Flow of EMHD nanofluid in curved channel through corrugated walls

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Abstract. The present examination deals with the effects of nanofluids on corrugated walls under the influence of electromagnetohydrodynamic (EMHD) in the curved channel. Investigation is carried out by water based nanofluids using copper nanoparticle. Firstly performed the mathematical modelling by applying the method of perturbation, we have evaluated analytical solutions for the velocity and temperature. For the corrugations of the two walls periodic sine waves are described for small amplitude either in phase or out of phase. By using numerical calculations we analyzed the corrugation effects on the velocity and temperature for EMHD flow. The physical effects of flow variables like Hartmann number, Volumetric concentration of nanoparticles, Grashof number, Curvature parameter and Heat absorption coefficient are graphically discussed. Moreover, effect of Curvature parameter on Stresses and Nusselt number are discussed through tables. The velocity and temperature decreases when the curvature parameter are increase. The electromagnetohydrodynamic (EMHD) velocity and temperature distributions show that $0^\circ$ is the phase difference between the two walls for in phase and the phase difference is equal to the $180^\circ$ between two walls for out of phase. The important conclusion is that reduce the unobvious wave effect on the velocity and temperature for small value of amplitude ratio parameter.

§1 Introduction

Nanoparticle investigation is presently a region of effective scientific interest because of a gigantic scope of potential applications in biomedical, optical and electronic field. Nanotechnology incorporates the creation and employments of a material with nanoscale levels 1-100nm to yield items that display peculiar properties. In designing and restorative sciences nanofluid keeps a
few remarkable features. In nanotechnology field a particle describes as small objects which acts like an individual in term of properties and transport. Nanoparticles with selective properties has much attention for the researcher due to different applications accessible in the literature [1-3]. Nanofluid dynamics is newly developed branch which finds sundry applications in science, energetics and medical science. Nanoparticles extensively used for analysis, treatment, tranquilize conveyance and therapeutic gadget covering, for example, in Refs [4-6]. "Nanofluid" was first discovered by Choi [7] in order to increase heat transfer fluid with significantly higher conductivities. Nanofluids consist of tiny particles having diameter lesser than 100nm. The essential idea is to profit the nanoparticles as the agent to raise the thermal conductivities of base liquid. Nanofluids used as working fluid in order to enhance thermal behaviour. Buongiorno [8] suggested that the thermophoresis and Brownian movement assume a key role in the elements of nanofluids. Akbar et al. [9] explore the effect of metallic nanoparticles on incompressible fluid in asymmetric channel. Nadeem et al. [10] dissected the two phase nanofluid in a curved channel. Sheikholeslamic and shehzad [11] selected simple model demonstrating of nanofluids development through a permeable enclosure. Ellahi et al. [12-15] discussed the effect of nanofluids on magnetohydrodynamics (MHD) flow.

Microfluidics perform an crucial role in numerous mechanical procedures and applications, including detection, division and examination of concoction and organ examples and plan of heat and mass exchange frameworks. Microfluidics devices frequently utilized for manipulating and controlling fluids flows with length scales less than a millimeter. Investigations of such phenomena related to the mechanical component of colloid science and plant science and numerous established of the dynamics flows [16,17]. Microfluidics field gradually more consideration in both scholarly world and industry due to plausibility and productivity for controlling flows in microscale devices [18]. One of the important research areas in microfluidics is microelectro mechanical system because of its potential applications as an instrument for concentrate crucial physical and biochemical procedures and organ measures [19]. Most microfluidic framework require an independent dynamic pump of a size practically identical with the volume of liquid to be pumped. The key contemplations for them incorporate their reliability, control utilization, activation voltage, cost of fabrication and a dosing exactness similar with fuel pump [20].

The EMHD micropump is the essential microfluidic systems generates by a constant flow and has no moving part. Lorentz flow is used as pumping source for EMHD micropumps. Lorentz force is produced by connection of electric and magnetic fields. The EMHD micropump have numerous applications, for example, fluid blending, flow control in fluidic systems, fluid pumping, liquid chromatography and thermal reactors [21]. Chakraborty and paul [22] studies the fluid flow in parallel plates under the influence of EMHD force. The impact of nanofluid on MHD flow in microchannel was examined by Sarkar and Ganguly [23].

Ellahi at el. [24] analysed the effects of radiative electro-magnetohydrodynamics on entropy generation. Utilizing annoyance technique flow between microparallel plates through corrugated walls analyzed by Buren et al. [27]. Nanofluid effect on EMHD flow in microchannel is examined by Rashid et al.[26].

Recently corrugated walls have been utilized in improvement of heat and mass transfer. One
basic subject among these investigations is geometrical impacts because of wall corrugations on the flow opposition or pressure drop in the channel. The surface roughness is also mimics by wavy boundary, where roughness impacts are magnified by small scale of channel. Surface roughness incurred during creation process or adsorption. Chu [27] examined slip flow in corrugated walls in the annulus. Shu et al. [28] investigated about wall corrugations on wavy microchannel. Nadeem et al. [29] discussed the corrugation effect on EMHD flow in porous medium. Corrugation have great consideration for researcher and various applications are in the literature such as [30-31].

In all above indicated efforts flow considered between two dimensional or asymmetric channels. In real world issues curved channel are essential significance. In simple channel most of the practical application do not encounter, curved channel increase significantly more important in veins, intestines and arteries. Sato et al. [32] firstly analyzed the flow in a rectangular curved channel for a viscous fluid. In curved channel unsteady transport inspected by Ramanamurthy et al. [33]. Hina et al. [34-36] discussed the effect of nanofluid on curved channel on ciliary motion. Nadeem et al. [37,38] investigated the effect of nanoparticles in the curved channel. Fluid motion in a curved channel has been discussed by Wi dean et al. [39]. The mathematical analysis for peristaltic flow of hyperbolic tangent fluid in a curved channel has been examined by Nadeem et al. [40]. In the curved channel flow of the pseudoplastic fluid using wall properties and slip conditions were explored by Hina et al. [41].

Inspiration of present analysis is to explore the effect of corrugations on EMHD nanofluid model in curved channel with no slip effects. Significant modeling for nanofluid in a curved channel is obtainable with the help of dimensionless parameters and using the perturbation technique. Plots for different flow quantities of interest are investigated and displayed.

§2 Mathematical Formulation

We consider a EMHD flow of laminar, an incompressible and electrically conducting fluid between corrugated wall in the curved channel separated by a distance $2H$, center at $O$ and radius $R'$ is considered. The flow induced in the channel by sinusoidal waves with small amplitude $\varepsilon$ along the corrugated walls of the channel. The wavy upper and lower walls are located at
$$r_u^* = H + \varepsilon H \sin (\lambda^* x^*) \quad \text{and} \quad r_l^* = -H \pm \varepsilon H \sin (\lambda^* x^*)$$

The corrugations of wavy walls are periodic sinusoidal where $\lambda^*$ is wave number and $\varepsilon$ is amplitude either in phase or out of phase. Lorenz force $\boldsymbol{J} \times \boldsymbol{B}^*$ is generated by interaction between electric and magnetic field, where $\boldsymbol{J} = \sigma (\nabla^* + \nabla^* \times \boldsymbol{B}^*)$ symbolize the current density.

Defining the velocity field as
$$\nabla^* = (0, 0, w^*(X^*, R^*)) \quad (1)$$

We assumed liquid is incompressible fluid in the curved channel and only taken in the $z^*$ direction. For present flow we considered the following basic equations [42-43].

Continuity equation
\( \nabla^* \hat{u}^* = 0. \)  

(2)

Equation of motion

\[
\frac{1}{R + R^*} \frac{\partial}{\partial R^*} \left( R^* + \dot{R} \right) \tau_{Z^* R^*}^* + \frac{\dot{R}}{R + R^*} \tau_{Z^* X^*}^* + \sigma_{nf} B^* (E^* - (\frac{\dot{R}}{R + R^*})^2 B^* w^*) \\
+ g(\rho \zeta)_{nf} (T^* - T_\text{ref}^*) = 0.
\]

(3)

Energy equation

\[
k_{nf} \left( \frac{1}{R + R^*} \frac{\partial}{\partial R^*} \left( R^* + \dot{R} \right) \tau_{Z^* R^*}^* + \frac{\dot{R}}{R + R^*} \tau_{Z^* X^*}^* + \sigma_{nf} B^* (E^* - (\frac{\dot{R}}{R + R^*})^2 B^* w^*) \\
+ g(\rho \zeta)_{nf} (T^* - T_\text{ref}^*) \right) + Q_0 = 0.
\]

(4)

In addition, assuming the channel is open in \( z^* \) direction and pressure gradient can be ignored, the Navier-Stokes Eq. (3) along the \( z^* \) direction \[44\] can be written as

\[
\frac{1}{R + R^*} \frac{\partial}{\partial R^*} \left( R^* + \dot{R} \right) \tau_{Z^* R^*}^* + \frac{\dot{R}}{R + R^*} \tau_{Z^* X^*}^* + \sigma_{nf} B^* (E^* - (\frac{\dot{R}}{R + R^*})^2 B^* w^*) \\
+ g(\rho \zeta)_{nf} (T^* - T_\text{ref}^*) = 0,
\]

(5)

where

\[
\tau_{Z^* R^*}^* = \mu_{nf} \frac{\partial w^*}{\partial R^*}, \quad \tau_{Z^* X^*}^* = \mu_{nf} \frac{\dot{R}}{R + R^*} \frac{\partial w^*}{\partial X^*}.
\]

(6)

Corresponding no-slip boundary conditions expressed as

\[
w^* (X^*, R_u^*) = 0 \text{ at } R_u^* = H + \varepsilon H \sin (\lambda^* X^*),
\]
\[
w^* (X^*, R_l^*) = 0 \text{ at } R_l^* = -H + \varepsilon H \sin (\lambda^* X^*),
\]
\[
T^* (X^*, R_u^*) = T_{\text{ref}}^* (X^*, R_u^*) \text{ at } R_u^* = H + \varepsilon H \sin (\lambda^* X^*),
\]
\[
T^* (X^*, R_l^*) = T_{\text{ref}}^* (X^*, R_l^*) \text{ at } R_l^* = -H + \varepsilon H \sin (\lambda^* X^*).
\]

(7)
Bring out the following dimensionless variables

\[ (r, x) = \left( \frac{r^*}{H}, \frac{x^*}{H} \right), \lambda = \lambda^* H, \quad w = \frac{w^*}{H}, \quad k = \frac{k^*}{H}, \quad Ha = B^* H \left( \frac{\sigma}{\mu} \right)^{1/2}, \]

\[ \beta = E_0 \left( \frac{\sigma}{\mu} \right)^{1/2} / U, \quad \theta = \frac{T-T_i}{T_u-T_i}, \quad G_r = \frac{\theta(H(T_u-T_i))}{\mu^* U}, \quad \phi = \frac{Q_0 H^2}{\mu^* U}. \]

Here \( Ha \) is the Hartman number, \( Gr \) is Grashof number, \( \beta \) is non-dimensional parameter, \( \theta \) is the temperature, \( \phi \) is heat absorption coefficient and \( k \) is the curvature parameter. The dimensionless momentum and temperature equation stated in the following form

\[
\frac{\mu_f}{\mu} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r + k} \frac{\partial w}{\partial r} + \left( \frac{k}{r + k} \right)^2 \frac{\partial^2 w}{\partial x^2} \right) + \frac{\sigma_{nf}}{\sigma_f} (Ha \beta + \left( \frac{k}{r + k} \right)^2 Ha^2 w) + \frac{\rho_f \beta_f}{\rho_f} Gr \theta = 0, \tag{9}
\]

\[
\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r + k} \frac{\partial \theta}{\partial r} + \left( \frac{k}{r + k} \right)^2 \frac{\partial^2 \theta}{\partial x^2} + \phi \frac{k_f}{k_{nf}} = 0. \tag{10}
\]

The corresponding non-dimensional boundary conditions define a

\[
w = \begin{cases} 0, & r_u = 1 + \varepsilon \sin(\lambda x), & r_l = -1 \pm \varepsilon \sin(\lambda x), \end{cases}, \tag{11}
\]

\[
\theta = \begin{cases} 1, & r_u = 1 + \varepsilon \sin(\lambda x), & \theta = 0, & r_l = -1 \pm \varepsilon \sin(\lambda x). \end{cases} \tag{12}
\]

§3 Solution of Problem

By using regular perturbation technique in above equation:

\[
w (r, x) = w_0 (r) + \varepsilon w_1 (r, x) + \varepsilon^2 w_2 (r, x) + \ldots \tag{13}
\]

\[
\theta (r, x) = \theta_0 (r) + \varepsilon \theta_1 (r, x) + \varepsilon^2 \theta_2 (r, x) + \ldots \tag{14}
\]

Equating the like power of \( \varepsilon \) after using the Eq. (13), Eq. (14) into Eqs. (9) to (12) we get the following systems as

3.1 Zeroth Order System

\[
\frac{\mu_f}{\mu} \left( \frac{\partial^2 w_0}{\partial r^2} + \frac{1}{r + k} \frac{\partial w_0}{\partial r} + \frac{\sigma_{nf}}{\sigma_f} (Ha \beta + \left( \frac{k}{r + k} \right)^2 Ha^2 w_0) + \frac{\rho_f \beta_f}{\rho_f} Gr \theta_0 = 0, \right. \tag{15}
\]

\[
\left. \frac{\partial^2 \theta_0}{\partial r^2} + \frac{1}{r + k} \frac{\partial \theta_0}{\partial r} + \phi \frac{k_f}{k_{nf}} = 0, \right. \tag{16}
\]

\[
\begin{align*}
\theta_0 & | \quad r=1, \quad \theta_0 |_{r=-1} = 0, \tag{17} \\
w_0 & | \quad r=1, \quad w_0 |_{r=-1} = 0. \tag{18}
\end{align*}
\]

3.2 First Order System

\[
\frac{\mu_f}{\mu} \left( \frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r + k} \frac{\partial w_1}{\partial r} + \left( \frac{k}{r + k} \right)^2 \frac{\partial^2 w_1}{\partial x^2} \right) + \frac{\sigma_{nf}}{\sigma_f} \left( \frac{k}{r + k} \right)^2 Ha^2 w_1 + \frac{\rho_f \beta_f}{\rho_f} Gr \theta_1 = 0, \tag{19}
\]

\[
\frac{\mu_f}{\mu} \left( \frac{\partial^2 \theta_1}{\partial r^2} + \frac{1}{r + k} \frac{\partial \theta_1}{\partial r} + \left( \frac{k}{r + k} \right)^2 \frac{\partial^2 \theta_1}{\partial x^2} \right) = 0. \tag{20}
\]
\[ \theta_1 |_{r=1} = - \sin(\lambda x) \left( \frac{d\theta_0}{dr} \right)_{r=1}, \quad \theta_1^+ |_{r=-1} = \mp \sin(\lambda x) \left( \frac{d\theta_0}{dr} \right)_{r=-1}. \] (21)

\[ w_1 |_{r=1} = - \sin(\lambda x) \left( \frac{dw_0}{dr} \right)_{r=1}, \quad w_1^+ |_{r=-1} = \mp \sin(\lambda x) \left( \frac{dw_0}{dr} \right)_{r=-1}. \] (22)

### 3.3 Second Order System

\[ \frac{\partial^2 \theta_2}{\partial r^2} + \frac{1}{r + k} \frac{\partial \theta_2}{\partial r} + \left( \frac{k}{r + k} \right)^2 \frac{\partial^2 \theta_2}{\partial x^2} = 0, \] (23)

\[ \frac{\mu_{nf}}{\mu_f} \left( \frac{\partial^2 w_2}{\partial r^2} + \frac{1}{r + k} \frac{\partial w_2}{\partial r} + \left( \frac{k}{r + k} \right)^2 \frac{\partial^2 w_2}{\partial x^2} \right) + \frac{\sigma_{nf} (k/\sigma_f)^2 \Lambda^2}{(\rho \zeta)^2} w_2 + \frac{(\rho \zeta)_{nf}}{(\zeta)^f} G \theta_2 = 0, \] (24)

\[ \theta_2 |_{r=1} = - \sin(\lambda x) \left( \frac{d\theta_0}{dr} \right)_{r=1} - \frac{1}{2} \sin^2(\lambda x) \left( \frac{d^2\theta_0}{dr^2} \right)_{r=1}, \] (25)

\[ \theta_2^+ |_{r=-1} = \mp \sin(\lambda x) \left( \frac{d\theta_0}{dr} \right)_{r=-1} - \frac{1}{2} \sin^2(\lambda x) \left( \frac{d^2\theta_0}{dr^2} \right)_{r=-1}, \] (26)

### 3.3.1 Solution of zeroth order system

Under the boundary conditions the zero-order solution can be expressed as:

\[ \theta_0 (r) = A_2 + \left( -\frac{1}{2} \phi (r(2k + r) - 2k^2 \ln(k + r))k_f + 2A_1 \ln(k + r) \right) k_{nf} / 2k_{nf}, \] (27)

with

\[ w_0(y) = \left\{ \begin{array}{l}
(-Gr(k + r)^2 \phi k_f \mu_f (\rho \zeta)_{nf} \sigma_f (16(-5k^2 - 2kr - 2r^2 + 8k^3 \ln(k + r)))u_{nf}^2 \sigma_f^2 \ Theta^2 + k^2 \Lambda^2 \\
(k^2 - 4kr - 2r^2 + 10k^2 \ln(k + r)) \mu_f \mu_{nf} \sigma_f \sigma_{nf} + k^4 \Lambda^4 (-r(2k + r) + 2 \ln(k + r))
\end{array} \right. \] (28)

\[ k^2 u_{nf}^2 \sigma_f^2 \]

\[ + 4k_{nf} f (B_1 \cos(a_1 \ln(k + r))) b(\rho \zeta) f + B_2 \sin(a_1 \ln(k + r)) (\rho \zeta) f b + \mu_f \\
(1 + r)^2 (4 \mu_{nf} \sigma_f (-Gr(-A_1 + A_2 + A_1 \ln(k + r)) (\rho \zeta)_{nf} \sigma_f - c) - k^2 \Lambda^2 \mu_f \sigma_f \sigma_{nf} (Gr \\
(A_2 + A_1 \ln(k + r)) (\rho \zeta)_{nf} \sigma_f + a_1))) / (4(\rho \zeta) f k_{nf} b a), \] (28)
3.3.2 Solution of first order system

Under the boundary conditions (21) and (22), we can assume the solution of the first order system as

\[
\begin{align*}
\theta_1(r, x) &= \sin(\lambda x) f(r), \\
w_1(r, x) &= \sin(\lambda x) g(r),
\end{align*}
\]

where \( f(r) \) and \( g(r) \) is function of \( r \). Using Eq.(29) into Eq.(19) to (22), we get

\[
\frac{d^2 f(r)}{dr^2} + \frac{1}{(k + r)} \frac{df(r)}{dr} - \frac{k^2}{(k + r)^2} \lambda^2 f(r) = 0,
\]

(30)

Correspondingly boundary conditions are transformed as

\[
f_{r=1} = \frac{d\theta_0}{dr}, \quad f_{r=-1} = \pm \frac{d\theta_0}{dr},
\]

(32)

and

\[
g_{r=1} = \frac{dw_0}{dr}, \quad g_{r=-1} = \pm \frac{dw_0}{dr}.
\]

(33)

The solution of first order problem can be expressed as

\[
\begin{align*}
\theta_1^\pm(r, x) &= \sin(\lambda x) \left\{ \begin{array}{ll}
C_1 \cosh(k \lambda \ln(k + r)) + iC_2 \sinh(k \lambda \ln(k + r)), \\
C_1' \cosh(k \lambda \ln(k + r)) + iC_2' \sinh(k \lambda \ln(k + r)).
\end{array} \right.
\end{align*}
\]

(34)

\[
\begin{align*}
D_1 \cosh(a_2 \ln(k + r)) + iD_2 \sinh(a_2 \ln(k + r)) + (Gr(k + r)^2 \mu_f (\rho \zeta)_n \sigma_f & \\
4((C_1' - iC_2' k \lambda) \cosh(k \lambda \ln(k + r)) + i(C_2' + iC_1 k \lambda) \sinh(k \lambda \ln(k + r))) \mu_{n_f} & \\
\sigma_f \sinh(k \lambda \ln(k + r)) + k^2 Ha^2 C_1 \cosh(k \lambda \ln(k + r)) + \sinh(k \lambda \ln(k + r))) & \\
iC_2 \mu_f(k \lambda \sigma_f)) / ((\rho \zeta)_f 16(-1 + k^2 \lambda^2) w_n^2 f^2 - 8k^2 Ha^2 \mu_\sigma_f \sigma_f \sigma_n f - k^4 Ha^4 \mu_\sigma_f^2 \sigma_n^2) & \\
D_1' \cosh(a_2 \ln(k + r)) + iD_2' \sinh(a_2 \ln(k + r)) + (Gr(k + r)^2 \mu_f (\rho \zeta)_n \sigma_f & \\
4((C_1' - iC_2' k \lambda) \cosh(k \lambda \ln(k + r)) + i(C_2' + iC_1 k \lambda) \sinh(k \lambda \ln(k + r))) \mu_{n_f} & \\
\sigma_f + k^2 Ha^2 C_1' \cosh(k \lambda \ln(k + r)) + iC_2' \sinh(k \lambda \ln(k + r))) \mu_{n_f} \sigma_f & \\
((\rho \zeta)_f 16(-1 + k^2 \lambda^2) w_n^2 \sigma_f^2 - 8k^2 Ha^2 \mu_\sigma_f \sigma_f \sigma_n f - k^4 Ha^4 \mu_\sigma_f^2 \sigma_n^2) & \\
a_2 = \frac{k \sqrt{\lambda^2 \mu_{n_f} \sigma_f - Ha^2 \mu_f \sigma_n f}}{\sqrt{\mu_{n_f} \sigma_f}}.
\end{align*}
\]

(35)

3.3.3 Solution of second order system

The boundary conditions (25) and (26) of the second order system can be simplified by using the solution of first and second order system. Under the boundary conditions, we can suppose the solution of second order system as

\[
\begin{align*}
\theta_2^\pm(r, x) &= h^\pm(r) + \cos(2\lambda x) k^\pm(r), \\
w_2^\pm(r, x) &= m^\pm(r) + \cos(2\lambda x) n^\pm(r),
\end{align*}
\]

(36)

(37)
where $h^\pm(r)$, $k^\pm(r)$, $m^\pm(r)$ and $n^\pm(r)$ are function of $r$ only. By utilizing Eq. (36) and Eq. (37) into Eq. (23) and Eq. (24), we get the following forms

\begin{equation}
\frac{d^2 h^\pm(r)}{dr^2} + \frac{1}{k + r} \frac{dh^\pm(r)}{dr} = 0,
\end{equation}

\begin{equation}
\frac{d^2 k^\pm(r)}{dr^2} + \frac{1}{k + r} \frac{dk^\pm(r)}{dr} - 4\lambda^2 k^2 = 0,
\end{equation}

\begin{equation}
\frac{\mu_n f}{\mu_f} \left( \frac{d^2 m^\pm(r)}{dr^2} + \frac{1}{(k + r)} \frac{dm^\pm(r)}{dr} \right) + \frac{\sigma_n f}{\sigma_f} \left( \frac{k}{r + k} \right)^2 H a^2 m^\pm(r) + \frac{(\rho c)_n f}{(\rho c)_f} Gr h^\pm(r) = 0,
\end{equation}

\begin{equation}
\frac{\mu_n f}{\mu_f} \left( \frac{d^2 n^\pm(r)}{dr^2} + \frac{1}{(k + r)} \frac{dn^\pm(r)}{dr} \right) - 4\lambda^2 \left( \frac{k}{r + k} \right)^2 n^\pm(r) + \frac{\sigma_n f}{\sigma_f} \left( \frac{k}{r + k} \right)^2 H a^2 n^\pm(r)
\end{equation}

\begin{equation}
+ \frac{(\rho c)_n f}{(\rho c)_f} Gr k^\pm(r) = 0.
\end{equation}

The boundary conditions of the functions $h^\pm(r)$, $k^\pm(r)$, $m^\pm(r)$ and $n^\pm(r)$ are respectively

\begin{equation}
h^\pm = \begin{cases} -\frac{1}{2} \left( \frac{df}{dr} + \frac{1}{2} \frac{d^2 \theta_0}{dr^2} \right) & \text{at } r = 1, \\
-\frac{1}{2} \left( \frac{df}{dr} - \frac{1}{2} \frac{d^2 \theta_0}{dr^2} \right) & \text{at } r = 1,
\end{cases}
\end{equation}

\begin{equation}
k^\pm = \begin{cases} \frac{1}{2} \left( \frac{df}{dr} + \frac{1}{2} \frac{d^2 \theta_0}{dr^2} \right) & \text{at } r = 1, \\
-\frac{1}{2} \left( \frac{df}{dr} - \frac{1}{2} \frac{d^2 \theta_0}{dr^2} \right) & \text{at } r = 1,
\end{cases}
\end{equation}

\begin{equation}
m^\pm = \begin{cases} \frac{1}{2} \left( \frac{df}{dr} + \frac{1}{2} \frac{d^2 \theta_0}{dr^2} \right) & \text{at } r = 1, \\
-\frac{1}{2} \left( \frac{df}{dr} - \frac{1}{2} \frac{d^2 \theta_0}{dr^2} \right) & \text{at } r = 1,
\end{cases}
\end{equation}

\begin{equation}
n^\pm = \begin{cases} \frac{1}{2} \left( \frac{df}{dr} + \frac{1}{2} \frac{d^2 \theta_0}{dr^2} \right) & \text{at } r = 1, \\
-\frac{1}{2} \left( \frac{df}{dr} - \frac{1}{2} \frac{d^2 \theta_0}{dr^2} \right) & \text{at } r = 1.
\end{cases}
\end{equation}

Thus the solution of second order system can be obtained as

\begin{equation}
\theta^\pm_2 = \begin{cases} E_2 + E_1 \ln(k + r) + \cos(2\lambda x) \left( G_1 \cosh(2k\lambda \ln(k + r)) + iG_2 \sinh(2k\lambda \ln(k + r)) \right), \\
E_2' + E_1' \ln(k + r) + \cos(2\lambda x) \left( G_1' \cosh(2k\lambda \ln(k + r)) + iG_2' \sinh(2k\lambda \ln(k + r)) \right),
\end{cases}
\end{equation}

\begin{equation}
w^\pm_2 = \begin{cases} F_1 \cosh(a_1 \ln(k + r)) + F_2 \sinh(a_1 \ln(k + r)) + \left( (Gr(k + r))^2 \mu_f (\rho c)_n f \sigma_f \left( -4( -E_1 + E_2') E_1' \ln(k + r)) \mu_n f \sigma_f - k^2 Ha^2 (E_2 + E_1 \ln(k + r)) \mu_f \sigma_n f) / ((\rho c)_f b) \right) \\
+ \cos(2\lambda x) \left( H_1 \cosh(a_3 \ln(k + r)) + iH_2 \sinh(a_3 \ln(k + r)) + \left( (Gr(\rho c)_n f \mu_f \sigma_f + (cosh(2k\lambda \ln(k + r)) + i sinh(2k\lambda \ln(k + r))) (4G_2 + 2iG_1 k\lambda)) \mu_n f \sigma_f + G_1 k^2 \\
Ha^2 \mu_f \sigma_n f) / ((\rho c)_f b) \right) \right) / (16(1 + 4k^2 \lambda^2) u^2_{n f} + 8k^2 Ha^2 \mu_f \mu_n f \sigma_f \sigma_{n f} - k^4 Ha^4 \mu^2_f \sigma^2_f) \right), \\
F_1' \cosh(a_1' \ln(k + r)) + F_2' \sinh(a_1 \ln(k + r)) + \left( (Gr(k + r))^2 \mu_f (\rho c)_n f \sigma_f \left( -4( -E_1' + E_2') E_1' \ln(k + r)) \mu_n f \sigma_f - k^2 Ha^2 (E_2' + E_1' \ln(k + r)) \mu_f \sigma_n f) / ((\rho c)_f b) \right) \\
+ \cos(2\lambda x) \left( H_1' \cosh(a_3 \ln(k + r)) + iH_2' \sinh(a_3 \ln(k + r)) + \left( (Gr(k + r))^2 (\rho c)_n f \mu_f \sigma_f (cosh(2k\lambda \ln(k + r)) + i sinh(2k\lambda \ln(k + r))) (4G_2' + 2iG_1' k\lambda) \mu_n f \sigma_f + G_1' k^2 Ha^2 \mu_f \sigma_n f) + i sinh(2k\lambda \ln(k + r)) \right) (4G_2' + 2iG_1' k\lambda) \mu_n f \sigma_f + G_1' k^2 Ha^2 \mu_f \sigma_n f) / ((\rho c)_f b) \right) \right) / (16(1 + 4k^2 \lambda^2) u^2_{n f} + 8k^2 Ha^2 \mu_f \mu_n f \sigma_f \sigma_{n f} - k^4 Ha^4 \mu^2_f \sigma^2_f) \right). 
\end{cases}
\end{equation}
with

\[
a_3 = \frac{k\sqrt{4\lambda^2\mu_n\sigma_f - Ha^2\mu_f\sigma_{nf}}}{\sqrt{\mu_n\sigma_f}}.
\]

Collecting Eqs. (27), (28), (34), (35), (46), and (47), the approximate velocity and temperature solution can be written as

\[
\theta^\pm (r, x) = \theta_0 (r) + \varepsilon\theta_1^\pm (r, x) + \varepsilon^2\theta_2^\pm (r, x) + ...
\]

and

\[
w^\pm (r, x) = w_0 (r) + \varepsilon w_1^\pm (r, x) + \varepsilon^2 w_2^\pm (r, x) + ...
\]

Evaluation of constants have been done by using Mathematica 9.

§4 Heat transfer rate

We can determine the Nusselt number depicting the strength of convective heat exchange. The Nusselt number is defined as follows

\[
Nu = \frac{Hq_w}{k_f(T_u - T_l^*)},
\]

On upper and lower walls we defined

\[
q_w = -k_n f \frac{\partial T^*}{\partial r^*} |_{r^* = r_u^*},
\]

From Eqs. (50) and (51), the Nusselt number can be expressed as

\[
Nu = -\frac{k_n f \frac{\partial \theta}{\partial r}}{k_f} |_{r = r_u}.
\]

Table 1 Thermophysical properties of the engine oil and the nanoparticles (SWCNT, MWCNT).

<table>
<thead>
<tr>
<th>Physical Properties</th>
<th>Water</th>
<th>Copper</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_p(J/kgK))</td>
<td>4179</td>
<td>385</td>
</tr>
<tr>
<td>(\rho(kg/m^3))</td>
<td>997.1</td>
<td>8933</td>
</tr>
<tr>
<td>(k(W/mK))</td>
<td>0.613</td>
<td>400</td>
</tr>
<tr>
<td>(\zeta \times 10^{-5}\ (1/K))</td>
<td>21.0</td>
<td>1.67</td>
</tr>
<tr>
<td>(\sigma(S/m))</td>
<td>5.0 \times 10^{-2}</td>
<td>5.96 \times 10^7</td>
</tr>
<tr>
<td>(\mu(kg/m.sec))</td>
<td>8.90 \times 10^{-4}</td>
<td>-</td>
</tr>
</tbody>
</table>

§5 Results and discussion

In this part, the graphical impacts of viscous fluid is investigated by EMHD flow in a curved channel with corrugated walls. All graphical outcomes are achieved by utilizing the MATLAB programming. This segment is explicitly arranged to investigate the impact of various inserted parameters on the different flow quantities. Plots 2D variations of velocity and contour are appeared dismembered through Figs. 2 to 9 respectively.

The contour plots obtained from the solutions of the velocity \(w^\pm\) and temperature \(\theta^\pm\) are exhibited in Eqs.48 and 49 for various estimations of parameters. In Figs. 2 to 5, the velocities \(w^\pm\) and temperature \(\theta^\pm\) are plotted against \(r\) for various estimations of parameters...
the Hartman number $Ha$, Volumetric concentration of nanoparticles $\Phi$, Grashof number $Gr$, Curvature parameter $k$ and Heat absorption coefficient $\phi$ when we take $\epsilon = 0.1$ and $\beta = 5$. Specifically, it can be found in Figs.2 to 4, that the velocity amplitude accomplishes the maximum value at the middle of the channel. Fig. 2(a) shows that the velocity $w^\pm$ increases with expanding Hartman number. Fig. 2(b) demonstrates that the velocity $w^\pm$ decline for various estimations of Volumetric concentration of nanoparticles $\Phi$. Fig. 3(a) shows that the velocity $w^\pm$ increases with Grashof number $Gr$. Fig. 3(b) shows that the velocity $w^\pm$ decrease for different values of curvature parameter $k$. Fig. 4 admits that EMHD velocity $w^\pm$ increases for different values of Heat absorption coefficient $\phi$. We can find that, the EMHD velocities in phase are weaker than out of phase. Figs. 5(a) shows that profile of temperature $\theta^\pm$ decreases when the curvature parameter $k$ are increase. Fig. 5(b) depicts the effect of Heat absorption coefficient $\phi$ on temperature profile. Profile of temperature increases when the $\phi$ increases.

The contour plots acquired from the solutions of the velocity $w^\pm$ and temperature $\theta^\pm$ for different values of parameters shown in Figs. 6 to 9. Contour for velocity $w^\pm$ shown in Figs. 6 to 7. In Figs. 6(a) and (b) the phase difference among the two corrugated walls equals $0^\circ$. Figs. 6(a) and (b) shows that the number of bolus increase with increasing the value of curvature parameter $k$. In Figs. 7(a) and (b), the phase difference between walls equals $180^\circ$. It is found that trapped bolus increase with increasing the value of curvature parameter $k$ out-of-phase corrugations when $\epsilon$ is small. The contour plots for temperature $\theta^\pm$ for different values of parameters shown in Figs.8(a) and 9. In Figs. 8(a) and (b) the phase difference among the two corrugated walls equals $0^\circ$ and in Figs. 9(a) and (b), the phase difference between walls equals $180^\circ$. As shows in Figs. 2 to 9, the wave phenomenon of the flow shape becomes obvious with the expansion of the corrugation. The wavy pattern increases by increase the value of parameters.

5.1 Tables Description

In this section, the impact of stress components and Nusselt number on EMHD flow of nanofluid discussed in a curved channel through corrugated walls. This section expressed the

![Figure 2](image-url)  
Figure 2. Show the effect of Hartman number (a) $Ha$ and Volumetric concentration of nanoparticles (b) $\Phi$ on velocity.
Figure 3. Show the effect of Grashof number (a) $Gr$ and Curvature parameter (b) $k$ on velocity.

Figure 4. Shows the effect of Heat absorption coefficient $\phi$ on velocity.

Figure 5. Show the effect of Curvature parameter (a) $k$ and Heat absorption coefficient (b) $\phi$ on temperature.
Figure 6. Velocity contour for Curvature parameter (a) $k = 1.5$ and (b) $k = 3.5$ in phase.

Figure 7. Velocity contour for Curvature parameter (a) $k = 1.5$ and (b) $k = 3.5$ out of phase.

Figure 8. Temperature contour for Curvature parameter (a) $k = 1.5$ and (b) $k = 3.5$ in phase.
Figure 9. Temperature contour for Curvature parameter (a) $k = 1.5$ and (b) $k = 3.5$ out of phase.

behavior of curvature parameter $k$ on the stress components $\tau_{zr}^\pm$ and $\tau_{zx}^\pm$ and Nusselt number $Nu^\pm$. Table 2 demonstrate that the stress component $\tau_{zr}^+$ and $\tau_{zr}^-$ decrease with the increasing value of $r$ and furthermore decrease with the raise of the curvature parameter $k$. Table 3 shown that the stress component $\tau_{zx}^+$ and $\tau_{zx}^-$ increase with the increasing value of $r$ and also increase with the increment in the value of curvature parameter $k$. The impact of Nusselt number $Nu=-\frac{\kappa_n}{\kappa_f}\theta(y_0)$ on EMHD flow of nanofluid discussed in microchannel through corrugated walls. This section expressed the behavior of curvature parameter $k$ on the Nusselt number $Nu^\pm$. Table 4 and Table 5 demonstrate that the Nusselt number $Nu^\pm$ decrease with the increasing value of $x$ and increase with increment in the value of curvature parameter $k$.

Table 2. Effect of curvature parameter $k$ on Stress components $\tau_{rz}^\pm = \frac{\mu_n \tau}{\mu_f \frac{\partial w}{\partial r}}$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$k = 2.5$</th>
<th>$k = 4.0$</th>
<th>$k = 5.0$</th>
<th>$k = 2.5$</th>
<th>$k = 4.0$</th>
<th>$k = 5.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>5.00097</td>
<td>4.38566</td>
<td>3.28585</td>
<td>4.90409</td>
<td>4.34458</td>
<td>4.20501</td>
</tr>
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<td>-0.8</td>
<td>3.77671</td>
<td>3.52612</td>
<td>2.56966</td>
<td>3.67526</td>
<td>3.4262</td>
<td>3.35829</td>
</tr>
<tr>
<td>-0.6</td>
<td>2.64766</td>
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<td>1.84091</td>
<td>2.54358</td>
<td>2.50167</td>
<td>2.48748</td>
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<tr>
<td>-0.4</td>
<td>1.61499</td>
<td>1.78832</td>
<td>1.11609</td>
<td>1.50944</td>
<td>1.58999</td>
<td>1.61143</td>
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<tr>
<td>-0.2</td>
<td>0.67171</td>
<td>0.943975</td>
<td>0.407727</td>
<td>0.565348</td>
<td>0.703732</td>
<td>0.744382</td>
</tr>
<tr>
<td>0</td>
<td>-0.191463</td>
<td>0.129656</td>
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§6 Conclusion

The effect of corrugated wall roughness on the viscous EMHD flow in a curved channel is determined in this paper. Perturbation technique is applied to inspect the issue. From the
above outcomes, the accompanying reasonings are drawn. The wavy phenomenon in the center becomes obvious when the amplitude \( \varepsilon \) is small with in and out of phase corrugations. The contour plots from the solutions of the velocity, it is found that trapped bolus are appear for out-of-phase corrugations. The wavy phenomenon increases by increase the estimation of parameters. The velocity amplitude accomplishes the maximum value at the center of the channel. The velocity increase for different values of Hartman number \( Ha \) and \( Gr \). The velocity \( w^\pm \) decline for various estimations of Volumetric concentration of nanoparticles \( \Phi \), and curvature parameter \( k \). The EMHD velocity \( w^\pm \) increases for with increment in the value of Heat absorption coefficient \( \phi \). The profile of temperature \( \theta^\pm \) decreases when the curvature parameter \( k \). The temperature increases when the \( \phi \) increases. The number of bolus increase with increasing the value of curvature parameter \( k \) in phase and out of phase corrugations when \( \varepsilon \) is small. The EMHD velocities in phase are weaker than out of phase. Stress component \( \tau^\pm_{r^2} \) decrease with the raise of the curvature parameter while stress component \( \tau^\pm_{r^2} \) increase with the increasing value of the curvature parameter. The Nusselt number \( Nu^\pm \) decrease with the

---

**Table 3. Effect of curvature parameter \( k \) on Stress components \( \tau^\pm_{r^2} = \frac{u_n f_k k}{r^2} \frac{\partial w}{\partial z} \).**

<table>
<thead>
<tr>
<th>( r )</th>
<th>( k = 2.5 )</th>
<th>( k = 4.0 )</th>
<th>( k = 5.0 )</th>
<th>( k = 2.5 )</th>
<th>( k = 4.0 )</th>
<th>( k = 5.0 )</th>
</tr>
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<tr>
<td>-1</td>
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<td>-0.20903</td>
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<tr>
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<td>0.161743</td>
<td>0.147063</td>
<td>0.143193</td>
</tr>
</tbody>
</table>

**Table 4. Effect of curvature parameter \( k \) on Nusselt number \( Nu^\pm \).**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( k = 1.5 )</th>
<th>( k = 2.5 )</th>
<th>( k = 3.5 )</th>
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<tbody>
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<tr>
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<td>1</td>
<td>-0.154804</td>
<td>-0.0911357</td>
<td>-0.12229</td>
</tr>
</tbody>
</table>
Table 5. Effect of curvature parameter $k$ on Nusselt number $Nu$. 

<table>
<thead>
<tr>
<th>$x$</th>
<th>$k = 1.5$</th>
<th>$k = 2.5$</th>
<th>$k = 3.5$</th>
</tr>
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<td>-0.059316</td>
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</tbody>
</table>

Increasing value of $x$ and increase with increment in the value of curvature parameter $k$.

References


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