X is r-fuzzy nearly ideal approximation compact.*

Proof. Similar to the proof of Theorem 6.1. □

Theorem 6.3. *Let $f : (X, R, \mathcal{I}_1) \to (Y, R^*, \mathcal{I}_2)$ be injective fuzzy approximation continuous mapping between two fuzzy ideal approximation spaces associated with $\lambda \in I^X$, $\mu \in I^Y$ respectively, $r \in I_0$ and $\mathcal{I}_1(\nu) \ge r \Longrightarrow \mathcal{I}_2(f(\nu)) \ge r\ \forall \nu \in I^X$, and $\eta \in I^X$ is r-fuzzy ideal approximation compact set. Then, $f(\eta)$ is r-fuzzy ideal approximation compact as well.*

Proof. Let $\{\xi_j \in I^Y : \mathrm{int}^{\mu}_{R^*}(\xi_j) = \xi_j, j \in J\}$ be a family with $f(\eta) \le \bigvee_{j\in J} \xi_j$.
By fuzzy approximation continuity of f, $\mathrm{int}^{\lambda}_{R}(f^{-1}(\xi_j), r) = f^{-1}(\xi_j)$ and $\eta \le \bigvee_{j\in J} f^{-1}(\xi_j)$. By r-fuzzy ideal approximation compactness of η, there exists a finite subset J_0 of J such that

$$\mathcal{I}_1(\eta \,\overline{\wedge}\, (\bigvee_{j\in J_0} (f^{-1}(\xi_j)))) \ge r.$$

Since $\mathcal{I}_1(\nu) \ge r \Longrightarrow \mathcal{I}_2(f(\nu)) \ge r\ \forall \nu \in I^X$, then

$$\mathcal{I}_2(f(\eta \,\overline{\wedge}\, (\bigvee_{j\in J_0} (f^{-1}(\xi_j))))) \ge r.$$

From f is injective, then $f(\eta \,\overline{\wedge}\, (\bigvee_{j\in J_0} (f^{-1}(\xi_j)))) \ = \ f(\eta) \,\overline{\wedge}\, (\bigvee_{j\in J_0} (\xi_j))$. Thus,

$$\mathcal{I}_2(f(\eta) \,\overline{\wedge}\, (\bigvee_{j\in J_0} (\xi_j))) \ge r.$$

Hence, $f(\eta)$ is r-fuzzy ideal approximation compact. □

§7 Conclusion

Let X be a non empty set and let $\mathcal{I}, \mathcal{G} : I^X \to I$ be two mappings satisfying the following conditions:

$$\mathcal{I}_{\mathcal{G}}(\lambda) \ = \ \bigvee\{r \ : \ \mathcal{G}(\lambda) < r \ ; \ r \in I_0\} \ \ \forall \lambda \in I^X, \tag{12}$$

$$\mathcal{G}_{\mathcal{I}}(\lambda) \ = \ \bigwedge\{r \ : \ \mathcal{I}(\lambda) \ge r \ ; \ r \in I_0\} \ \ \forall \lambda \in I^X. \tag{13}$$

If $\mathcal{G}$ is a fuzzy grill on X ([1]), then $\mathcal{I}_{\mathcal{G}}$ is a fuzzy ideal on X generated by $\mathcal{G}$. Also, if $\mathcal{I}$ is a fuzzy ideal on X, then $\mathcal{G}_{\mathcal{I}}$ is a fuzzy grill on X generated by $\mathcal{I}$. This correspondence is given by (12), (13).

That is, studying topological properties in fuzzy ideal approximation spaces will be the same if we studied these properties in fuzzy grill approximation spaces in Šostak sense, and consequently in Chang sense. So, r-fuzzy approximation separation axioms or r-fuzzy approximation connectedness, or r-fuzzy approximation compactness introduced in this paper could be redefined and give us the same results if we replaced $\mathcal{I}$ with the notion of fuzzy grill $\mathcal{G}$.

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