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Certain averaging operators on Triebel-Lizorkin spaces

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Abstract. In this article, we study the boundedness properties of the averaging operator S_t^γ on Triebel-Lizorkin spaces $\dot{F}_{p,q}^{\alpha}(\mathbb{R}^n)$ for various p, q. As an application, we obtain the norm convergence rate for $S_t^\gamma(f)$ on Triebel-Lizorkin spaces and the relation between the smoothness imposed on functions and the rate of norm convergence of S_t^γ is given.

§1 Introduction

For $f \in \mathscr{S}(\mathbb{R}^n)$ and $\gamma > 0$, we consider the spherical mean

$$S_t^\gamma(f)(x) = \frac{\Gamma(\gamma + n/2)}{\pi^{n/2}\Gamma(\gamma)} \int_{|y|<1} \left(1 - |y|^2\right)^{\gamma-1} f(x - ty)dy,$$

where $x \in \mathbb{R}^n$, $t > 0$. For brevity, we denote $S_t^\gamma(f)(x) = S^\gamma(f)(x)$ for $t = 1$. In view of the formula in [13, Theorem 3.3, p. 155] and the identity in [9, p. 576], we see that the Fourier transform of $S_t^\gamma(f)$ is

$$\widehat{S_t^\gamma(f)}(\xi) = m_\gamma(t\xi)\widehat{f}(\xi)$$

with the multiplier

$$m_\gamma(\xi) = \Gamma(\gamma + \frac{n}{2})2^{\frac{n-2}{2}+\gamma} V_{\frac{n-2}{2}+\gamma}(2\pi|\xi|), \tag{1.1}$$

where $V_\nu(u) = \mathcal{J}_\nu(u)u^{-\nu}$, and $\mathcal{J}_\nu(x)$ denotes the Bessel function of order ν (see [13, Appendix B, p. 573]). Since $\mathcal{J}_\nu(u)$ is an analytic function on the domain $\{\nu \in \mathbb{C} : \mathrm{Re}(\nu) > -1/2\}$ for any fixed $u \geq 0$, one may extend $\{S_t^\gamma\}$ to be a family of Fourier multiplier operators with symbols $m_\gamma(t\xi)$ on the region

$$\left\{\gamma \in \mathbb{C} : \mathrm{Re}(\gamma) > -\frac{n-1}{2}\right\}.$$

In this paper, we consider the means S_t^γ for all real γ satisfying $\gamma > -\frac{n-1}{2}$. Thus, $\{S_t^\gamma\}$ is a family of convolution operators with $\gamma > -\frac{n-1}{2}$. It is well-known that $S_t^1(f)$ is the ball average

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of f and $S_t^0(f)$ is the spherical average of f (see Chap XI in [12]). Consider the following Cauchy problem of the wave equation

$$\begin{cases} (\partial_t^2 - \Delta)u(x,t) = 0,\ (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = 0,\ \partial_t u(x,0) = f(x). \end{cases}$$

Its solution $u(x,t)$ is formally given by

$$u(x,t) = c_n t S_t^{\frac{3-n}{2}}(f)(x),$$

where c_n is a constant depending only on n.

This family of operators $\{S_t^\gamma\}$ was extensively studied by many authors in the literature (see e.g. [1, 4, 11]). One can also see [12, Chapter XI] for more information. Particularly, the following theorem can be found in Proposition 4.1 and Remark 4.1 in [7].

Theorem A. ([7, p. 86-88]) *Let $n \geq 2$, $\gamma \geq 0$ and $t > 0$. Then $S_t^\gamma(f)$ is bounded on the real Hardy space $H^p(\mathbb{R}^n)$ and*

$$\lim_{t\to 0^+} \|S_t^\gamma(f) - f\|_{H^p(\mathbb{R}^n)} = 0$$

for $f \in H^p(\mathbb{R}^n)$, provided $p \geq \frac{n-1}{\gamma+n-1}$.

We notice that the Triebel-Lizorkin space $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ is a more general frame of function spaces which takes the space $H^p(\mathbb{R}^n)$ as a special case

$$\dot{F}^0_{p,2}(\mathbb{R}^n) = H^p(\mathbb{R}^n).$$

The first aim of this paper is to extend the known result on $H^p(\mathbb{R}^n)$ by studying the boundedness of S_t^γ on the Triebel-Lizorkin space $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ for different indices p, q. We establish the following result.

Theorem 1.1. *Let $\gamma, \alpha \in \mathbb{R}, \gamma > -\frac{n-1}{2}$ and $0 < p, q < \infty$. One then has the following boundedness properties of the averaging operator S_t^γ on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$:*

(1) Assume $\gamma > 0$.

- *(a) S_t^γ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ if $1 \leq p, q < \infty$;*
- *(b) S_t^γ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ if $\gamma \geq (n-1)(1/p-1)$ for $0 < p < 1 < q < \infty$;*
- *(c) For $0 < q \leq p < 1$, S_t^γ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ if $\gamma > (n-1)(1/p-1)$.*

(2) Let $\gamma = 0$. S_t^0 is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ if $1 \leq p < \infty$ and $1 < q < \infty$.

(3) Let $-\frac{n-1}{2} < \gamma < 0$. For $1 < p, q < \infty$, S_t^γ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ if

$$\gamma > (n-1)\left[\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}\right].$$

Moreover,

- *(a) for $1 < p \leq q \leq 2$, S_t^γ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ if*

$$\gamma \geq (n-1)\left(\frac{1}{p} - 1\right);$$

(b) for $2 \le q \le p < \infty$, S_t^γ is bounded on $\dot F^\alpha_{p,q}(\mathbb{R}^n)$ if

$$\gamma \ge -\frac{(n-1)}{p}.$$

Remark 1.2. *Since $\dot F^0_{p,2}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$, Theorem 1.1 tells that when $n \ge 2$, $\gamma > -\frac{n-1}{2}$ and $t > 0$, $S_t^\gamma(f)$ is bounded on the space $H^p(\mathbb{R}^n)$ if*

$$\gamma \ge (n-1)\left[\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}\right]$$

for all $0 < p < \infty$, which recovers Theorem A. Thus, Theorem 1.1 is a natural extension of Theorem A. Precisely, for $q = 2, \alpha = 0$, (1a) in Theorem 1.1 means that if $\gamma > 0$ then $m_\gamma(\xi)$ is an $L^p(\mathbb{R}^n)$ multiplier for any $p \ge 1$; (1b) in Theorem 1.1 means that if $\gamma > 0$ and $0 < p < 1$ then $m_\gamma(\xi)$ is an $H^p(\mathbb{R}^n)$ multiplier if $\gamma \ge (n-1)(1/p-1)$; (2) in Theorem 1.1 means that if $\gamma = 0$ and $1 \le p < \infty$, then $m_\gamma(\xi)$ is an $L^p(\mathbb{R}^n)$ multiplier; (3) in Theorem 1.1 means that if $-\frac{n-1}{2} < \gamma < 0$ and $1 < p < \infty$, then $m_\gamma(\xi)$ is an $L^p(\mathbb{R}^n)$ multiplier if

$$\gamma \ge (n-1)\left[\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}\right].$$

Collecting all above results, we conclude that Theorem 1.1 recovers Theorem A.

On the other hand, Fan and Zhao [7] studied the convergence rate of $S_t^\gamma(f)$ in the H^p norm and established its relation to the K-functional $K\left(f,\Delta,t^2\right)_{H^p}$. They obtained the following result.

Theorem B. ([7, Proposition 4.2, p. 89]) *Let $n \ge 2$, $\gamma \ge 0$ and $t > 0$. If $\beta \le 2$, then*

$$\|t^{-\beta}\left(S_t^\gamma(f) - f\right)\|_{H^p(\mathbb{R}^n)} \preceq \|I_{-\beta}(f)\|_{H^p(\mathbb{R}^n)},$$

for any $f \in I_\beta(H^p)(\mathbb{R}^n)$ provided $\frac{n-1}{n-1+\beta+\gamma} \le p < \infty$, where $I_\beta(L^p)(\mathbb{R}^n)$ is the L^p-Sobolev space defined below.

The Sobolev type spaces mentioned above were introduced by Strichartz [14, 15] in a more general setting. Let I_β denote the Riesz potential of order β. For any function space or a space of tempered distributions X, one defines the Sobolev space based on X using $I_\beta(X)$, to be the image of X under I_β [15]. By this definition, it is easy to check that $I_\beta(\dot F^\alpha_{p,q})(\mathbb{R}^n) = \dot F^{\alpha+\beta}_{p,q}(\mathbb{R}^n)$ for all $0 < p, q < \infty$.

In order to obtain the convergence rate of $S_t^\gamma(f) - f$ in the space $\dot F^\alpha_{p,q}$, our second aim is to study the boundedness of the operator $t^{-\beta}\left(S_t^\gamma(f) - f\right)$ on the Triebel-Lizorkin space $\dot F^\alpha_{p,q}$ for different p, q. Precisely, we will study the inequality

$$\|t^{-\beta}\left(S_t^\gamma(f) - f\right)\|_{\dot F^\alpha_{p,q}} \preceq \|I_{-\beta}(f)\|_{\dot F^\alpha_{p,q}} \tag{1.2}$$

for $f \in I_\beta(\dot F^\alpha_{p,q})$.

Note that the Fourier transform of $t^{-\beta}\left(S_t^\gamma(f) - f\right)$ is

$$\frac{m_\gamma(t\xi)-1}{|t\xi|^\beta}|\xi|^\beta \widehat{f}(\xi)$$

and write $|\xi|^\beta \widehat{f}(\xi) = \widehat{g}(\xi)$. Then (1.2) is equivalent to the statement that the function

$$\mu_{\gamma,\beta}(\xi) = \frac{m_\gamma(\xi)-1}{|\xi|^\beta}$$

is an $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier.

First, we observe that $\mu_{\gamma,\beta}(\xi)$ is not an $L^p(\mathbb{R}^n)$ multiplier for any p if $\beta > 2$. In fact, a similar calculation of (1.1) and Lemma 3.2 in [6] enable us to get

$$\frac{1-m_\gamma(\xi)}{|\xi|^\beta} \approx \int_0^1 \frac{(1-s^2)^{\frac{n-3}{2}+\gamma}\sin^2(\pi s|\xi|)}{|\xi|^\beta}ds,$$

where and in the following, $A \approx B$ means that there exist positive constants c and C independent of all essential variables such that $c|B| \le |A| \le C|B|$. Thus, the Taylor expansion the sine function yields, for small $|\xi|$,

$$\frac{1-m_\gamma(\xi)}{|\xi|^\beta} \approx |\xi|^{2-\beta}.$$

It says that $\mu_{\gamma,\beta}(\xi)$ is not a bounded function if $\beta > 2$, which implies that it is not an $L^p(\mathbb{R}^n)$ multiplier for any p. For this reason, we will mainly concern with the case $\beta \in (0,2]$ in the rest of this paper.

We establish the following result.

Theorem 1.3. *Let $n \ge 2$, $\alpha,\beta,\gamma \in \mathbb{R}$, $\gamma > -\frac{n-1}{2}$, $0 < \beta \le 2$ and $0 < p,q < \infty$.*

(1) Assume $\gamma > -\beta$.

(a) $\mu_{\gamma,\beta}(\xi)$ is an $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier if $1 \le p,q < \infty$;

(b) $\mu_{\gamma,\beta}(\xi)$ is an $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier if $\gamma \ge (n-1)(1/p-1)-\beta$, where $0 < p < 1 < q < \infty$;

(c) $\mu_{\gamma,\beta}(\xi)$ is an $\dot{F}^{\alpha}_{p,q}(\mathbb{R})$ multiplier if $\gamma > (n-1)(1/p-1)-\beta$ and $0 < q \le p < 1$.

(2) For $\gamma = -\beta$, if $1 \le p < \infty$ and $1 < q < \infty$ then $\mu_{\gamma,\beta}(\xi)$ is an $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier.

(3) Assume $-\frac{n-1}{2} < \gamma < -\beta$. For $1 < p,q < \infty$, $\mu_{\gamma,\beta}(\xi)$ is an $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier if

$$\gamma > (n-1)\left[\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}\right]-\beta.$$

Moreover,

(a) for $1 < p \le q \le 2$, $\mu_{\gamma,\beta}(\xi)$ is an $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier if

$$\gamma \ge (n-1)\left(\frac{1}{p}-1\right)-\beta;$$

(b) for $2 \le q \le p < \infty$, $\mu_{\gamma,\beta}(\xi)$ is a $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier if

$$\gamma \ge -\frac{(n-1)}{p}-\beta.$$

Remark 1.4. *Since $\dot{F}^{0}_{p,2}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$, Theorem 1.3 tells that when $n \ge 2$, $\gamma > -\frac{n-1}{2}$ and $t > 0$ then, for $0 < \beta \le 2$, $\mu_{\gamma,\beta}(\xi)$ is an $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier if*

$$\gamma > (n-1)\left[\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}\right]-\beta,$$

which recovers Theorem B. Thus, Theorem 1.3 is an extension of Theorem B.

As an application of Theorem 1.3, we have the following theorem about the convergence rate, that is

$$\|S_t^{\gamma}(f)-f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}=o(t^{\beta}),\ \text{as } t\to 0 \tag{1.3}$$

for $f\in I_{\beta}(\dot{F}^{\alpha}_{p,q})(\mathbb{R}^n)$.

Theorem 1.5. *Let $n\geq 2$, $\alpha\in\mathbb{R}$, $\gamma>-\frac{n-1}{2}$, $0<\beta<2$, $0<p,q<\infty$ and $f\in I_{\beta}(\dot{F}^{\alpha}_{p,q})(\mathbb{R}^n)$.*

(1) Assume $\gamma>-\beta$.

(a) (1.3) holds for any $1\leq p,q<\infty$;

(b) For $0<p<1<q<\infty$, (1.3) holds if $\gamma\geq(n-1)(1/p-1)-\beta$;

(c) For $0<q\leq p<1$, (1.3) holds if $\gamma>(n-1)(1/p-1)-\beta$.

(2) For $\gamma=-\beta$, (1.3) holds for any $1\leq p<\infty$ and $1<q<\infty$.

(3) Assume $-\frac{n-1}{2}<\gamma<-\beta$ and $1<p,q<\infty$. (1.3) holds if

$$\gamma>(n-1)\left[\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}\right]-\beta.$$

Moreover,

(a) for $1<p\leq q\leq 2$, (1.3) holds if

$$\gamma\geq(n-1)\left(\frac{1}{p}-1\right)-\beta;$$

(b) for $2\leq q\leq p<\infty$, (1.3) holds if

$$\gamma\geq-\frac{(n-1)}{p}-\beta.$$

Before ending up this section, we give an application of Theorem 1.5 to the wave equations. Recall that $u(x,t)=tS_t^{-\frac{n-3}{2}}(f)(x)$ solves the Cauchy problem for the wave equation

$$\begin{cases}(\partial_t^2-\Delta)u(x,t)=0,\ (x,t)\in\mathbb{R}^n\times\mathbb{R},\\ u(x,0)=0,\ \partial_t u(x,0)=f(x).\end{cases}$$

We have the following corollary.

Corollary 1.6. *Let $n\geq 2$, $\gamma>-\frac{n-1}{2}$, $0<\beta<2$ and $0<p<\infty$. If $f\in I_{\beta}(\dot{F}^{\alpha}_{p,q})(\mathbb{R}^n)$ with p satisfying*

$$\left|\frac{1}{p}-\frac{1}{2}\right|\leq\frac{\beta+1}{n-1},$$

then we have

$$\left\|\frac{u(\cdot,t)}{t}-f\right\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}=o(t^{\beta}),\ \text{as } t\to 0^+.$$

This paper is organized as follows. In the second section we will introduce some preliminaries and necessary lemmas that will be used throughout this paper. Then we will prove Theorem 1.1 in Section 3 and Theorem 1.3 in Section 4 respectively. Finally, Theorem 1.5 will be proved in Section 5. Throughout this article, we use the symbol $A\preceq B$ to mean that there exists a constant $C > 0$ independent of all essential variables such that $A \leq CB$. We use the notation $A \approx B$ if $|A| \preceq |B|$ and $|B| \preceq |A|$ and use the symbol $A \simeq B$ to mean that there exists a constant C independent of all essential variables such that $A = CB$. Also, a_k, b_k, c_k, $k = 1, 2, \ldots$, represent some constants that may be different at each of their appearances. For quasi-normed spaces A_1 and A_2, $A_1 \subset A_2$ means that A_1 is continuously embedded in A_2. i.e. there exists a constant c such that $\|a\|_{A_2} \leq c\|a\|_{A_1}$ holds for all $a \in A_1$.

§2 Preliminaries

Let $\Phi : \mathbb{R}^n \to [0, 1]$ be a smooth radial cut-off function, say,

$$\Phi(\xi) = \begin{cases} 1 & |\xi| \leq 1, \\ smooth & 1 < |\xi| < 2, \\ 0 & |\xi| \geq 2. \end{cases}$$

Denote $\varphi(\xi) = \Phi(\xi) - \Phi(2\xi)$, and we introduce the function sequence $\{\varphi_k\}_{k=0}^{\infty}$:

$$\begin{cases} \varphi_k(\xi) = \varphi(2^{-k}\xi), \ k \in \mathbb{N}, \\ \varphi_0(\xi) = 1 - \sum_{k=1}^{\infty} \varphi_k(\xi) = \Phi(\xi). \end{cases}$$

Since $\mathrm{supp}(\varphi) \subset \{\xi : 2^{-1} \leq |\xi| \leq 2\}$, we easily see that $\mathrm{supp}(\varphi_k) \subset \{\xi : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$, $k \in \mathbb{N}$, and $\mathrm{supp}(\varphi_0) \subset \{\xi : |\xi| \leq 2\}$. Let

$$\Psi(\xi) = 1 - \Phi(\xi) = \sum_{k=1}^{\infty} \varphi_k(\xi).$$

Ψ is a nonnegative and radial Schwartz function supported in the set $\{\xi \in \mathbb{R}^n : |\xi| > 1\}$, and equals to 1 on the smaller set $\{\xi \in \mathbb{R}^n : |\xi| \geq 2\}$. Define

$$\Delta_k = \mathscr{F}^{-1}\varphi_k\mathscr{F}, \quad k \in \mathbb{Z},$$

where $\{\Delta_k\}_{k=-\infty}^{\infty}$ is the Littlewood-Paley (or dyadic) decomposition operator. Let

$$\dot{\mathscr{S}}(\mathbb{R}^n) = \{\psi | \psi \in \mathscr{S}(\mathbb{R}^n) : \partial^\alpha(\hat{\psi})(0) = 0 \text{ for every multi-index } \alpha\},$$

which is equivalent to

$$\int_{\mathbb{R}^n} x^\alpha \psi(x) dx = 0$$

for every multi-index α. $\dot{\mathscr{S}}(\mathbb{R}^n)$ is the subspace of $\mathscr{S}(\mathbb{R}^n)$ that inherits the same topology as $\mathscr{S}(\mathbb{R}^n)$ and the dual space of $\dot{\mathscr{S}}(\mathbb{R}^n)$ under the topology inherited from $\mathscr{S}(\mathbb{R}^n)$ is

$$\dot{\mathscr{S}}'(\mathbb{R}^n) = \mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n),$$

where $\mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ denote the space of tempered distributions modulo polynomials.

For $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, the Triebel-Lizorkin space $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ is the set of all f in $\dot{\mathscr{S}}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} = \left\| \left(\sum_{j \in \mathbb{Z}} (2^{j\alpha}|\Delta_j f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty. \tag{2.1}$$

It is well-known that $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space if $-\infty < \alpha < \infty$, $0 < p < \infty$, $0 < q \leq \infty$ and that the function φ in the above definition is flexible in the sense that any two different functions φ give the equivalent norms (2.1). By this definition, $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) \simeq I_\alpha(\dot{F}^0_{p,q})(\mathbb{R}^n)$, $\dot{F}^0_{p,2}(\mathbb{R}^n) \simeq H^p(\mathbb{R}^n)$ and $\dot{F}^{\alpha}_{p,2}(\mathbb{R}^n) \simeq I_\alpha(H^p)(\mathbb{R}^n)$ for $0 < p, q < \infty$. Furthermore,

$$\dot{\mathscr{S}}(\mathbb{R}^n) \subset \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) \subset \dot{\mathscr{S}}'(\mathbb{R}^n)$$

for $-\infty < \alpha < \infty$, $0 < p < \infty$ and $0 < q \le \infty$. If $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, then $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ is complete, $\dot{\mathscr{S}}(\mathbb{R}^n)$ is dense in $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, and

$$\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \approx \left\| \left[\int_0^\infty \left(s^{-\alpha} \left| (f * \varphi_s)(\cdot) \right| \right)^q \frac{ds}{s} \right]^{1/q} \right\|_p. \tag{2.2}$$

For brevity, we denote by $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) = \dot{F}^{\alpha}_{p,q}$.

The Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ has the following imbedding and lifting properties.

Lemma 2.1 (Imbedding). *([16]). The space $\dot{F}^{\alpha}_{p,q}$ has the imbedding relationship:*

(1) For $\alpha \in \mathbb{R}$, $0 < p < \infty$. if $q_1 \le q_2$, then $\dot{F}^{\alpha}_{p,q_1} \subset \dot{F}^{\alpha}_{p,q_2}$.

(2) Given reals $-\infty < \alpha_2 < \alpha_1 < \infty$ and $0 < p_1 < \infty$, $0 < q_1, q_2 \le \infty$, let $0 < p_2 \le \infty$ be determined by $\alpha_1 - \frac{n}{p_1} = \alpha_2 - \frac{n}{p_2}$. Then

$$\dot{F}^{\alpha_1}_{p_1,q_1} \subset \dot{F}^{\alpha_2}_{p_2,q_2}.$$

Lemma 2.2 (Lifting). *(see [2, p. 2073]). The space $\dot{F}^{\alpha}_{p,q}$ has the lifting property:*

$$\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \simeq \|I_{-\alpha}(f)\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},$$

where $(I_{-\alpha}f)^\wedge(\xi) = |\xi|^\alpha \widehat{f}(\xi)$.

By the lifting property, as is pointed out in [2], to prove that a convolution operator T is bounded on the space $\dot{F}^{\alpha}_{p,q}$, it suffices to show its boundedness on $\dot{F}^0_{p,q}$.

Let T_μ be a convolution operator and $\widehat{T_\mu(f)}(\xi) = \mu(\xi)\widehat{f}(\xi)$, where μ is called the multiplier of T_μ. If T_μ is a bounded operator on the Triebel-Lizorkin Space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, then we say that μ is an $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier and denote by

$$\|\mu(\cdot)\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) \to \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$$

the operator norm of T_μ on the space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$. By a scaling argument, it is not difficult to show the following lemma.

Lemma 2.3. *Let $0 < p < \infty$ and let μ be an $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier. Then for any $t > 0$,*

$$\|\mu(\cdot)\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) \to \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} = \|\mu(t\cdot)\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) \to \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}.$$

The following $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ multiplier theorem will be frequently used in this article.

Lemma 2.4 (Hörmander multiplier theorem). *([5, Theorem 5.1, pp. 851]) Let $\alpha, \beta \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. Suppose that ℓ is a nonnegative integer and $\mu \in C^\ell(\mathbb{R}^n \setminus \{0\})$ satisfies the condition*

$$\sup_{R>0} \left(R^{-n+2\alpha+2|\sigma|} \int_{R<|\xi|<2R} \left| \partial_\xi^\sigma \mu(\xi) \right|^2 d\xi \right) \le A_\sigma, \quad |\sigma| \le \ell \tag{2.3}$$

with $\ell > \max\{n/p, n/q\} + n/2$. Then

$$\|T_\mu(f)\|_{\dot{F}^{\alpha+\beta}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\beta}_{p,q}(\mathbb{R}^n)}.$$

When $\alpha = 0$, (2.3) is known as the Hörmander condition (see [9, Theorem 6.2.7, p. 446]).

The Bessel function $\mathcal{J}_\nu(r)$ has the following asymptotic expansion as $r \to \infty$.

Lemma 2.5. *(see [7, Proposition 5.1, p. 93]). Let $r > 0$ and $\nu > -1/2$. For any positive integer L, on the interval $[1,\infty)$ we have*

$$\mathcal{J}_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \sum_{j=1}^{L} a_j e^{ir} r^{-\frac{1}{2}-j} + \sum_{j=1}^{L} b_j e^{-ir} r^{-\frac{1}{2}-j} + E_L(r),$$

where a_j and b_j are constants for all natural numbers j, and $E_L(r)$ is a C^∞ function satisfying

$$E_L^{(k)}(r) = O\left(r^{-\frac{1}{2}-L-1}\right), \quad as \quad r \to \infty,$$

for all nonnegative integer k.

Let Ψ be the function defined as in the definition of $\dot{F}^\alpha_{p,q}$. W_ν is a wave operator if it is a Fourier multiplier operator with symbol $\Psi(\xi)e^{ic|\xi|}|\xi|^{-\nu}$ for a fixed nonzero constant c. One has the following estimate on W_ν.

Lemma 2.6. *(see [3, p. 760]) Let $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$. One has the following estimates for wave operators.*

(1) For $1 < p \le q \le 2$ or $2 \le q \le p < \infty$, $W_\nu(f)$ is bounded on the Triebel-Lizorkin space $\dot{F}^\alpha_{p,q}$ if

$$\alpha \ge (n-1)\left|\frac{1}{2} - \frac{1}{p}\right|.$$

(2) For $1 < p, q < \infty$ or $0 < q \le p \le 1$, $W_\nu(f)$ is bounded on the Triebel-Lizorkin Space $\dot{F}^\alpha_{p,q}$ if

$$\alpha > (n-1)\left|\frac{1}{2} - \frac{1}{p}\right|.$$

(3) For $0 < p \le 1 < q < \infty$, $W_\nu(f)$ is bounded on the Triebel-Lizorkin Space $\dot{F}^\alpha_{p,q}$ if

$$\alpha \ge (n-1)\left(\frac{1}{p} - \frac{1}{2}\right).$$

§3 Proof of Theorem 1.1

By Lemma 2.2, to prove S^γ_t is bounded on $\dot{F}^\alpha_{p,q}$, we only need to show its boundedness on $\dot{F}^0_{p,q}$. Without loss of generality, we may assume $t = 1$ by the scaling argument in Lemma 2.3.

First, when $\dot{F}^0_{p,q}$ is a normed space (i.e. $1 \le p, q < \infty$), we observe that Theorem 1.1 is trivially true for the case $\gamma > 0$. In fact, by the Minkowski integral inequality, it is easy to obtain that

$$\begin{aligned}
\|S^\gamma_1(f)(x)\|_{\dot{F}^0_{p,q}} &\le \frac{\Gamma(\gamma + n/2)}{\pi^{n/2}\Gamma(\gamma)} \int_{|y|<1} \left(1 - |y|^2\right)^{\gamma-1} \|f(\cdot - y)\|_{\dot{F}^0_{p,q}} dy \\
&= \left(\frac{\Gamma(\gamma + n/2)}{\pi^{n/2}\Gamma(\gamma)} \int_{|y|<1} \left(1 - |y|^2\right)^{\gamma-1} dy\right) \|f\|_{\dot{F}^0_{p,q}} \\
&\preceq \|f\|_{\dot{F}^0_{p,q}},
\end{aligned}$$

since $\int_{|y|<1}\left(1-|y|^2\right)^{\gamma-1}dy<\infty$ if $\gamma>0$. By this observation, in the following we mainly concern with the case either $\gamma\le 0$ or the case that $\dot F^0_{p,q}$ is not a normed space. More precisely, our purpose is to find the optimal relation between γ and p,q to ensure that $m_\gamma(\xi)$ is an $\dot F^0_{p,q}$ multiplier in the case when $\gamma\le 0$ or in the case that $\dot F^0_{p,q}$ is not a normed space.

If either $\gamma\le 0$ or $\dot F^0_{p,q}$ is not a normed space, then we let Ψ and Φ be the same as above, that is to say Ψ and Ψ are $C^\infty(\mathbb{R}^n)$ radial functions with $\Phi(\xi)=1-\Psi(\xi)$ and Ψ satisfying $\Psi(\xi)\equiv 0$ if $|\xi|\le 1$, $\Psi(\xi)\equiv 1$ if $|\xi|\ge 2$. We write

$$\begin{aligned}m_\gamma(|\xi|)&=\Gamma(\gamma+\frac{n}{2})2^{\frac{n-2}{2}+\gamma}V_{\frac{n-2}{2}+\gamma}(2\pi|\xi|)\\&=\Gamma(\gamma+\frac{n}{2})2^{\frac{n-2}{2}+\gamma}\left(V_{\frac{n-2}{2}+\gamma}(|\xi|)\Phi(\xi)+V_{\frac{n-2}{2}+\gamma}(|\xi|)\Psi(\xi)\right).\end{aligned}$$

Using the derivative formula for the Bessel function

$$\frac{dV_\nu(t)}{dt}=-tV_{\nu+1}(t), \tag{3.1}$$

and the well-known formula

$$|V_\nu(|\xi|)|\preceq 1 \quad\text{if } |\xi|\le 1,$$

by the condition of Lemma 2.4, we can easily see that $V_{\frac{n-2}{2}+\gamma}(|\xi|)\Phi(\xi)$ is an $\dot F^0_{p,q}$ multiplier for any $p,q>0$. By Lemma 2.5, the second multiplier can be written as

$$\begin{aligned}&V_{\frac{n-2}{2}+\gamma}(|\xi|)\Psi(\xi)\\&=\sum_{j=0}^{L}a_j\Psi(\xi)e^{i|\xi|}|\xi|^{-\frac{n-1}{2}-\gamma-j}+\sum_{j=0}^{L}b_j\Psi(\xi)e^{-i|\xi|}|\xi|^{-\frac{n-1}{2}-\gamma-j}+\tilde E_L(\xi)\Psi(\xi),\end{aligned}\tag{3.2}$$

where $\tilde E_L(\xi)$ is a C^∞ function satisfying

$$\left|\partial_\xi^\sigma\tilde E_L(\xi)\right|\preceq|\xi|^{-(\frac{n+1}{2}+L+\gamma)},\ \text{ whenever }\ |\xi|>1$$

for any multi-index σ. Noting $\Psi(\xi)=0$ if $|\xi|\le 1$, we choose a suitably large L such that

$$\left|\partial_\xi^\sigma\left(\tilde E_L\Psi\right)(\xi)\right|\le C_\sigma|\xi|^{-|\sigma|}$$

for any multi-index σ, which satisfying (2.3) with $\alpha=0$. Invoking Lemma 2.4, it follows that $\tilde E_L(\xi)\Psi(\xi)$ is an $\dot F^0_{p,q}$ multiplier for any $p,q>0$. Furthermore, we know that, for each j,

$$m_j^+(\xi)=\Psi(\xi)e^{i|\xi|}|\xi|^{-\frac{n-1}{2}-\gamma-j}\ \text{ or }\ m_j^-(\xi)=\Psi(\xi)e^{-i|\xi|}|\xi|^{-\frac{n-1}{2}-\gamma-j}$$

is a multiplier of the wave operator W_ν with $\nu=\frac{n-1}{2}+\gamma+j$. By Lemma 2.6, we know that in the expression

$$V_{\frac{n-2}{2}+\gamma}(|\xi|)\Psi(\xi)=\sum_{j=0}^{L}a_jm_j^+(\xi)+\sum_{j=0}^{L}b_jm_j^-(\xi)+E_L(\xi)\Psi(\xi),$$

$m_0^+(\xi)$ and $m_0^-(\xi)$ are $\dot F^0_{p,q}$ multipliers for any $p,q>0$ satisfying the condition in Theorem 1.1. Also, it is easy to see that if $\Psi(\xi)e^{\pm i|\xi|}|\xi|^{-\nu}$ is an $\dot F^0_{p,q}$ multiplier, then $\Psi(\xi)e^{\pm i|\xi|}|\xi|^{-\nu-\varepsilon}$, for any positive ε, is also an $\dot F^0_{p,q}$ multiplier for the same p and q. Thus, all $m_j^+(\xi)$ and $m_j^-(\xi)$, $j=1,2,\dots,L$, are $\dot F^0_{p,q}$ multipliers for $p,q>0$ satisfying the condition in Theorem 1.1. As a consequence, we obtain that, for $\gamma\le 0$ or $\dot F^0_{p,q}$ is not a normed space, $V_{\frac{n-2}{2}+\gamma}(|\xi|)\Psi(\xi)$ is an $\dot F^0_{p,q}$ multiplier for $0<p,q<\infty$ satisfying the condition in Theorem 1.1. The proof of Theorem 1.1 is completed.

§4 Proof of Theorem 1.3

In this section, we devote to prove Theorem 1.3. Also, in this section and the rest of the paper, we always assume $\beta \in (0,2]$.

By Lemma 2.2, to prove Theorem 1.3, it suffices to show that $\mu_{\gamma,\beta}(\xi)$ is an $\dot{F}^0_{p,q}$ multiplier under the assumption in the theorem.

Recall that Φ is a radial $C^\infty(\mathbb{R}^n)$ function satisfying $\Phi(\xi) \equiv 1$ if $|\xi| \le 1$ and supp $\Phi \subset \{\xi : |\xi| \le 2\}$ and $\Psi(\xi) = 1 - \Phi(\xi)$, we decompose

$$\mu_{\gamma,\beta}(\xi) = \mu_{\gamma,\beta,1}(\xi) + \mu_{\gamma,\beta,2}(\xi),$$

where

$$\mu_{\gamma,\beta,1}(\xi) = \Phi(\xi)\mu_{\gamma,\beta}(\xi), \quad \mu_{\gamma,\beta,2}(\xi) = \Psi(\xi)\mu_{\gamma,\beta}(\xi).$$

Let

$$m_{\gamma,\beta}(\xi) = m_\gamma(\xi)|\xi|^{-\beta}\Psi(\xi), \tag{4.1}$$

where m_γ is the multiplier of S_1^γ. We may write

$$\mu_{\gamma,\beta,2}(\xi) = m_{\gamma,\beta}(\xi) - \Psi(\xi)|\xi|^{-\beta}.$$

The following lemma will play a crucial role in the proof of Theorem 1.3.

Lemma 4.1. *For $\beta \in (0,2]$ and $0 < p, q < \infty$, $\mu_{\gamma,\beta}$ is an $\dot{F}^0_{p,q}$ multiplier if and only if $m_{\gamma,\beta}$ is an $\dot{F}^0_{p,q}$ multiplier.*

Proof of Lemma 4.1. Using Lemma 2.4, it is easy to see that $\Psi(\xi)|\xi|^{-\beta}$ is an $\dot{F}^0_{p,q}$ multiplier for any $p, q > 0$. A direct calculation shows that

$$\mu_{\gamma,\beta,1}(\xi) = \Phi(\xi)|\xi|^{-\beta}\left(\int_0^1 (1-s^2)^{\frac{n-3}{2}+\gamma}\sin^2(\pi s|\xi|)ds\right). \tag{4.2}$$

By Taylor's expansion

$$\sin(\pi|\xi|s) = \sum_{k=0}^N c_k(|\xi|s)^{2k+1} + \Theta(|\xi|s),$$

where, in the support of Φ, $\Theta(t)$ is a C^∞ function satisfying that, for $k \le 2N+3$,

$$\Theta^{(k)}(t) = O(t^{2N+3-k}) \quad \text{as } t \to 0.$$

Thus, invoking (4.2), we obtain

$$\mu_{\gamma,\beta,1}(\xi) = \Phi(\xi)|\xi|^{-\beta}\left(\int_0^1 (1-s^2)^{\frac{n-3}{2}+\gamma}\left\{\sum_{k=0}^N c_k(|\xi|s)^{2k+1} + \Theta(|\xi|s)\right\}^2 ds\right). \tag{4.3}$$

To obtain the estimate on $\partial_\xi^\sigma(\mu_{\gamma,\beta,1})(\xi)$, by choosing a large N in (4.3), it suffices to work with each term

$$\mu^k(\xi) := \Phi(\xi)|\xi|^{2(2k+1)-\beta}\left(\int_0^1 (1-s^2)^{\frac{n-3}{2}+\gamma}s^{2(2k+1)}ds\right).$$

We may write

$$\mu^k(\xi) \simeq \Phi(\xi)|\xi|^{2(2k+1)-\beta},$$

since

$$\int_0^1 (1-s^2)^{\frac{n-3}{2}+\gamma}s^{2(2k+1)}ds = B\left(2k+\frac{3}{2}, \frac{n-1}{2}+\gamma\right) < \infty$$

for any $\gamma > -\frac{n-1}{2}$. Noting that $\beta \in (0,2]$, $k \in \mathbb{N}\bigcup\{0\}$, we have $2(2k+1) - \beta \ge 4k \ge 0$. It is easy to check that

$$\left|\partial_{\xi}^{\sigma}\left(\mu^{k}(\xi)\right)\right| \simeq \left|\partial_{\xi}^{\sigma}\left(\Phi(\xi)|\xi|^{2(2k+1)-\beta}\right)\right| \leq C_{\sigma}\,|\xi|^{-|\sigma|}$$

for any multi-index σ satisfying (2.3) with $\alpha = 0$. Invoking Lemma 2.4, it follows that μ^k is an $\dot{F}^0_{p,q}$ multiplier for any $p, q > 0$. Combining (4.2) with (4.3), we obtain that $\mu_{\gamma,\beta,1}$ is an $\dot{F}^0_{p,q}$ multiplier for any $p, q > 0$. On the other hand, we have

$$\mu_{\gamma,\beta}(\xi) = \mu_{\gamma,\beta,1}(\xi) + m_{\gamma,\beta}(\xi) - \Psi(\xi)|\xi|^{-\beta}.$$

Hence, $\mu_{\gamma,\beta}$ is an $\dot{F}^0_{p,q}$ multiplier if and only if $m_{\gamma,\beta}$ is an $\dot{F}^0_{p,q}$ multiplier, as desired. □

By Lemma 4.1, to prove Theorem 1.3, it suffices to consider the multiplier $m_{\gamma,\beta}$. We need the following proposition to complete the proof.

Proposition 4.2. *Let $n \geq 2$, $\gamma > -\frac{n-1}{2}$ and $0 \leq \beta \leq 2$. If $\gamma + \beta > 0$ and $\dot{F}^0_{p,q}$ is a normed space, then $m_{\gamma,\beta}$ is an $\dot{F}^0_{p,q}$ multiplier.*

We postpone the proof for Proposition 4.2 to the end of this section. First, let us describe how to conclude the proof of Theorem 1.3 by virtue of the proposition.

From Proposition 4.2, to prove Theorem 1.3, we only need to concern with the case either $\gamma + \beta \leq 0$ or the case that $\dot{F}^0_{p,q}$ is not a normed space. More precisely, we aim to find some relation between γ and p, q to ensure that $m_{\gamma,\beta}(\xi)$ is an $\dot{F}^0_{p,q}$ multiplier for $\gamma + \beta \leq 0$ or in the case that $\dot{F}^0_{p,q}$ is not a normed space.

Assume that $\gamma + \beta \leq 0$ or $\dot{F}^0_{p,q}$ is not a normed space. Using Lemma 2.5, we write

$$\begin{aligned} m_{\gamma,\beta}(\xi) &= m_{\gamma}(\xi)\,|\xi|^{-\beta}\,\Psi(\xi) \simeq |\xi|^{-\frac{n-2}{2}-\gamma-\beta}\mathcal{J}_{\frac{n-2}{2}+\gamma}(2\pi|\xi|)\Psi(|\xi|) \\ &\simeq \frac{\cos\left(2\pi|\xi| - \frac{\gamma\pi}{2} - \frac{(n-1)\pi}{4}\right)\Psi(|\xi|)}{|\xi|^{\frac{n-1}{2}+\gamma+\beta}} + \sum_{j=1}^{L}\frac{a_j e^{i2\pi|\xi|}\Psi(|\xi|)}{|\xi|^{\frac{n-1}{2}+\gamma+\beta+j}} \\ &\quad + \sum_{j=1}^{L}\frac{b_j e^{-i2\pi|\xi|}\Psi(|\xi|)}{|\xi|^{\frac{n-1}{2}+\gamma+\beta+j}} + \frac{E_L(2\pi|\xi|)\Psi(|\xi|)}{|\xi|^{\frac{n-1}{2}+\gamma+\beta}}. \end{aligned} \tag{4.4}$$

Here we choose a positive L such that the error term $E_L(2\pi|\xi|)\,\Psi(|\xi|)\,|\xi|^{-\frac{n-1}{2}-\gamma-\beta}$ satisfying

$$\left|\partial_{\xi}^{\sigma}\left(\frac{E_L(2\pi|\xi|)\Psi(|\xi|)}{|\xi|^{\frac{n-1}{2}+\gamma+\beta}}\right)\right| \leq C_{\sigma}\,|\xi|^{-|\sigma|}$$

for any multi-index σ, which satisfies (2.3) with $\alpha = 0$. Invoking Lemma 2.4, it follows that $E_L(2\pi|\xi|)\,\Psi(|\xi|)\,|\xi|^{-\frac{n-1}{2}-\gamma-\beta}$ is an $\dot{F}^0_{p,q}$ multiplier for any $p, q > 0$. So, we need to study the rest terms in (4.4). This procedure can be done by employing the same proof as that for Theorem 1.1 in Section 3. Thus, Theorem 1.3 is proved. □

Finally, it remains to prove Proposition 4.2.

Proof of Proposition 4.2.

Let $T_{m_{\gamma,\beta}}$ be the convolution operator $K_{\gamma,\beta} * f$, where $K_{\gamma,\beta}$ is the function defined as

$$K_{\gamma,\beta}(x) \simeq \int_{\mathbb{R}^n} m_{\gamma,\beta}(\xi)\,e^{i2\pi\xi\cdot x}d\xi. \tag{4.5}$$

To prove Proposition 4.2, it is equivalent to prove $T_{m_{\gamma,\beta}}$ is bounded on $\dot{F}^0_{p,q}$ for $\gamma + \beta > 0$ and

$p, q \geq 1$. By (4.4), we know that $T_{m_{\gamma,\beta}}$ is bounded on $\dot{F}^0_{p,q}$ if and only if

$$\upsilon(\xi) = \frac{\cos\left(2\pi|\xi| - \frac{\gamma\pi}{2} - \frac{(n-1)\pi}{4}\right)\Psi(|\xi|)}{|\xi|^{\frac{n-1}{2}+\gamma+\beta}}$$

is an $\dot{F}^0_{p,q}$ multiplier.

• *Case 1.* $\gamma + \beta > 1$. Using Theorem 3.3 of Chapter 4 in [13], the inverse Fourier transform of υ is given by

$$\begin{aligned}\upsilon^{\vee}(x) &= \int_{\mathbb{R}^n} \frac{\Psi(|\xi|)\cos\left(2\pi|\xi| - \frac{\gamma\pi}{2} - \frac{n-1}{4}\pi\right)}{|\xi|^{\frac{n-1}{2}+\gamma+\beta}} e^{2\pi i x\cdot\xi} d\xi \\ &= \int_0^\infty \frac{\Psi(t)\cos\left(2\pi t - \frac{\gamma\pi}{2} - \frac{n-1}{4}\pi\right)}{t^{\frac{n-1}{2}+\gamma+\beta}} V_{\frac{n-2}{2}}(t|x|)dt.\end{aligned}$$

We want to show that $\upsilon^{\vee}(x)$ is an integrable function. To this end, from Lemma 2.5, it suffices to estimate its leading term

$$L_\upsilon(x) = \frac{1}{|x|^{\frac{n-1}{2}}}\int_0^\infty \frac{\Psi(t)}{t^{\gamma+\beta}}\cos\left(2\pi t - \frac{\gamma\pi}{2} - \frac{n-1}{4}\pi\right)\cos\left(2\pi t|x| - \frac{n-1}{4}\pi\right)dt.$$

If $|x| < 1$ then we have

$$|L_\upsilon(x)| \preceq \frac{1}{|x|^{\frac{n-1}{2}}}\left|\int_1^\infty \frac{\Psi(t)}{t^{\gamma+\beta}}dt\right| \preceq \frac{1}{|x|^{\frac{n-1}{2}}}. \tag{4.6}$$

If $|x| \geq 1$ we keep using integration by parts N times (N sufficiently large, say $N > \left[\frac{n+1}{2}\right]+1$,). It is easy to get

$$\begin{aligned}&|L_\upsilon(x)| \simeq \\ &= \frac{1}{|x|^{\frac{n-1}{2}+1}}\left|\int_0^\infty \sin\left(2\pi t|x| - \frac{n-1}{4}\pi\right)\frac{d}{dt}\left(\frac{\Psi(t)\cos\left(2\pi t - \frac{\gamma\pi}{2} - \frac{n-1}{4}\pi\right)}{t^{\gamma+\beta}}\right)dt\right| \\ &\preceq \frac{1}{|x|^{\frac{n-1}{2}+N}}.\end{aligned} \tag{4.7}$$

(4.6) and (4.7) conclude that, if $\gamma+\beta > 1$, $\upsilon^{\vee}(x) \in L^1$. Thus, if $\dot{F}^0_{p,q}$ is a normed space, by the Minkowski integral inequality, it is easy to obtain that $\upsilon(\xi)$ is an $\dot{F}^0_{p,q}$ multiplier.

• *Case 2.* $0 < \gamma + \beta \leq 1$. In this case, we need an auxiliary lemma, and its proof can be found in [8, p. 171].

Lemma 4.3. *(see page 171 in [8]). Let* $\varepsilon > 0$ *and* $\gamma \neq -1$. *Then*

$$\int_0^\infty e^{-\varepsilon r} r^\gamma e^{isr} dr = ie^{i\frac{\gamma\pi}{2}}\Gamma(\gamma+1)(s+i\varepsilon)^{-\gamma-1},$$

where $\Gamma(\gamma+1)$ *is the Gamma function.*

For $\varepsilon > 0$, the Abel mean $G_\varepsilon(f)$ of f is

$$\widehat{G_\varepsilon(f)}(\xi) = e^{-\varepsilon|\xi|}\widehat{f}(\xi).$$

The kernel of G_ε is an $L^1(\mathbb{R}^n)$ function and its $L^1(\mathbb{R}^n)$ norm is independent of $\varepsilon > 0$ (see [13, p. 10]). Thus, the Minkowski integral inequality yields that for $1 \leq p, q \leq \infty$

$$\|G_\varepsilon(f)\|_{\dot{F}^0_{p,q}} \preceq \|f\|_{\dot{F}^0_{p,q}}, \tag{4.8}$$

and

$$\lim_{\varepsilon\to 0+} \|G_\varepsilon(f) - f\|_{\dot{F}^0_{p,q}} = 0.$$

for all $f \in \dot{F}^0_{p,q}$. To show that $\upsilon(\xi)$ is an $\dot{F}^0_{p,q}$ multiplier for $0 < \gamma + \beta \le 1$ and $p, q \ge 1$, it suffices to show that $e^{-\varepsilon|\xi|}\upsilon(\xi)$ is an $\dot{F}^0_{p,q}$ multiplier uniformly on $\varepsilon > 0$. To this end, we need to prove that the kernel

$$\Re_\varepsilon(x) = \int_{\mathbb{R}^n} \frac{\cos\left(2\pi|\xi| - \frac{\gamma\pi}{2} - \frac{(n-1)\pi}{4}\right)\Psi(|\xi|)\, e^{-\varepsilon|\xi|}}{|\xi|^{\frac{n-1}{2}+\gamma+\beta}} e^{2\pi i x\cdot\xi} d\xi$$

is an integrable function for $0 < \gamma + \beta \le 1$ and its $L^1(\mathbb{R}^n)$ norm is independent of $\varepsilon > 0$. In fact, suppose this is true, since $\dot{F}^0_{p,q}$ is a normed space, by the Minkowski integral inequality we have that

$$\begin{aligned}\|\Re_\varepsilon * f\|_{\dot{F}^0_{p,q}} &\le \int_{\mathbb{R}^n} |\Re_\varepsilon(y)|\, \|f(\cdot - y)\|_{\dot{F}^0_{p,q}} dy \\ &= \left(\int_{\mathbb{R}^n} |\Re_\varepsilon(y)|\, dy\right) \|f\|_{\dot{F}^0_{p,q}} \\ &\preceq \|f\|_{\dot{F}^0_{p,q}}.\end{aligned}$$

Combining this with the fact that for $p, q \ge 1$

$$\|T_\upsilon(f)\|_{\dot{F}^0_{p,q}} \le \lim_{\varepsilon\to 0+} \|\Re_\varepsilon * f\|_{\dot{F}^0_{p,q}},$$

we conclude that T_υ is bounded on $\dot{F}^0_{p,q}$, that is, $\upsilon(\xi)$ is an $\dot{F}^0_{p,q}$ multiplier for $0 < \gamma + \beta \le 1$ and $p, q \ge 1$.

Hence, to complete the proof of Proposition 4.2, it only remains to prove that $\Re_\varepsilon(x)$ is an integrable function for $0 < \gamma + \beta \le 1$. The method is analogous to that in the proof of integrability of $\Re_\varepsilon(x)$ for the case $\gamma + \beta = 0$ and $\beta = 2$ in [17]. For the sake of completeness, we present its proof below.

By using the formula in [13, Theorem 3.3, p. 155], a computation of the Fourier transform yields

$$\begin{aligned}\Re_\varepsilon(x) &= \int_{\mathbb{R}^n} \frac{\Psi(|\xi|)\cos\left(2\pi|\xi| - \frac{\gamma\pi}{2} - \frac{n-1}{4}\pi\right)}{|\xi|^{\frac{n-1}{2}+\gamma+\beta}} e^{-\varepsilon|\xi|} e^{2\pi i x\cdot\xi} d\xi \\ &= \int_0^\infty \frac{\Psi(t)\, e^{-\varepsilon t}\cos\left(2\pi t - \frac{\gamma\pi}{2} - \frac{n-1}{4}\pi\right)}{t^{\frac{n-1}{2}+\gamma+\beta}} V_{\frac{n-2}{2}}(t|x|) dt.\end{aligned} \tag{4.9}$$

From Lemma 2.5, since the estimates of all terms in the above expansion of $\Re_\varepsilon$ are the same, it suffices to estimate the leading term

$$\begin{aligned}&\mathcal{L}_\varepsilon(x) \\ &= \frac{1}{|x|^{\frac{n-1}{2}}} \int_0^\infty \frac{\Psi(t)\, e^{-\varepsilon t}}{t^{\gamma+\beta}} \cos\left(2\pi t - \frac{\gamma\pi}{2} - \frac{n-1}{4}\pi\right) \cos\left(2\pi t|x| - \frac{n-1}{4}\pi\right) dt \\ &\simeq \frac{1}{|x|^{\frac{n-1}{2}}} \int_0^\infty \frac{\Psi(t)\, e^{-\varepsilon t}}{t^{\gamma+\beta}} (a_1 e^{2\pi i t(1+|x|)} + a_2 e^{-2\pi i t(1+|x|)} \\ &\quad + a_3 e^{2\pi i t(1-|x|)} + a_4 e^{-2\pi i t(1-|x|)}) dt.\end{aligned} \tag{4.10}$$

For $|x| < 1/2$, we have

$$|\mathcal{L}_\varepsilon(x)| \preceq \frac{1}{|x|^{\frac{n-1}{2}}} \left| \int_1^\infty \frac{\Psi(t)}{t^{\gamma+\beta}} dt \right| \preceq \frac{1}{|x|^{\frac{n-1}{2}}} \in L^1. \tag{4.11}$$

For $|x| > 2$, integration by parts for n times yields

$$|\mathcal{L}_\varepsilon(x)| \preceq \frac{(|x|-1)^{-n}}{|x|^{\frac{n-1}{2}}} \preceq |x|^{-n-1} \in L^1. \tag{4.12}$$

For $1/2 \le |x| \le 2$, if $\gamma + \beta = 1$, invoking (4.10), to estimate $\mathcal{L}_\varepsilon(x)$, we only need to deal with the term

$$\frac{a_3}{|x|^{\frac{n-1}{2}}} \int_0^\infty \frac{\Psi(t)\, e^{-\varepsilon t}}{t} e^{2\pi i t(1-|x|)} dt,$$

since the estimate of other terms are similar. Using integration by parts we now have

$$\int_0^\infty \frac{e^{it[(1-|x|)+i\varepsilon]}}{t} \Psi(t)\, dt = \frac{i}{[(1-|x|)+i\varepsilon]} \int_0^\infty e^{it[(1-|x|)+i\varepsilon]} \frac{d}{dt}\left(\frac{\Psi(t)}{t}\right) dt.$$

By the definition of $\Psi(t)$, it is easy to see

$$\left| \int_0^\infty \frac{e^{it[(1-|x|)+i\varepsilon]}}{t} \Psi(t)\, dt \right| \preceq \frac{1}{|(1-|x|)+i\varepsilon|} \preceq \frac{1}{|1-|x||}.$$

Hence, if $\gamma + \beta = 1$, then $|\mathcal{L}_\varepsilon(x)| \preceq \frac{1}{|x|^{\frac{n-1}{2}}} \frac{1}{|1-|x||}$, which is integrable on $\mathbb{R}^n$ for $1/2 \le |x| \le 2$.

On the other hand, if $0 < \gamma + \beta < 1$, for $1/2 \le |x| \le 2$, we write

$$\begin{aligned} &\frac{1}{|x|^{\frac{n-1}{2}}} \int_0^\infty \frac{\Psi(t)\, e^{-\varepsilon t}}{t^{\gamma+\beta}} (a_3 e^{2\pi i t(1-|x|)} + a_4 e^{-2\pi i t(1-|x|)}) dt \\ \simeq &\int_0^\infty \frac{e^{-\varepsilon t}}{t^{\gamma+\beta}} (a_3 e^{2\pi i t(1-|x|)} + a_4 e^{-2\pi i t(1-|x|)}) dt \\ &- \int_0^\infty \frac{\Phi(t)\, e^{-\varepsilon t}}{t^{\gamma+\beta}} (a_3 e^{2\pi i t(1-|x|)} + a_4 e^{-2\pi i t(1-|x|)}) dt. \end{aligned}$$

Here, by the definition of Φ, we know that

$$\int_0^\infty \frac{\Phi(t)\, e^{-\varepsilon t}}{t^{\gamma+\beta}} (a_3 e^{2\pi i t(1-|x|)} + a_4 e^{-2\pi i t(1-|x|)}) dt = O(1)$$

uniformly for $1/2 \le |x| \le 2$ and $\varepsilon > 0$. Invoking Lemma 4.3, by the fact that $\Gamma(1-\gamma-\beta) > 0$ when $0 < \gamma + \beta < 1$, we obtain that

$$\begin{aligned} &\int_0^\infty \frac{e^{-\varepsilon t}}{t^{\gamma+\beta}} (a_3 e^{2\pi i t(1-|x|)} + a_4 e^{-2\pi i t(1-|x|)}) dt \\ \simeq &\int_0^\infty t^{-(\gamma+\beta)} e^{it(1-|x|+i\varepsilon)} + t^{-(\gamma+\beta)} e^{-it(1-|x|-i\varepsilon)} dt \\ = &\, i\Gamma(1-\gamma-\beta) \left[\frac{1}{(1-|x|+i\varepsilon)^{1-\gamma-\beta}} - \frac{1}{(1-|x|-i\varepsilon)^{1-\gamma-\beta}} \right] \\ = &\, i\Gamma(1-\gamma-\beta) \frac{(1-|x|-i\varepsilon)^{1-\gamma-\beta} - (1-|x|+i\varepsilon)^{1-\gamma-\beta}}{((1-|x|)^2+\varepsilon^2)^{1-\gamma-\beta}} \\ \simeq &\frac{\sin\left((1-\gamma-\beta)\arctan \frac{\varepsilon}{1-|x|}\right)}{((1-|x|)^2+\varepsilon^2)^{\frac{1-\gamma-\beta}{2}}} \\ \preceq &\frac{\varepsilon}{1-|x|} \cdot \frac{1}{((1-|x|)^2+\varepsilon^2)^{\frac{1-\gamma-\beta}{2}}} \end{aligned}$$

for sufficiently small ε. Thus, we have

$$|\mathcal{L}_\varepsilon(x)| \preceq \frac{\varepsilon}{1-|x|} \cdot \frac{1}{((1-|x|)^2+\varepsilon^2)^{\frac{1-\gamma-\beta}{2}}} + O(1), \tag{4.13}$$

uniformly for $1/2 \le |x| \le 2$ and $\varepsilon > 0$. This together with (4.9) and (4.11) yields

$$|\Re_\varepsilon(x)| \preceq \frac{\varepsilon}{1-|x|} \cdot \frac{1}{((1-|x|)^2+\varepsilon^2)^{\frac{1-\gamma-\beta}{2}}} + O(1),$$

uniformly for $\varepsilon > 0$, where $1/2 \le |x| \le 2$ and $0 < \gamma + \beta < 1$.

With (4.11), (4.12), (4.13) and (4.9) in hand, we can derive the following estimate on $\Re_\varepsilon(x)$.

Lemma 4.4. *Assume* $0 < \gamma + \beta \le 1$. *If* $|x| < 1/2$ *then*

$$|\Re_\varepsilon(x)| \preceq |x|^{-\frac{n-1}{2}}.$$

If $|x| > 2$, *then*

$$|\Re_\varepsilon(x)| \preceq |x|^{-n-1}.$$

For $1/2 \le |x| \le 2$, *we have if* $\gamma + \beta = 1$, *then* $|\Re_\varepsilon(x)| \preceq \frac{1}{|x|^{\frac{n-1}{2}}} \frac{1}{|1-|x||}$; *if* $0 < \gamma + \beta < 1$,

$$|\Re_\varepsilon(x)| \preceq \frac{\varepsilon}{1-|x|} \cdot \frac{1}{((1-|x|)^2+\varepsilon^2)^{\frac{1-\gamma-\beta}{2}}} + O(1),$$

uniformly for $\varepsilon > 0$.

Now we are able to prove that $\Re_\varepsilon(x)$ is an integrable function for $0 < \gamma + \beta \le 1$. By Lemma 4.4, if $\gamma + \beta = 1$, then

$$\begin{aligned}\|\Re_\varepsilon\|_{L^1(\mathbb{R}^n)} &\preceq \int_{|x|\le 1/2} \frac{dx}{|x|^{\frac{n-1}{2}}} + \int_{|x|>2} \frac{dx}{|x|^{n+1}} + \int_{\{1/2\le|x|\le 2\}} \frac{1}{|x|^{\frac{n-1}{2}}} \frac{1}{|1-|x||} dx \\ &\preceq 1,\end{aligned}$$

if $0 < \gamma + \beta < 1$,

$$\begin{aligned}\|\Re_\varepsilon\|_{L^1(\mathbb{R}^n)} \preceq\ & \int_{|x|\le 1/2} \frac{dx}{|x|^{\frac{n-1}{2}}} + \int_{|x|>2} \frac{dx}{|x|^{n+1}} \\ &+ \int_{\{1/2\le|x|\le 2\}\cap\{|(1-|x|)|>10\varepsilon\}} \frac{\varepsilon}{(1-|x|)^{2-\gamma-\beta}} dx \\ &+ \varepsilon^{\gamma+\beta} \int_{\{1/2\le|x|\le 2\}\cap\{|(1-|x|)|<10\varepsilon\}} \frac{1}{|1-|x||} dx + \int_{\{1/2\le|x|\le 2\}} dx \\ \preceq\ & 1,\end{aligned}$$

as required. □

§5 Proof of Theorem 1.5

The proof of Theorem 1.5 is based on Theorem 1.3 and a standard dense argument. We only prove the case $0 < \beta < 2$. Once Theorem 1.3 is established, Theorem 1.5 will be a direct consequence of it and the fact that the means $t^{-\beta}(S_t^\gamma(h) - h)$ converge to 0 in $\dot{F}^\alpha_{p,q}$ norm for h in a dense subclass of $I_\beta(\dot{F}^\alpha_{p,q})(\mathbb{R}^n)$. Such a dense class is $\dot{\mathscr{S}}(\mathbb{R}^n)$, where $\dot{\mathscr{S}}(\mathbb{R}^n)$ is the space of all Schwartz functions h whose Fourier transform satisfying

$$\dot{\mathscr{S}}(\mathbb{R}^n) = \{h | h \in \mathscr{S}(\mathbb{R}^n) : \partial^\alpha(\hat{h})(0) = 0 \text{ for every multi-index } \alpha\}.$$

For a function h in this class, we easily see that $t^{-\beta}\left(S_t^\gamma(h)-h\right) \to 0$ pointwise as $t \to 0^+$. Indeed, noting the fact that

$$\begin{aligned} t^{-\beta}\left(S_t^\gamma(h)-h\right)(x) &= t^{-\beta}\int_{\mathbb{R}^n}(1-m_\gamma)(t\xi))\widehat{h}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &\approx t^{-\beta}\int_{\mathbb{R}^n}\int_0^1 \sin^2(\pi st|\xi|)(1-s^2)^{\frac{n-3}{2}+\gamma}ds\widehat{h}(\xi)e^{2\pi i\xi\cdot x}d\xi, \end{aligned}$$

since $h \in \dot{\mathscr{S}}$ and $|\sin s| \le |s|$ for $s \in \mathbb{R}$, we have

$$\begin{aligned} t^{-\beta}\left(S_t^\gamma(h)-h\right)(x) &\preceq t^{-\beta}\int_{\mathbb{R}^n}\int_0^1 (t|\xi|s)^2\left(1-s^2\right)^{\frac{n-3}{2}+\gamma}ds\widehat{h}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &\preceq t^{2-\beta}B\left(\frac{3}{2},\frac{n-1}{2}+\gamma\right)\int_{\mathbb{R}^n}|\xi|^2\widehat{h}(\xi)e^{2\pi i\xi\cdot x}d\xi \qquad (5.1) \\ &=: t^{2-\beta}B\left(\frac{3}{2},\frac{n-1}{2}+\gamma\right)g(x), \end{aligned}$$

where $B(\cdot,\cdot)$ is the beta function, and $g \in \dot{\mathscr{S}}$ since we know the fact that if $h \in \dot{\mathscr{S}}$ then so does $(|\xi|^z\hat{h}(\xi))^\vee$ for all $z \in \mathbb{C}$ (see [10, p. 4]). Also,

$$\left|t^{-\beta}\left(S_t^\gamma(h)-h\right)(x)\right| \preceq t^{2-\beta}B\left(\frac{3}{2},\frac{n-1}{2}+\gamma\right),$$

Then $t^{-\beta}\left(S_t^\gamma(h)-h\right) \to 0$ pointwise as $t \to 0^+$ for $0<\beta<2$ and $\gamma > -\frac{n-1}{2}$. On the other hand, from (5.1) we know that if $t<1$, the functions $t^{-\beta}\left(S_t^\gamma(h)-h\right)(x)$ are pointwise controlled by the function $g \in \dot{\mathscr{S}}$. Invoking (2.2), the Lebesgue dominated convergence theorem implies that $t^{-\beta}\left(S_t^\gamma(h)-h\right)$ converges to 0 in $\dot{F}^\alpha_{p,q}$. Finally, using (1.2) and the results in Theorem 1.3, a standard $\varepsilon/3$ argument yields $t^{-\beta}\left(S_t^\gamma(f)-f\right) \to 0$ in $\dot{F}^\alpha_{p,q}$ for general $I_\beta(\dot{F}^\alpha_{p,q})$ functions f. This completes the proof.

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