Appl. Math. J. Chinese Univ.
2022, 37(4): 530-545

Inverse resonance problems with the discontinuous conditions

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Abstract. In this paper, we consider the inverse resonance problems for the discontinuous and non-selfadjoint Sturm-Liouville problem. We prove the uniqueness theorem and provide a reconstructive algorithm for the potential by using the Cauchy data and Weyl function.

§1 Introduction

Let us consider the following problem:

$$ly := -y''(x) + q(x)y(x) = \lambda^2 y(x), \qquad x \in (0, d) \cup (d, \pi), \tag{1}$$

with the boundary conditions

$$y(0) = 0, \qquad y'(\pi) = i\lambda y(\pi), \tag{2}$$

and the jump conditions

$$\begin{cases} y(d+0) = ay(d-0), \\ y'(d+0) = a^{-1}y'(d-0) \end{cases} \tag{3}$$

at the point $d \in (0, \pi)$. Here λ is the spectral parameter, $q(x) \neq 0$ is a real-valued function in $L^\infty(0, \pi)$, $a > 0$ and $0 < d \leq \frac{\pi}{2}$. Denoted by $L = L(q, d, a)$ the problem (1)-(3).

The inverse Sturm-Liouville problem has been studied by many authors (see, e.g., [4, 6, 8–11, 18]). After the separation of variables, the problem (1)-(2) arises in the hyperbolic problem with the absorbing boundary condition at $x = \pi$. Coming from the motivation of the scattering theory for the Schrödinger equation, we can extend $q(x)$ on $(0, +\infty)$ to be zero for $x > \pi$ and regard it as a central potential on $\mathbb{R}^3$.

For the continuous problem, the result that the potential $q(x)$ can be constructed by resonance parameters, i.e. the complex eigenvalues, was proposed by Regge [14, 15]. For example,

Received: 2019-12-20. Revised: 2020-12-28.
MR Subject Classification: 34A55, 34B24, 47E05.
Keywords: inverse resonance problem, discontinuous conditions, Gelfand-Levitan kernel, Weyl function.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-022-4004-x.
The research work was supported in part by the National Natural Science Foundation of China (11871031) and by the Natural Science Foundation of the Jiangsu Province of China(BK 20201303). Ran Zhang was supported in part by Postgraduate Research and Practice Innovation Program of Jiangsu Province (KYCX20_0245).
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in [14], it was shown that many of the characters of Jost function can be derived from general theorems and indicated a theorem on the existence of infinitely many zeros of the Jost function; in [15], the author simplified the Gelfand and Levitan's procedure for constructing the potential by the spectral measure function.

Brown ([5]) presented a new technique that the potential is uniquely determined from the location of eigenvalues and resonances in the context of a Schrödinger operator on a half line. Aktosun constructed the potential on the half line from the Jost function in [2]. He gave various uniqueness and non-uniqueness conclusions and illustrated the recovery with some explicit examples. In [12], Pivovarchik et al. studied the inverse eigenvalue problem for Sturm-Liouville problem with nonselfadjoint boundary conditions depending on the spectral parameter and recovered the real coefficients from the eigenvalues using entire function theory and the solution of a Marchenko integral equation.

In [17], Rundell and Sacks, motivated by the work of Regge, reconstructed a radial potential in $\mathbb{R}^3$ from its resonance parameters for the non-selfadjoint Sturm-Liouville problem. By recovering a function which is related to the boundary values of the corresponding Gelfand-Levitan kernel and using a particular computational technique, they gave the method of reconstruction for the potential.

For the discontinuous problem, Akcay ([1]) gave some information of the kernel function and the asymptotic formulas of the eigenvalues and eigenfunctions. In [19], Yang and Bondarenko studied the local solvability and stability of the discontinuous problem. They proved the uniqueness theorem and gave the constructive algorithm for the potential.

In this paper, we generalize the work of Rundell and Sacks [17] to the case with a discontinuity point d on a finite interval $(0,\pi)$. Section 2 gives the known information for the discontinuous case. In Section 3, we prove that one spectrum can uniquely determine the Cauchy data and then we can also show that the Cauchy data can determine the potential $q(x)$ on (d,π). For convenience, in Section 4, we shall convert the discontinuous problem to a new problem and get some results in later sections. In Sections 5 and 6, we prove the uniqueness theorem for $q(x)$ on $(0,d)$ and provide the algorithms to reconstruct the potential $q(x)$ on $(0,\pi)$ when $d=\frac{\pi}{2}$ and $0<d<\frac{\pi}{2}$, respectively.

§2 Preliminaries

Denote by $\phi(x,\lambda)$ the solution of (1) satisfying the discontinuity condition (3) and the initial conditions

$$\phi(0,\lambda)=1, \qquad \phi'(0,\lambda)=i\lambda. \tag{4}$$

We know that when $q(x)=0$ in (1), the solution $\phi_0(x,\lambda)$ of (1) satisfying (3) and (4), has the following form [1]:

$$\phi_0(x,\lambda)=\begin{cases} e^{i\lambda x}, & x<d,\\ a^+e^{i\lambda x}+a^-e^{i\lambda(2d-x)}, & x>d, \end{cases} \tag{5}$$

where $a^+=\frac{a+a^{-1}}{2}$ and $a^-=\frac{a-a^{-1}}{2}$.

Lemma 1. *[1] The solution $\phi(x,\lambda)$ of (1) satisfying the initial conditions (4) and the discontinuity conditions (3) can be expressed by the following formula:*

$$\phi(x,\lambda) = \phi_0(x,\lambda) + \int_{-x}^{x} K(x,t)e^{i\lambda t}dt \tag{6}$$

and the kernel function $K(x,t)$ has the following properties:

$$K(x,x) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^x q(t)dt, & 0 < x < d, \\ \dfrac{a^+}{2}\displaystyle\int_0^x q(t)dt, & d < x < \pi, \end{cases} \tag{7}$$

$$K(x,t)|_{t=2d-x-0}^{t=2d-x+0} = -\frac{a^-}{2}\Big(\int_0^d q(t)dt - \int_d^x q(t)dt\Big), \qquad x > d \tag{8}$$

and

$$K(x,-x) = 0, \qquad x \in [0,d) \cup (d,\pi]. \tag{9}$$

Moreover, if $q(x)$ is differentiable, then the kernel function $K(x,t)$ also satisfies the following relations:

$$K_{xx}(x,t) - K_{tt}(x,t) = q(x)K(x,t), \quad x \in (0,d) \cup (d,\pi) \textit{ and } |t| < x, \tag{10}$$

$$\frac{d}{dx}K(x,x) = \begin{cases} \dfrac{1}{2}q(x), & x < d, \\ \dfrac{a^+}{2}q(x), & x > d, \end{cases} \tag{11}$$

$$\frac{d}{dx}\Big\{K(x,t)|_{t=2d-x-0}^{t=2d-x+0}\Big\} = \frac{a^-}{2}q(x), \qquad x > d. \tag{12}$$

Let $\varphi(x,\lambda)$ be the solution of (1) satisfying the initial condition $\varphi(0,\lambda) = 0, \varphi'(0,\lambda) = 1$ and jump condition (3). We can rewrite $\varphi(x,\lambda)$ as

$$\varphi(x,\lambda) = \frac{\phi(x,\lambda) - \phi(x,-\lambda)}{2i\lambda}. \tag{13}$$

Then, substituting the representations of $\phi(x,\lambda)$ and $\phi(x,-\lambda)$ into (13) and using the Euler's formula, we can obtain that $\varphi(x,\lambda)$ satisfies the following integral equation: for $0 < x < d$,

$$\begin{aligned} \varphi(x,\lambda) =& \frac{1}{2i\lambda}\Big(e^{i\lambda x} + \int_{-x}^{x} K(x,t)e^{i\lambda t}dt - e^{-i\lambda x} - \int_{-x}^{x} K(x,t)e^{-i\lambda t}dt\Big) \\ =& \frac{1}{2i\lambda}\Big(\cos\lambda x + i\sin\lambda x - \cos\lambda x + i\sin\lambda x + \int_{-x}^{x} K(x,t)(e^{i\lambda t} - e^{-i\lambda t})dt\Big) \\ =& \frac{\sin\lambda x}{\lambda} + \int_{-x}^{x} K(x,t)\frac{\sin\lambda t}{\lambda}dt \\ =& \frac{\sin\lambda x}{\lambda} + \int_0^x \Big(K(x,t) - K(x,-t)\Big)\frac{\sin\lambda t}{\lambda}dt \end{aligned} \tag{14}$$

and for $d < x < \pi$,

$$\begin{aligned} \varphi(x,\lambda) =& \frac{1}{2i\lambda}\Big(a^+e^{i\lambda x} + a^-e^{i\lambda(2d-x)} + \int_{-x}^{x} K(x,t)e^{i\lambda t}dt \\ & - a^+e^{-i\lambda x} - a^-e^{-i\lambda(2d-x)} - \int_{-x}^{x} K(x,t)e^{-i\lambda t}dt\Big) \\ =& a^+\frac{\sin\lambda x}{\lambda} + a^-\frac{\sin\lambda(2d-x)}{\lambda} + \int_{-x}^{x} K(x,t)\frac{\sin\lambda t}{\lambda}dt \end{aligned}$$

$$=a^{+}\frac{\sin\lambda x}{\lambda}+a^{-}\frac{\sin\lambda(2d-x)}{\lambda}+\int_0^x\Big(K(x,t)-K(x,-t)\Big)\frac{\sin\lambda t}{\lambda}dt. \tag{15}$$

We can also obtain by (9) and (15) that for $d<x<\pi$,

$$\begin{aligned}\varphi'(x,\lambda)=&a^{+}\cos\lambda x-a^{-}\cos\lambda(2d-x)+\int_0^x\Big(K_x(x,t)-K_x(x,-t)\Big)\frac{\sin\lambda t}{\lambda}dt\\&+\frac{a^{+}}{2}\int_0^x q(t)dt\frac{\sin\lambda x}{\lambda},\end{aligned} \tag{16}$$

where $K(x,t)$ is above-mentioned and has the following forms [1]:

$$K(x,t)=\frac{1}{2}\int_0^{\frac{x+t}{2}}q(\xi)d\xi+\frac{1}{2}\int_0^x q(\xi)\int_{t-(x-\xi)}^{t+(x-\xi)}K(\xi,s)dsd\xi,\ 0<x<d, \tag{17}$$

$$\begin{aligned}K(x,t)=&K_0(x,t)+\frac{a^{+}}{2}\int_0^d q(\xi)\int_{t-(x-\xi)}^{t+(x-\xi)}K(\xi,s)dsd\xi\\&+\frac{a^{-}}{2}\int_0^{2d-x}q(\xi)\int_{t+x+\xi-2a}^{t-x-\xi+2a}K(\xi,s)dsd\xi\\&-\frac{a^{-}}{2}\int_{2d-x}^{d}q(\xi)\int_{t-x-\xi+2a}^{t+x+\xi-2a}K(\xi,s)dsd\xi\\&+\frac{1}{2}\int_d^x q(\xi)\int_{t-(x-\xi)}^{t+(x-\xi)}K(\xi,s)dsd\xi,\qquad d<x<\pi,\end{aligned} \tag{18}$$

and

$$K_0(x,t)=\begin{cases}\dfrac{a^{+}}{2}\displaystyle\int_0^{\frac{x+t}{2}}q(\xi)d\xi, & -x<t<x-2d,\\ \dfrac{a^{+}}{2}\displaystyle\int_0^{\frac{x+t}{2}}q(\xi)d\xi+\dfrac{a^{-}}{2}\int_0^{\frac{t-x+2d}{2}}q(\xi)d\xi, & x-2d\le t<2d-x,\\ \dfrac{a^{+}}{2}\displaystyle\int_0^{d}q(\xi)d\xi-\dfrac{a^{-}}{2}\int_{\frac{t-x+2d}{2}}^{d}q(\xi)d\xi+\dfrac{a^{+}}{2}\int_d^{\frac{x+t}{2}}q(\xi)d\xi & \\ \qquad+\dfrac{a^{-}}{2}\displaystyle\int_d^{\frac{x-t+2d}{2}}q(\xi)d\xi, & 2d-x<t<x.\end{cases}$$

Denote

$$\Delta(\lambda)=\varphi'(\pi,\lambda)-i\lambda\varphi(\pi,\lambda). \tag{19}$$

The function $\Delta(\lambda)$ is called the characteristic function of L, which is entire in λ. The zeros $\{\lambda_n^2\}_{n=-\infty}^{\infty}$ of $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem L.

From the representations of (15) and (16), we have that

$$\begin{aligned}\Delta(\lambda)=&a^{+}e^{-i\lambda\pi}-a^{-}e^{i\lambda(2d-\pi)}+K(\pi,\pi)\frac{\sin\lambda\pi}{\lambda}+\int_0^\pi K_x(\pi,t)\frac{\sin\lambda t}{\lambda}dt\\&-\int_0^\pi K_x(\pi,-t)\frac{\sin\lambda t}{\lambda}dt-i\int_0^\pi K(\pi,t)\sin\lambda t dt+i\int_0^\pi K(\pi,-t)\sin\lambda t dt.\end{aligned} \tag{20}$$

Integration by parts gives

$$\begin{aligned}\Delta(\lambda) =&a^+e^{-i\lambda\pi} - a^-e^{i\lambda(2d-\pi)} + K(\pi,\pi)\frac{\sin\lambda\pi}{\lambda} + \int_0^\pi K_x(\pi,t)\frac{\sin\lambda t}{\lambda}dt\\ &- \int_0^\pi K_x(\pi,-t)\frac{\sin\lambda t}{\lambda}dt + \frac{i}{\lambda}K(\pi,\pi)\cos\lambda\pi\\ &- \frac{i}{\lambda}\int_0^\pi K_t(\pi,t)\cos\lambda t dt - \frac{i}{\lambda}\int_0^\pi K_t(\pi,-t)\cos\lambda t dt.\end{aligned}\tag{21}$$

Theorem 1. *The asymptotic formula of the eigenvalues* $\{\lambda_n^2\}_{n=-\infty}^{\infty}$ *is as follows:*

$$\lambda_n = \lambda_n^0 + \frac{a^+(ie^{-i\lambda_n^0\pi}) + ia^-\cos\lambda_n^0(2d-\pi)}{2\dot{\Delta}(\lambda_n^0)\lambda_n^0}\int_0^\pi q(t)dt + O\Big(\frac{1}{\lambda_n^0}\Big),\tag{22}$$

where $\{\lambda_n^0\}_{n=-\infty}^{\infty}$ *are zeros of the function* $\Delta_0(\lambda) := a^+e^{-i\lambda\pi} - a^-e^{i\lambda(2d-\pi)}$ *and has the following representation:*

$$\lambda_n^0 = \frac{n\pi}{d} - \frac{\ln\frac{a^-}{a^+}}{2di},\quad n\in\mathbb{Z}\tag{23}$$

and $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$.

To prove the theorem, we firstly give the following lemma.

Lemma 2. *There holds* $\inf|\lambda_n^0 - \lambda_m^0| = \gamma > 0$ *for* $n\neq m$, *i.e., the roots of* $\Delta_0(\lambda)$=*0 are separate.*

Proof. The proof is similar to that in [3]. So we omit it. □

Proof of Theorem 1. Denote $G_n := \Big\{\lambda : |\lambda| = |\lambda_n^0| + \frac{\gamma}{2},\ n\in\mathbb{N}\Big\}$ and $G_\delta := \Big\{\lambda : |\lambda - \lambda_n^0| \geq \delta,\ n\in \mathbb{N}\Big\}$, where $\delta\in(0,\frac{\gamma}{2})$. It follows from the representation of $\Delta_0(\lambda)$ that $|\Delta_0(\lambda)| \geq C_\delta e^{|\Im\lambda|\pi}$ for $\lambda\in\bar{G}_\delta$.

In view of

$$\begin{aligned}\Delta(\lambda)-\Delta_0(\lambda)=&K(\pi,\pi)\frac{\sin\lambda\pi}{\lambda} + \int_0^\pi K_x(\pi,t)\frac{\sin\lambda t}{\lambda}dt - \int_0^\pi K_x(\pi,-t)\frac{\sin\lambda t}{\lambda}dt\\ &+ \frac{i}{\lambda}K(\pi,\pi)\cos\lambda\pi - \frac{i}{\lambda}\int_0^\pi K_t(\pi,t)\cos\lambda t dt\\ &- \frac{i}{\lambda}\int_0^\pi K_t(\pi,-t)\cos\lambda t dt,\end{aligned}$$

thus we can obtain that for $\lambda\in\bar{G}_\delta$,

$$|\Delta(\lambda)-\Delta_0(\lambda)| \leq \frac{Ce^{|\Im\lambda|\pi}}{|\lambda|}.$$

That is,

$$\lim_{|\lambda|\to\infty}\frac{\Delta(\lambda)-\Delta_0(\lambda)}{e^{|\Im\lambda|\pi}} = 0.$$

For sufficiently large n and $\lambda\in G_n$, we have

$$|\Delta(\lambda)-\Delta_0(\lambda)| < \frac{C_\delta}{2}e^{|\Im\lambda|\pi}.$$

According to Rouche's theorem, it follows that for sufficiently large n, $\Delta_0(\lambda)$ and $\Delta_0(\lambda) + (\Delta(\lambda)-\Delta_0(\lambda))$ have the same number of zeros inside contour G_n. Similarly, it is shown by

Rouche's theorem again that for sufficiently large n, $\Delta(\lambda)$ has a unique of zero inside G_δ. Since $\delta > 0$ is arbitrary, there exists small ε_n such that

$$\lambda_n = \lambda_n^0 + \varepsilon_n.$$

Using the similar method in [3], we get

$$\varepsilon_n = \frac{a^+(ie^{-i\lambda_n^0\pi}) + ia^- \cos\lambda_n^0(2d-\pi)}{2\dot{\Delta}(\lambda_n^0)\lambda_n^0}\int_0^\pi q(t)dt + O\Big(\frac{1}{\lambda_n^0}\Big).$$

Next, we shall show the representation of λ_n^0. Since $\Delta_0(\lambda_n^0) = 0$,
namely, $a^+e^{-i\lambda_n^0\pi} = a^-e^{i\lambda_n^0(2d-\pi)}$. That is,

$$\frac{a^-e^{2id\lambda_n^0}}{a^+} = 1 = e^{2i\pi n} \Rightarrow ln\frac{a^-}{a^+} = 2i\pi n - 2id\lambda_n^0.$$

Thus (23) holds. This completes the proof. □

§3 Uniqueness theorem for $q(x)$ on (d, π)

For the continuous Sturm-Liouville problem, we know that the Cauchy data $\Big\{K_x(\pi,t), K_t(\pi, t)\Big\}$ can uniquely determine the potential $q(x)$ on the whole line [16]. Using the same method we know that $\Big\{K_x(d,t), K_t(d,t)\Big\}$ can uniquely determine the potential $q(x)$ on $(0, d)$. So we only show that for the discontinuous problem, the uniqueness of $q(x)$ on (d, π) can be determined by the Cauchy data $\Big\{K_x(\pi,t), K_t(\pi,t)\Big\}$.

Theorem 2. *For the discontinuous problem L, the Cauchy data* $\Big\{K_x(\pi,t), K_t(\pi,t)\Big\}$ *can uniquely determine* $q(x)$ *on* (d, π).

Proof. In view of (18), here we only prove the case when $t \in (-x, x-2d)$. For $-x < t < x-2d$, deriving (18) with respect to x and t, respectively, we get

$$\begin{aligned}K_x(x,t) =&\frac{a^+}{4}q(\frac{x+t}{2}) + \frac{a^+}{2}\int_0^d q(\xi)\Big(K(\xi,t+x-\xi) + K(\xi,t-x+\xi)\Big)d\xi\\ &+\frac{a^-}{2}\int_0^{-x+2d} q(\xi)\Big(-K(\xi,t-x+2d-\xi) - K(\xi,t+x-2d+\xi)\Big)d\xi\\ &-\frac{a^-}{2}\int_{-x+2d}^d q(\xi)\Big(K(\xi,t+x-2d+\xi) + K(\xi,t-x+2d-\xi)\Big)d\xi\\ &+\frac{1}{2}\int_d^x q(\xi)\Big(K(\xi,t+x-\xi) + K(\xi,t-x+\xi)\Big)d\xi \qquad (24)\end{aligned}$$

and

$$\begin{aligned}K_t(x,t) =&\frac{a^+}{4}q(\frac{x+t}{2}) + \frac{a^+}{2}\int_0^d q(\xi)\Big(K(\xi,t+x-\xi) - K(\xi,t-x+\xi)\Big)d\xi\\ &+\frac{a^-}{2}\int_0^{-x+d} q(\xi)\Big(K(\xi,t-x+2d-\xi) - K(\xi,t+x-2d+\xi)\Big)d\xi\end{aligned}$$

$$-\frac{a^-}{2}\int_{-x+2d}^{d} q(\xi)\Big(K(\xi,t+x-2d+\xi)-K(\xi,t-x+2d-\xi)\Big)d\xi$$
$$+\frac{1}{2}\int_{d}^{x} q(\xi)\Big(K(\xi,t+x-\xi)-K(\xi,t-x+\xi)\Big)d\xi. \tag{25}$$

Putting $x=\pi, t=2x-\pi$ and adding (24) and (25), it yields

$$q(x)=\frac{2}{a^+}\Big(K_x(\pi,2x-\pi)+K_t(\pi,2x-\pi)\Big)-2\int_0^d q(\xi)K(\xi,2x-\xi)d\xi$$
$$-\frac{2}{a^+}\int_d^{\pi} q(\xi)K(\xi,2x-\xi)d\xi+\frac{2a^-}{a^+}\int_0^d q(\xi)K(\xi,2x-2d+\xi)d\xi. \tag{26}$$

Then, we will use (26) to prove the uniqueness on the interval (d,π). Denote by M a mapping which has the following form [16]:

$$M: \quad q \rightarrow Mq,$$

where

$$Mq:=P(x)-2\int_0^d q(\xi)K(\xi,2x-\xi,q)d\xi-\frac{2}{a^+}\int_d^{\pi} q(\xi)K(\xi,2x-\xi,q)d\xi$$
$$+\frac{2a^-}{a^+}\int_0^d q(\xi)K(\xi,2x-2d+\xi,q)d\xi$$

with

$$P(x)=\frac{2}{a^+}\Big(K_x(\pi,2x-\pi)+K_t(\pi,2x-\pi)\Big).$$

Once we show that the mapping M has at most one fixed point in $L^\infty(d,\pi)$, then, the uniqueness on (d,π) can be obtained. This method is the same as which introduced in [13] and [16].

For a fixed $C>0$, put $L_C=\Big\{q\in L^2(d,\pi)\Big|\|q(x)\|_\infty\le C \text{ a.e. on } (d,\pi)\Big\}$ and denote by P_C the operator of projection onto L_C, i.e.,

$$P_C q(x)=\begin{cases} q(x), & \text{when } |q(x)|\le C,\\ \pm C, & \text{when } \pm|q(x)|\ge C.\end{cases}$$

Assume that q and $\tilde q$ ($q\ne\tilde q$) are all fixed points of M and select C such that $\|q\|_\infty,\|\tilde q\|_\infty<C$. It follows that q and $\tilde q$ are also fixed points of $P_C M$. It suffices to show that $P_C M$ is contracting on L_C with the following norm for some sufficiently large λ [13],

$$\|q\|_\lambda^2=\int_0^{\pi} q^2(x)e^{2\lambda(x-\pi)}dx. \tag{27}$$

We know that

$$\|P_C M(q)-P_C M(\tilde q)\|_\lambda\le\|M(q)-M(\tilde q)\|_\lambda \tag{28}$$

and also obtain that by (26),

$$M(q)(x)-M(\tilde q)(x)$$
$$=2\int_0^d(\tilde q(\xi)-q(\xi))K(\xi,2x-\xi,q)d\xi$$
$$+2\int_0^d \tilde q(\xi)\Big[K(\xi,2x-\xi,\tilde q)-K(\xi,2x-\xi,q)\Big]d\xi$$

$$
\begin{aligned}
&+\frac{2}{a^+}\int_d^\pi (\tilde{q}(\xi)-q(\xi))K(\xi,2x-\xi,q)d\xi\\
&+\frac{2}{a^+}\int_d^\pi \tilde{q}(\xi)\Big[K(\xi,2x-\xi,\tilde{q})-K(\xi,2x-\xi,q)\Big]d\xi\\
&-\frac{2a^-}{a^+}\int_0^d (\tilde{q}(\xi)-q(\xi))K(\xi,2x-2d+\xi,q)d\xi\\
&-\frac{2a^-}{a^+}\int_0^d \tilde{q}(\xi)\Big[K(\xi,2x-2d+\xi,\tilde{q})-K(\xi,2x-2d+\xi,q)\Big]d\xi. \qquad (29)
\end{aligned}
$$

Using the Riemann function, we can write the second, sixth and fourth terms on the right-hand side of the equality as the following formulas, respectively:

$$\int_0^d N_1(x,y)(\tilde{q}(\xi)-q(\xi))d\xi \quad \text{or} \quad \int_d^\pi N_2(x,y)(\tilde{q}(\xi)-q(\xi))d\xi, \qquad (30)$$

where $N_1(x,y)$ and $N_2(x,y)$ are bounded kernel functions depending on q and $\tilde{q}$. Then

$$|M(q)(x)-M(\tilde{q})(x)| \le C_1\int_0^\pi |q(\xi)-\tilde{q}(\xi)|d\xi. \qquad (31)$$

Thus, by a accurate calculation, we have

$$\|M(q)(x)-M(\tilde{q})(x)\|_\lambda \le \frac{C_1}{\sqrt{\lambda}}\|q-\tilde{q}\|_\lambda. \qquad (32)$$

So, when λ is sufficiently large, we can get the necessary contracting property of $P_C M$. This completes the proof. □

Theorem 3. *Let $f(t)=K_t(\pi,t)+K_x(\pi,t)$ and $F(t)=-\int_t^\pi f(s)ds$. Then, the following formula holds:*

$$\Delta(\lambda)=e^{-i\lambda\pi}\int_0^{2\pi}\Big(a^+\delta(s)-F(s-\pi)\Big)e^{i\lambda s}ds-a^-e^{i\lambda(2d-\pi)}, \qquad (33)$$

where $\delta(s)$ is the Dirac-delta function.

Proof. By integration by parts, we can get

$$
\begin{aligned}
\Delta(\lambda)=&a^+e^{-i\lambda\pi}-a^-e^{i\lambda(2d-\pi)}+K(\pi,\pi)\frac{\sin\lambda\pi}{\lambda}+\int_0^\pi K_x(\pi,t)\frac{\sin\lambda t}{\lambda}dt\\
&-\int_0^\pi K_x(\pi,-t)\frac{\sin\lambda t}{\lambda}dt+\frac{iK(\pi,\pi)}{\lambda}\cos\lambda\pi\\
&-\frac{i}{\lambda}\int_0^\pi K_t(\pi,t)\cos\lambda t dt-\frac{i}{\lambda}\int_0^\pi K_t(\pi,-t)\cos\lambda t dt. \qquad (34)
\end{aligned}
$$

In characteristic triangle $\{(x,t):0<|t|<x<1\}$, we obtain that K_x and K_t may be extended as odd and even functions, respectively. Then,

$$
\begin{aligned}
\lambda\Delta(\lambda)=&\lambda a^+e^{-i\lambda\pi}-\lambda a^-e^{i\lambda(2d-\pi)}+K(\pi,\pi)\sin\lambda\pi+iK(\pi,\pi)\cos\lambda\pi\\
&+\frac{1}{2}\int_{-\pi}^\pi\Big(K_x(\pi,t)+K_t(\pi,t)\Big)(\sin\lambda t-i\cos\lambda t)dt\\
&+\frac{1}{2}\int_{-\pi}^\pi\Big(-K_x(\pi,-t)+K_t(\pi,-t)\Big)(\sin\lambda t-i\cos\lambda t)dt
\end{aligned}
$$

$$
\begin{aligned}
&=\lambda a^+ e^{-i\lambda\pi} - \lambda a^- e^{i\lambda(2d-\pi)} + K(\pi,\pi)\sin\lambda\pi + iK(\pi,\pi)\cos\lambda\pi \\
&\quad + \frac{1}{2}\int_{-\pi}^{\pi}\Big(K_x(\pi,t) + K_t(\pi,t)\Big)(\sin\lambda t - i\cos\lambda t)dt \\
&\quad + \frac{1}{2}\int_{-\pi}^{\pi}\Big(K_x(\pi,t) + K_t(\pi,t)\Big)(\sin\lambda t - i\cos\lambda t)dt. \qquad (35)
\end{aligned}
$$

Multiplying both sides of (35) by $2i$ and considering the expression of $f(t)$, it follows that

$$
\begin{aligned}
2i\lambda\Delta(\lambda) =&2i\lambda a^+ e^{-i\lambda\pi} - 2i\lambda a^- e^{i\lambda(2d-\pi)} - a^+ e^{-i\lambda\pi}\int_0^{\pi} q(\xi)d\xi \\
&+ 2\int_{-\pi}^{\pi}\Big(K_x(\pi,t)+ K_t(\pi,t)\Big)e^{i\lambda t}dt \\
=&2i\lambda a^+ e^{-i\lambda\pi} - 2i\lambda a^- e^{i\lambda(2d-\pi)} - a^+ e^{-i\lambda\pi}\int_0^{\pi} q(\xi)d\xi \\
&+ 2\int_{-\pi}^{\pi} f(t)e^{i\lambda t}dt. \qquad (36)
\end{aligned}
$$

Note that

$$
\int_{-\pi}^{\pi} f(t)dt = \int_{-\pi}^{\pi}\Big(K_x(\pi,t)+K_t(\pi,t)\Big)dt = \int_{-\pi}^{\pi} K_t(\pi,t)dt = \frac{a^+}{2}\int_0^{\pi} q(t)dt. \qquad (37)
$$

It follows that

$$
2i\lambda\Delta(\lambda) = 2\int_{-\pi}^{\pi} f(t)e^{i\lambda t}dt - 2i\lambda a^- e^{i\lambda(2d-\pi)} + a^+ e^{-i\lambda\pi}\Big(2i\lambda - \int_0^{\pi} q(\xi)d\xi\Big). \qquad (38)
$$

Integrating by parts once again and using

$$
F(-\pi) = -\int_{-\pi}^{\pi} f(s)ds = -\frac{a^+}{2}\int_0^{\pi} q(t)dt,
$$

we can get (33). This completes the proof. □

Theorem 4. *The eigenvalues* $\{\lambda_n^2\}_{n=-\infty}^{\infty}$ *can uniquely determine* $q(x)$ *on* (d,π).

Proof. From Theorem 3, we know that

$$
\Delta(\lambda) + a^- e^{i\lambda(2d-\pi)} = e^{-i\lambda\pi}\int_0^{2\pi}\Big(a^+\delta(s) - F(s-\pi)\Big)e^{i\lambda s}ds. \qquad (39)
$$

That is

$$
\frac{1}{e^{-i\lambda\pi}}\Big(\Delta(\lambda) + a^- e^{i\lambda(2d-\pi)}\Big) = \int_0^{2\pi}\Big(a^+\delta(s) - F(s-\pi)\Big)e^{i\lambda s}ds \qquad (40)
$$

Taking the limit of both sides of (40), we can get in the upper-half of the complex plane by the Riemann–Lebesgue Lemma,

$$
\lim_{|\lambda|\to\infty}\frac{1}{e^{-i\lambda\pi}}\Big(\Delta(\lambda) + a^- e^{i\lambda(2d-\pi)}\Big) = a^+ - \lim_{|\lambda|\to\infty}\int_0^{2\pi} F(s-\pi)e^{i\lambda s}ds = a^+. \qquad (41)
$$

So $\Delta(\lambda)$ and $f(t) = F'(t)$ can be uniquely determined by the eigenvalues and then $q(x)$ can be determined by $\{\lambda_n^2\}_{n=-\infty}^{\infty}$. □

Remark 1. *It follows from the above discussions that* $\{\lambda_n^2\}_{n=-\infty}^{\infty}$ *can uniquely determine* $f(t)$ *which can be split into its odd part* $K_x(\pi,t)$ *and even part* $K_t(\pi,t)$. *That is, the pair of* $\Big\{K_t(\pi,t), K_x(\pi,t)\Big\}$ *can also be determined by* $\{\lambda_n^2\}_{n=-\infty}^{\infty}$. *It follows from Theorem 2 that* $q(x)$ *can be uniquely determined on* (d,π) *by* $\{\lambda_n^2\}_{n=-\infty}^{\infty}$.

§4 Conversion of the inverse problem on $(0, d)$

In this section, we convert the problem L to a new form of problem L_1 which be given below. Then, we can prove the uniqueness of $q(x)$ on $(0, d)$ and give the procedure of reconstructing the potential in the Sections 5 and 6.

For convenience, we can rewrite the problem (1)-(3) in the following form:

$$-y_i''(x) + q_i(x)y_i(x) = \lambda^2 y_i(x), \quad x \in (0, d_i), \quad i = 1, 2, \tag{42}$$

$$y_1(0) = 0, \qquad -y_2'(0) = i\lambda y_2(0), \tag{43}$$

$$y_2(d_2) = a y_1(d_1), \qquad y_2'(d_2) = -a^{-1} y_1'(d_1), \tag{44}$$

where $d_1 = d$, $d_2 = \pi - d$, $q_i \in L^\infty(0, d_i)$ for $i = 1, 2$ and $q_1(x) = q(x)|_{[0,d_1]}$, $q_2(x) = q(\pi - x)|_{[0,d_2]}$. The problem (42)-(44) can be denoted by $L_1 = L_1(q_1, q_2, d_1, d_2, a)$.

Denote by $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$ the solution of (42) satisfying the initial conditions $\varphi_1(0, \lambda) = 0, \varphi_1'(0, \lambda) = 1$ and $\varphi_2(0, \lambda) = 1, \varphi_2'(0, \lambda) = -i\lambda$, respectively. For any $\alpha > 0$, let $\mathcal{L}^\alpha$ be the class of entire functions of exponential type not greater than α which belongs to $L^2(\mathbb{R})$ for real λ. From the above conditions, we can get that

$$\varphi_1(d_1, \lambda) = \frac{\sin \lambda d_1}{\lambda} + D_{d_1} \frac{\cos \lambda d_1}{\lambda^2} + \frac{A_1(\lambda)}{\lambda^2}; \tag{45}$$

$$\varphi_1'(d_1, \lambda) = \cos \lambda d_1 - D_{d_1} \frac{\sin \lambda d_1}{\lambda} - \frac{A_2(\lambda)}{\lambda}; \tag{46}$$

$$\varphi_2(d_2, \lambda) = e^{-i\lambda d_2} + \int_0^{d_2} K_1(t) e^{-i\lambda t} dt, \tag{47}$$

where $D_{d_1} = -\frac{1}{2}\int_0^{d_1} q_1(t)dt$ and the functions $A_1(\lambda) := \int_0^{d_1} N_1(t) \cos \lambda t dt$, $A_2(\lambda) := \int_0^{d_1} N_2(t)$ $\times \sin \lambda t dt$ belong to the class of $\mathcal{L}^{d_1}$. Here $N_1, N_2 \in L^2(0, d_1)$. Note that the eigenvalues $\{\lambda_n^2\}_{n=-\infty}^{\infty}$ is also the squared zeros of the characteristic function $\Delta_1(\lambda)$:

$$\Delta_1(\lambda) = a\varphi_1(d_1, \lambda)\varphi_2'(d_2, \lambda) + a^{-1}\varphi_1'(d_1, \lambda)\varphi_2(d_2, \lambda). \tag{48}$$

Substituting (45) and (46) into (48) and taking $\lambda = \lambda_n$, it follows that

$$\begin{aligned} 0 = &\Big(\frac{a}{\lambda_n}\varphi_2'(d_2, \lambda_n)\Big)\Big(\lambda_n \sin \lambda_n d_1 + D_{d_1} \cos \lambda_n d_1 + \int_0^{d_1} N_1(t) \cos \lambda_n t dt\Big) \\ &+ \Big(a^{-1}\varphi_2(d_2, \lambda_n)\Big)\Big(\lambda_n \cos \lambda_n d_1 - D_{d_1} \sin \lambda_n d_1 - \int_0^{d_1} N_2(t) \sin \lambda_n t dt\Big). \end{aligned} \tag{49}$$

Introduce the Hilbert space $H = L^2(0, d_1) \oplus L^2(0, d_1)$ with real-valued vector-functions $\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$, $\nu_i \in L^2(0, d_1)$ $(i = 1, 2)$ and define the scalar product and the norm in H as follows:

$$(f, \nu)_H = \int_0^{d_1} \Big(f_1(x)\nu_1(x) + f_2(x)\nu_2(x)\Big) dx, \ \|\nu\|_H^2 = \int_0^{d_1} \Big(\nu_1^2(x) + \nu_2^2(x)\Big) dx,$$

$$\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad f, \ \nu \in H.$$

It is obvious that the vector-functions

$$N(t)=\begin{pmatrix} N_1(t)\\ N_2(t)\end{pmatrix},$$

$$\omega_n(t):=\begin{pmatrix} \frac{a}{\lambda_n}\varphi_2'(d_2,\lambda_n)\cos\lambda_n t\\ a^{-1}\varphi_2(d_2,\lambda_n)\sin\lambda_n t\end{pmatrix}=\begin{pmatrix} \frac{a}{\lambda_n}\varphi_2'(\pi-d_1,\lambda_n)\cos\lambda_n t\\ a^{-1}\varphi_2(\pi-d_1,\lambda_n)\sin\lambda_n t\end{pmatrix} \tag{50}$$

all belong to H. So, we can rewrite (49) as the following form:

$$(N,\omega_n)_H=g_n, \tag{51}$$

here

$$\begin{aligned} g_n=&-\frac{a}{\lambda_n}\varphi_2'(d_2,\lambda_n)\Big(\lambda_n\sin\lambda_n d_1+D_{d_1}\cos\lambda_n d_1\Big)+a^{-1}\varphi_2(d_2,\lambda_n)\\ &\times\Big(\lambda_n\cos\lambda_n d_1+D_{d_1}\sin\lambda_n d_1\Big). \end{aligned} \tag{52}$$

In order to obtain the uniqueness and the method of reconstructing the potential function on $(0,\pi)$, we should investigate the main equation (51).

§5 The case $d_1=\frac{\pi}{2}$

We agree that if a certain symbol υ denotes an object related to L, then $\tilde{\upsilon}$ denote the analogous object related to $\tilde{L}$. Suppose that $\{\tilde{\lambda}_n\}_{n=-\infty}^{\infty}$ are the eigenvalues of the problem $\tilde{L}_1=\tilde{L}_1(\tilde{q}_1,\tilde{q}_2,d_1,d_2,a)$.

Theorem 5. *If $\lambda_n^2=\tilde{\lambda}_n^2$ for $n\in\mathbb{Z}$, then $q=\tilde{q}$ on the whole interval $(0,\pi)$ for $d_1=\frac{\pi}{2}$.*

Before proving this result, we shall mention the following Lemma which will be needed later.

Lemma 3. *The system of vector-functions $\{\omega_n\}_{n\in\mathbb{Z}}$ in (50) is complete in H.*

Proof. Suppose that the system $\{\omega_n\}_{n\in\mathbb{Z}}$ is not complete in H. Then there exists a element $0\neq\nu\in H$, such that $(\nu,\omega_n)_H=0$. Namely, there exist such functions $\nu_1(x),\nu_2(x)\in L^2(0,\frac{\pi}{2})$, s.t.

$$\int_0^{\frac{\pi}{2}}\Big(\nu_1(t)\frac{a}{\lambda_n}\varphi_2'(\frac{\pi}{2},\lambda_n)\cos\lambda_n t+\nu_2(t)a^{-1}\varphi_2(\frac{\pi}{2},\lambda_n)\sin\lambda_n t\Big)dt=0 \tag{53}$$

Corresponding (3), (48) and $\Delta_1(\lambda_n)=0$, the equality (53) can be rewritten as the following form:

$$\int_0^{\frac{\pi}{2}}\Big(\nu_1(t)\frac{1}{\lambda_n}\varphi_1'(\frac{\pi}{2},\lambda_n)\cos\lambda_n t+\nu_2(t)\varphi_1(\frac{\pi}{2},\lambda_n)\sin\lambda_n t\Big)dt=0 \tag{54}$$

Thus,

$$V(\lambda):=\int_0^{\frac{\pi}{2}}\Big(\nu_1(t)\varphi_1'(\frac{\pi}{2},\lambda)\cos\lambda t-\lambda\nu_2(t)\varphi_1(\frac{\pi}{2},\lambda)\sin\lambda t\Big)dt \tag{55}$$

is entire function and has the zeros $\{\lambda_n\}_{n=-\infty}^{\infty}$.

Define $G_\varepsilon:=\Big\{\lambda:|\lambda|>|\lambda^\star|,\ \varepsilon<|\arg\lambda|<\pi-\varepsilon\Big\}$, where $\lambda^\star$ is any number in the complex plane and ε is some positive number and $G(\lambda):=\frac{V(\lambda)}{\Delta_1(\lambda)}$ which is entire in the complex plane

$\mathbb{C}$. It follows from (45), (46), (48) and (55) that

$$|V(\lambda)| \le Ce^{|\Im\lambda|\pi} \quad \text{and} \quad \Delta_1(\lambda) \ge Ce^{|\Im\lambda|\pi}, \quad \text{for } \lambda \in G_\varepsilon \tag{56}$$

Thus, we can arrive at

$$|G(\lambda)| = \Big|\frac{V(\lambda)}{\Delta_1(\lambda)}\Big| \le C, \qquad \text{for } \lambda \in G_\varepsilon, \tag{57}$$

where C is some positive constant. It yields

$$\frac{V(\lambda)}{\Delta_1(\lambda)} = O(1) \qquad \text{for } \lambda \in G_\varepsilon.$$

By Phragmen-Lindelöf's and Liouville's theorem, we can conclude that $V(\lambda) \equiv C\Delta_1(\lambda)$. Using (55), it is shown that $V \in \mathcal{L}^\pi$. However, (21) implies $\Delta_1(\lambda) \notin \mathcal{L}^\pi$. Thus $C = 0$ and $V(\lambda) = 0$ for all λ in $\mathbb{C}$. Taking (55) into account, it implies that $\nu_1 = \nu_2 = 0$. Namely, the element $\nu = 0$ in $L^2(0, \frac{\pi}{2})$, which is a contradiction. □

Lemma 4. *The following relation holds:*

$$\omega_n(x) = \omega_n^0(x) + O\Big(\frac{1}{n}\Big), \quad n \to \infty,$$

where $\omega_n^0(x) = \begin{pmatrix} -iae^{-i\lambda_n^0 \frac{\pi}{2}} \cos\lambda_n^0 x \\ a^{-1}e^{-i\lambda_n^0\frac{\pi}{2}} \sin\lambda_n^0 x \end{pmatrix}$, $n \in \mathbb{Z}$. *And the $O-$ estimate is uniform with respect to $x \in [0, \frac{\pi}{2}]$.*

Proof. It follows from (22), (23) and (50) that we can get the result. □

It follows from (23) that $\lambda_n^0 t = (2n - \frac{\ln\frac{a^-}{a^+}}{\pi i})t$. And the system $\Big\{\cos\Big(2n - \frac{\ln\frac{a^-}{a^+}}{\pi i}\Big)t\Big\}_{n\in\mathbb{Z}}$ and $\Big\{\sin\Big(2n - \frac{\ln\frac{a^-}{a^+}}{\pi i}\Big)t\Big\}_{n\in\mathbb{Z}}$ are Riesz bases in $L^2(0, \frac{\pi}{2})$, respectively (see [7]). Next, we will show that the system $\{\omega_n^0\}_{n\in\mathbb{Z}}$ is a Riesz basis in H. From the above lemma, we know that

$$\omega_{2n+1}^0(x) = \begin{pmatrix} -iae^{-i\lambda_{2n+1}^0\frac{\pi}{2}}\cos\lambda_{2n+1}^0 t \\ a^{-1}e^{-i\lambda_{2n+1}^0\frac{\pi}{2}}\sin\lambda_{2n+1}^0 t\end{pmatrix}, \ \omega_{2n+2}^0(x) = \begin{pmatrix} -iae^{-i\lambda_{2n+1}^0\frac{\pi}{2}}\cos\lambda_{2n+2}^0 t \\ a^{-1}e^{-i\lambda_{2n+2}^0\frac{\pi}{2}}\sin\lambda_{2n+2}^0 t\end{pmatrix}, \quad n\in\mathbb{Z}.$$

Lemma 5. *The system $\{\omega_n^0\}_{n\in\mathbb{Z}}$ is a Riesz basis in H.*

Proof. First, we can construct a linear operator $T_1 : H \to H$ and $T_2 : H \to H$ with a bounded inverse. Let

$$T_1\omega = T_1\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\omega_1\\ \omega_2 - f_1\end{pmatrix}, \quad T_1^{-1}\omega = \begin{pmatrix}\omega_1\\ \omega_2 + f_1\end{pmatrix},$$

$$T_2\omega = T_2\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\omega_1 - f_2\\ \omega_2\end{pmatrix}, \quad T_2^{-1}\omega = \begin{pmatrix}\omega_1 + f_2\\ \omega_2\end{pmatrix},$$

where $f_1(t) = ia^{-2}\sum\limits_{n=-\infty}^{\infty} c_{1,n}\sin(2n - \frac{\ln\frac{a^-}{a^+}}{\pi i})t$ and $\{c_{1,n}\}_{n\in\mathbb{Z}}$ are the coordinates of ω_1 with respect to the Riesz basis $\{\cos(2n - \frac{\ln\frac{a^-}{a^+}}{\pi i})t\}_{n\in\mathbb{Z}}$:

$$\omega_1(t) = \sum_{n=-\infty}^{\infty} c_{1,n}\cos(2n - \frac{\ln\frac{a^-}{a^+}}{\pi i})t,$$

and $f_2(t) = -ia^2 \sum\limits_{n=-\infty}^{\infty} c_{2,n} \cos(2n - \frac{\ln \frac{a^-}{a^+}}{\pi i})t$ and $\{c_{2,n}\}_{n\in\mathbb{Z}}$ are the coordinates of ω_2 with respect to the Riesz basis $\{\sin(2n - \frac{\ln \frac{a^-}{a^+}}{\pi i})t\}_{n\in\mathbb{Z}}$:

$$\omega_2(t) = \sum_{n=-\infty}^{\infty} c_{2,n} \sin(2n - \frac{\ln \frac{a^-}{a^+}}{\pi i})t.$$

Then, we have that

$$T_1\omega^0_{2n+1}(t) = -iae^{-i\frac{\pi}{2}(4n+2+\frac{\ln \frac{a^+}{a^-}}{\pi i})} \begin{pmatrix} \cos(4n+2-\frac{\ln \frac{a^-}{a^+}}{\pi i})t \\ 0 \end{pmatrix},$$

and

$$T_2\omega^0_{2n+2}(t) = a^{-1}e^{-i\frac{\pi}{2}(4n+4+\frac{\ln \frac{a^+}{a^-}}{\pi i})} \begin{pmatrix} 0 \\ \sin(4n+4-\frac{\ln \frac{a^-}{a^+}}{\pi i})t \end{pmatrix}.$$

So $\{\omega^0_n\}_{n\in\mathbb{Z}}$ is the Riesz basis in H. □

Proof of Theorem 5. Since $\lambda_n^2 = \tilde{\lambda}_n^2$, it follows from (22) that $\int_0^\pi q(t)dt = \int_0^\pi \tilde{q}(t)dt$ and from Theorem 4 that $q(x) = \tilde{q}(x)$ on $(\frac{\pi}{2}, \pi)$. Next, we should only prove that $q(x) = \tilde{q}(x)$ on $(0, \frac{\pi}{2})$. In view of

$$\int_0^\pi q(t)dt = \int_0^{\frac{\pi}{2}} q_1(t)dt - \int_{\frac{\pi}{2}}^0 q(\pi - t)dt = \int_0^{\frac{\pi}{2}} q_1(t)dt + \int_0^{\frac{\pi}{2}} q_2(t)dt \tag{58}$$

and $q_2(x) = \tilde{q}_2(x)$, we can get that $\int_0^{\frac{\pi}{2}} q_1(t)dt = \int_0^{\frac{\pi}{2}} \tilde{q}_1(t)dt$. So

$$\omega_{\frac{\pi}{2}} = -\frac{1}{2}\int_0^{\frac{\pi}{2}} q_1(t)dt = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \tilde{q}_1(t)dt = \tilde{\omega}_{\frac{\pi}{2}}.$$

From $\varphi_2(\frac{\pi}{2}, \lambda_n) = \tilde{\varphi}_2(\frac{\pi}{2}, \lambda_n)$ and $\omega_{\frac{\pi}{2}} = \tilde{\omega}_{\frac{\pi}{2}}$, we can also obtain that $g_n = \tilde{g}_n$ by (52). By virtue of the completeness of ω_n and using (51), we have that $N(t) = \tilde{N}(t)$. Then we can find $\frac{\varphi_1'(\frac{\pi}{2},\lambda)}{\varphi_1(\frac{\pi}{2},\lambda)} = \frac{\tilde{\varphi}_1'(\frac{\pi}{2},\lambda)}{\tilde{\varphi}_1(\frac{\pi}{2},\lambda)}$. The function $\frac{\varphi_1'(\frac{\pi}{2},\lambda)}{\varphi_1(\frac{\pi}{2},\lambda)}$ is the Weyl function. Thus, the potential $q_1(x)$ can be uniquely determined by the Weyl function [7]. □

According to the proof of uniqueness for Theorem 5, we can give the reconstruction of $q(x)$ on $(0, \pi)$ and the algorithm is as follows.

Algorithm 1. *Suppose that $d_1 = \frac{\pi}{2}$. If $\{\lambda_n^2\}_{n=-\infty}^{\infty}$ and q_2, a are known a priori, then we can find q_1.*

1. Construct $\varphi_2'(\frac{\pi}{2}, \lambda_n)$ by using q_2 and $\omega_{\frac{\pi}{2}}$ can be obtained.
2. Construct $\omega_n(t)$ and g_n by using (50) and (52).
3. Use the Riesz basis $\{\omega_n\}_{n=-\infty}^{\infty}$ and (51) to find $N(t)$:

$$N(t) = \sum_{n=-\infty}^{\infty} g_n\omega_n^\star(t),$$

where $\{\omega_n^\star(t)\}_{n\in\mathbb{Z}}$ is the Riesz basis which is biorthonormal to $\{\omega_n(t)\}_{n\in\mathbb{Z}}$.
4. Using $N_1(t)$, $N_2(t)$ of $N(t)$, construct $\varphi_1(\frac{\pi}{2}, \lambda)$ and $\varphi_1'(\frac{\pi}{2}, \lambda)$ by (45) and (46), respectively.
5. Construct q_1 from the Weyl function $\frac{\varphi_1'(\frac{\pi}{2},\lambda)}{\varphi_1(\frac{\pi}{2},\lambda)}$, solving the classical inverse problem by the method of spectral mappings [7].

§6 The case $0 < d < \frac{\pi}{2}$

Let I be a fixed subset of $\mathbb{Z}$. First, we give the uniqueness for the potential $q(x)$ on $(0, \pi)$.

Theorem 6. *Suppose that the system $\{\exp(\pm i\lambda_n x)\}_{n\in I}$ is complete in $L^2(-2d, 2d)$. If $\lambda_n = \tilde{\lambda}_n (n \in I)$, then $q_1 = \tilde{q_1}$ in $L^2(0, d)$.*

Lemma 6. *Suppose $\{\exp(\pm i\lambda_n x)\}_{n\in I}$ is complete in $L^2(-2d, 2d)$, Then, the system $\{\omega_n\}_{n\in I}$ defined by (50) is complete in H.*

Proof. Similarly, suppose that the systems $\{\omega_n\}_{n\in I}$ is not complete in H. Then there exists a element $0 \neq \nu \in H$, such that $(\nu, \omega_n)_H = 0$. From (55), we know that

$$V(\lambda) := \int_0^d \Big(\nu_1(t)\frac{1}{\lambda}\varphi_1'(d,\lambda)\cos\lambda t - \nu_2(t)\varphi_1(d,\lambda)\sin\lambda t\Big)dt \tag{59}$$

is entire. It follows from (45) and (46) that $V \in \mathcal{L}^{2d}$. Since the system $\{\exp(\pm i\lambda_n x)\}_{n\in I}$ is complete in $L^2(-2d, 2d)$ and using Paley-Wiener Theorem, it yields $V(\lambda) \equiv 0$. We can also get that $\nu_1 = \nu_2 = 0$ in $L^2(0, d)$. Thus, the system $\{\omega_n\}_{n\in I}$ is complete in H. □

Proof of Theorem 6. Since $\lambda_n^2 = \tilde{\lambda}_n^2$, it follows from (22) that $\int_0^\pi q(t)dt = \int_0^\pi \tilde{q}(t)dt$ and from Theorem 4 that $q(x) = \tilde{q}(x)$ on (d, π). In view of

$$\int_0^\pi q(t)dt = \int_0^d q_1(t)dt - \int_{\pi-d}^0 q(\pi - t)dt = \int_0^d q_1(t)dt + \int_0^{\pi-d} q_2(t)dt \tag{60}$$

and $q_2(x) = \tilde{q}_2(x)$, we can get that $\int_0^d q_1(t)dt = \int_0^d \tilde{q}_1(t)dt$. So

$$D_d = -\frac{1}{2}\int_0^d q_1(t)dt = -\frac{1}{2}\int_0^d \tilde{q}_1(t)dt = \tilde{D}_d.$$

From $\varphi_2(\pi - d, \lambda_n) = \tilde{\varphi}_2(\pi - d, \lambda_n)$ and $D_d = \tilde{D}_d$, we can also obtain that $g_n = \tilde{g}_n$ by (52). By virtue of the completeness of ω_n and using (51), we have that $N(t) = \tilde{N}(t)$. Then we can find $\frac{\varphi_1'(d,\lambda)}{\varphi_1(d,\lambda)} = \frac{\tilde{\varphi}_1'(d,\lambda)}{\tilde{\varphi}_1(d,\lambda)}$. While the function $\frac{\varphi_1'(d,\lambda)}{\varphi_1(d,\lambda)}$ is the Weyl function. Thus, the potential $q_1(x)$ can be uniquely determined by the Weyl function [7]. □

Suppose that $\{\omega_n\}_{n\in I}$ is a Riesz basis in H. Then, similar to the case when $d = \frac{\pi}{2}$, we can construct the potential $q(x)$ by the following algorithm.

Algorithm 2. *Suppose that $0 < d < \frac{\pi}{2}$. Let $\{\lambda_n^2\}_{n\in I}$ and q_2, a are known a priori, then we can find q_1.*

1. Construct $\varphi_2'(d_2, \lambda_n)$ by using q_2 and D_d can be obtained.

2. Construct $\omega_n(t)$ and g_n by using (50) and (52).

3. Use the Riesz basis $\{\omega_n\}_{n\in I}$ and (51) to find $N(t)$:

$$N(t) = \sum_{n\in I} g_n\omega_n^\star(t),$$

where $\{\omega_n^\star(t)\}_{n\in I}$ is the Riesz basis which is biorthonormal to $\{\omega_n(t)\}_{n\in I}$.

4. Using $N_1(t)$, $N_2(t)$ of $N(t)$, construct $\varphi_1(d,\lambda)$ and $\varphi_1'(d,\lambda)$ by (45) and (46), respectively.

5. Construct q_1 from the Weyl function $\frac{\varphi_1'(d,\lambda)}{\varphi_1(d,\lambda)}$, solving the classical inverse problem by the method of spectral mappings (see [7]).

Acknowledgement

The authors would like to thank the referees for valuable comments.

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