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Application of Bernstein polynomials for solving Fredholm integro-differential-difference equations

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Abstract. In this paper, the Bernstein polynomials method is proposed for the numerical solution of Fredholm integro-differential-difference equation with variable coefficients and mixed conditions. This method is using a simple computational manner to obtain a quite acceptable approximate solution. The main characteristic behind this method lies in the fact that, on the one hand, the problem will be reduced to a system of algebraic equations. On the other hand, the efficiency and accuracy of the Bernstein polynomials method for solving these equations are high. The existence and uniqueness of the solution have been proved. Moreover, an estimation of the error bound for this method will be shown by preparing some theorems. Finally, some numerical experiments are presented to show the excellent behavior and high accuracy of this algorithm in comparison with some other well-known methods.

§1 Introduction

Applications of integral equations within applied mathematics and engineering problems such as scattering in quantum mechanics, water waves and spatial-temporal development of an epidemic become nowadays wide and flourishing [2, 6, 14, 24]. Many numerical methods have been used for solving these equations such as Taylor matrix method [26], Legendre collocation method [10], Taylor polynomial solution [17], Newton-Tau numerical solution [13], He's homotopy perturbation method [20], Taylor collocation method [28], Chebyshev polynomial method [1], Taylor collocation method [25], CAS wavelet operational matrix method [5], Legendre wavelets operational method [22], Sinc-Galerkin Technique [23], the contraction principle and quadrature formula [19] and so on. In recent years, Bernstein polynomials have been extensively used for solving problems formulated by mixed Volterra–Fredholm integral equations [12], Fredholm and Volterra integral equations of the second kind [18], Volterra integral

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equations [3], Volterra–Fredholm integral equations [9] and system of Volterra–Fredholm integral equations [11]. In this work, we apply the Bernstein polynomials method for the following form of Fredholm integro-differential-difference equation with variable coefficients

$$\sum_{k=0}^{m}\beta_k(x)u^{(k)}(x)+\sum_{j=0}^{n}\gamma_j(x)u^{(j)}(x-\tau)=f(x)+\int_a^b K(x,t)u(t-\tau)dt, \tag{1.1}$$

for $\tau \geq 0, n \leq m$. This equation is subjected to the following mixed conditions

$$\sum_{k=0}^{m-1}\left[a_{k,l}u^{(k)}(a)+b_{k,l}u^{(k)}(b)+c_{k,l}u^{(k)}(c)\right]=d_l,\ \ l=0,1,...,m-1, \tag{1.2}$$

where $f(x), \alpha(x), \beta_k(x), \gamma_j(x), K(x,t)$ are continuous functions and $a_{k,l}, b_{k,l}, c_{k,l}$ are appropriate constants. Several numerical methods such as Legendre spectral collocation method [21], Fibonacci collocation method [16], Boubaker polynomial bases method [27], Laguerre collocation method [8] and homotopy analysis method [15] were applied for solving this problem. In this work, we will extend the Bernstein polynomials method to approximate the solution of this equation. The properties of Bernstein polynomials are used to reduce the problem into a system of algebraic equations. Some theorems were performed to show the existence and uniqueness of the proposed method. Besides, an estimation of error bound for this method will be given. The obtained upper bound for the error indicates the convergence of this algorithm. Finally, we apply this method to several examples in order to show the efficiency of the presented method.

In Section 2, we will introduce the Bernstein polynomials and some of their properties. The existence and uniqueness of the proposed method were performed by some theorems in Section 3. In Section 4, the method for approximating the solution of problem (1.1) with the mixed conditions (1.2) will be discussed. Section 5 is devoted to the convergence analysis of this method. Section 6 offers some numerical examples to illustrate the efficiency of this algorithm. A brief conclusion is given in section 7.

§2 Preliminaries

In this section, we will introduce the Bernstein polynomials and explain some of their required properties .

Definition 2.1. [18] The well-known Bernstein polynomials of degree N are defined on the interval $[0,1]$ as follows:

$$B_{r,N}(x)=\binom{N}{r}x^r(1-x)^{N-r}, \quad r=0,1,2,...,N. \tag{2.1}$$

If $r<0$ or $r>N$, then we let $B_{r,N}=0$. By using the binomial expansion $(1-x)^{N-r}=\sum_{i=0}^{N-r}(-1)^i\binom{N-r}{i}x^i$, one obtains

$$B_{r,N}(x)=\sum_{i=0}^{N-r}(-1)^i\binom{N}{r}\binom{N-r}{i}x^{r+i}, \quad x\in[0,1].$$

In order to use these polynomials on the interval $[0,l]$ we define the Bernstein polynomials of degree N as follows:

$$B_{r,N}(x) = \binom{N}{r} \frac{x^r(l-x)^{N-r}}{l^N}, \ r = 0, 1, 2, ..., N. \tag{2.2}$$

Using the binomial expansion $(l-x)^{N-r} = \sum_{i=0}^{N-r}(-1)^i \binom{N-r}{i} l^{N-r-i}x^i$, we get

$$B_{r,N}(x) = \sum_{i=0}^{N-r}(-1)^i \binom{N}{r}\binom{N-r}{i}\frac{x^{r+i}}{l^{r+i}}, \quad x \in [0, l].$$

Eq. (2.2) can be written as

$$B_N(x) = [B_{0,N}(x)\ B_{1,N}(x) ...\ B_{n,N}(x)] = \mathbf{X}(x)\mathbf{D}^T, \tag{2.3}$$

in which $\mathbf{X}(x) = [1\ x\ x^2 ...\ x^N],\ \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & \dots & d_{1N} \\ d_{21} & d_{22} & d_{23} & \dots & d_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{N1} & d_{N2} & d_{N3} & \dots & d_{NN} \end{bmatrix}$ and

$$d_{ij} = \begin{cases} \frac{(-1)^{j-i}}{l^j}\binom{N}{i}\binom{N-i}{j-i} & \text{for } i \geq j, \\ 0 & \text{for } i < j. \end{cases} \tag{2.4}$$

§3 The existence and uniqueness of solution

The main object of this section is to study existence and uniqueness of the solution of problem (1.1) with the mixed conditions (1.2). Using the shooting method which required converting a boundary value problem to an initial value problem [4], the existence and uniqueness of solution will be discussed in the following theorems. Let

$$u^{(k)}(a) = d_k,\ k = 0, 1, ..., m-1 \tag{3.1}$$

Consider the Fredholm integro-differential-difference equation (1.1). Assume that $f, \beta_k, \gamma_j \in C[a, b]$, for $k = 0, 1, ..., m$, $j = 0, 1, ..., n$ and $K \in C[a, b]^2$. We define the following norm on $C^{m-1}[a, b]$ as follows:

$$\|u\|_M = \max\left\{\|u\|_\infty, \|u'\|_\infty, ..., \left\|u^{m-1}\right\|_\infty\right\}, \tag{3.2}$$

where $\|.\|_\infty$ is the uniform norm. To discuss about this concepts, at first we consider the following theorem.

Theorem 3.1. *Let $u \in C^m[a, b]$, then u is a solution of problem (1.1) with the mixed conditions (1.2) if and only if u is a solution of the following integral equation*

$$u(x) = \sum_{k=0}^{m-1}\frac{d_k}{k!}(x-a)^k - \int_a^x \frac{h\left(t, u, u', ..., u^{(m-1)}, (Tu)(t)\right)}{(m-1)!}dt, \tag{3.3}$$

where $(Tu)(x) = \int_a^b K(x, t)u(t-\tau)dt$, and

$$h\left(t, u, u', ..., u^{(m-1)}, (Tu)(t)\right) = f(t) - \sum_{k=0}^{m-1}\beta_k(t)u^{(k)}(t) - \sum_{j=0}^{n-1}\gamma_j(t)u^{(j)}(t-\tau) + (Tu)(t).$$

Proof. Since $u \in C^m[a,b]$, so, using Taylor's theorem, we have

$$u(x) = \sum_{k=0}^{m-1} \frac{d_k}{k!}(x-a)^k + \int_a^x \frac{(x-t)^{m-1}}{(m-1)!} dt. \tag{3.4}$$

Suppose that u be a solution of problem (1.1) with the mixed conditions (1.2), then

$$\begin{aligned} u(x) = \sum_{k=0}^{m-1} \frac{d_k}{k!}(x-a)^k + \int_a^x \frac{(x-t)^{m-1}u^{(m)}(t)}{(m-1)!} dt = \sum_{k=0}^{m-1} \frac{d_k}{k!}(x-a)^k \\ + \int_a^x \frac{(x-t)^{m-1}h\left(t, u, u', ..., u^{(m-1)}, (Tu)(t)\right)}{(m-1)!} dt. \end{aligned} \tag{3.5}$$

Thus, u satisfies the relation (3.3). To prove the other side, let u satisfies the relation (3.3), then we have

$$\begin{aligned} u'(x) &= \sum_{k=0}^{m-1} \frac{d_k(x-a)^{k-1}}{(k-1)!} + \int_a^x \frac{(x-t)^{m-2}h\left(t, u, u', ..., u^{(m-1)}, (Tu)(t)\right)}{(m-2)!} dt, \\ u''(x) &= \sum_{k=0}^{m-1} \frac{d_k(x-a)^{k-2}}{(k-2)!} + \int_a^x \frac{(x-t)^{m-3}h\left(t, u, u', ..., u^{(m-1)}, (Tu)(t)\right)}{(m-3)!} dt, \\ &\vdots \\ u^{(m)}(x) &= h\left(x, u, u', ..., u^{(m-1)}, (Tu)(x)\right). \end{aligned} \tag{3.6}$$

This implies that u satisfies Eq. (1.1). Direct substitution shows that $y^{(k)}(a) = d_k$ for $k = 0, 1, ..., m$ and the proof is completed. $\square$

Using theorem 3.1, to prove the existence and uniqueness of solution for problem (1.1) with the mixed conditions (1.2), it is enough to discuss about the existence and uniqueness of solution for the integral equation (3.3).

Theorem 3.2. *(existence) Let $f, \beta_k, \gamma_j \in C[a,b]$, for $k = 0, 1, ..., m$, $j = 0, 1, ..., n$ and $K \in C[a,b]^2$, then for any $\varepsilon > 0$, there exists a solution for problem (3.1) and (3.3) as a function $u : [a, \Psi] \longrightarrow \mathbb{R}$, such that*

$$\Psi = \min\left\{ a + \ln\left(\varepsilon m! \left(\sum_{k=0}^{m-1} \|\beta_k\|_\infty + \sum_{j=0}^{n-1} \|\gamma_j\|_\infty + (b-a)\|K\|_\infty \right)\right), b \right\}. \tag{3.7}$$

Proof. Let

$$\Omega = \left\{ u \in C^{m-1}[a,b] : \left\| u - \sum_{k=0}^{m-1} \frac{d_k}{k!}x^k \quad - \int_a^x (x-t)^{m-1} f(t)dt \right\|_M < \varepsilon \right\}. \tag{3.8}$$

Then, Ω is a closed subset of the Banach space of all $(m-1)$ continuous differentiable functions on $[a,b]$ equipped with the norm (3.2). Since

$$u(x) = \sum_{k=0}^{m-1} \frac{d_k}{k!}x^k - \int_a^x (x-t)^{m-1} f(t)dt,$$

is in Ω for $x \in [a,b]$, then $\Omega \neq \emptyset$. Define the operator L on Ω as follows:

$$L[u](x) = \sum_{k=0}^{m-1} \frac{d_k}{k!}(x-a)^k + \int_a^x \frac{(x-t)^{m-1} h\left(t, u, u', ..., u^{(m-1)}, (Tu)(t)\right)}{(m-1)!} dt, \tag{3.9}$$

then, Eq. (3.1) can be written as

$$L[u] = u. \tag{3.10}$$

We show that Eq. (3.9) has a fixed point in Ω. Since $f \in C[a,b], K \in C[a,b]^2$, then L is a continuous function. To prove that L is self-mapping on Ω, one can write

$$\begin{aligned}
&\left| L[u](x) - \sum_{k=0}^{m-1} \frac{d_k}{k!} x^k - \int_a^x (x-t)^{m-1} f(t) dt \right| \\
&= \left| \int_a^x \frac{(x-t)^{m-1} h\left(t, u, u', ..., u^{(m-1)}, (Tu)(t)\right)}{(m-1)!} dt - \int_a^x (x-t)^{m-1} f(t) dt \right| \\
&\leq \frac{(x-t)^m}{m!} \left(\sum_{k=0}^{m-1} \|\beta_k\|_\infty + \sum_{j=0}^{n-1} \|\gamma_j\|_\infty + (b-a) \|K\|_\infty \right) \|u\|_M \leq \varepsilon \left\| u^{(k)} \right\|_\infty,
\end{aligned} \tag{3.11}$$

for $k = 0, 1, ..., m-1$ and $x \in [a,b]$. we have $\left\| L[u](x) - \sum_{k=0}^{m-1} \frac{d_k}{k!} x^k - \int_a^x (x-t)^{m-1} f(t) dt \right\| \leq \varepsilon$. Thus, $L[u] \in \Omega$ if $u \in \Omega$. This means that L maps Ω into itself. Based on the Banach's fixed-point theorem, the proof is completed. □

Theorem 3.3. *(uniqueness) Let $f, \beta_k, \gamma_j \in C[a,b]$ and $K \in C[a,b]^2$. Then, problem* (3.1) *and* (3.3) *has a unique solution if*

$$\frac{(b-a)^m \left(\sum_{k=0}^{m-1} \|\beta_k\|_\infty + \sum_{j=0}^{n-1} \|\gamma_j\|_\infty + (b-a) \|K\|_\infty \right)}{m!} < 1. \tag{3.12}$$

Proof. Suppose that u_1 and u_2 are two solutions of problem (3.1) and (3.3), then we have

$$u_1(x) = \sum_{k=0}^{m-1} \frac{d_k}{k!}(x-a)^k + \int_a^x \frac{(x-t)^{m-1} h\left(t, u_1, u_1', ..., u_1^{(m-1)}, (Tu_1)(t)\right)}{(m-1)!} dt,$$

and

$$u_2(x) = \sum_{k=0}^{m-1} \frac{d_k}{k!}(x-a)^k + \int_a^x \frac{(x-t)^{m-1} h\left(t, u_2, u_2', ..., u_2^{(m-1)}, (Tu_2)(t)\right)}{(m-1)!} dt.$$

So, one can write

$$\begin{aligned}
&\left| h\left(t, u_1, u_1', ..., u_1^{(m-1)}, (Tu_1)(t)\right) - h\left(t, u_2, u_2', ..., u_2^{(m-1)}, (Tu_2)(t)\right) \right| \\
&\leq \left| \sum_{k=0}^{m-1} \beta_k(x) \left(u_1^{(k)}(x) - u_2^{(k)}(x) \right) \right| + \left| \sum_{j=0}^{n-1} \gamma_j(x) \left(u_1^{(j)}(x-\tau) - u_2^{(j)}(x-\tau) \right) \right| \\
&\quad + \left| \int_a^b K(x,t) u(t-\tau) dt \right| \\
&\leq \delta \|u_1 - u_2\|_M .
\end{aligned} \tag{3.13}$$

This implies that

$$|u_1(x)-u_2(x)| \leq \frac{\int_a^x (x-t)^{m-1}\left(\sum_{k=0}^{m-1}\|\beta_k\|_\infty+\sum_{j=0}^{n-1}\|\gamma_j\|_\infty+(b-a)\|K\|_\infty\right)\|u_1-u_2\|_M}{(m-1)!}$$

$$\leq \frac{(b-a)^m\left(\sum_{k=0}^{m-1}\|\beta_k\|_\infty+\sum_{j=0}^{n-1}\|\gamma_j\|_\infty+(b-a)\|K\|_\infty\right)}{m!}\|u_1-u_2\|_\infty . \tag{3.14}$$

Thus, we have

$$\|u_1-u_2\|_\infty \leq \frac{(b-a)^m\left(\sum_{k=0}^{m-1}\|\beta_k\|_\infty+\sum_{j=0}^{n-1}\|\gamma_j\|_\infty+(b-a)\|K\|_\infty\right)}{m!}\|u_1-u_2\|_\infty . \tag{3.15}$$

Since

$$\frac{(b-a)^m\left(\sum_{k=0}^{m-1}\|\beta_k\|_\infty+\sum_{j=0}^{n-1}\|\gamma_j\|_\infty+(b-a)\|K\|_\infty\right)}{m!}<1, \tag{3.16}$$

therefore, $u_1=u_2$ which completes the proof. □

Using theorem 3.1, problem (1.1) with considering condition (3.1) has a unique solution. To show the existence and uniqueness of solution for problem (1.1) with the mixed conditions (1.2), it is enough to prove that the obtained solution of this equation satisfies the conditions (1.2).

§4 Implementation of the method

In this section, the Bernstein polynomials method to approximate the solution of problem (1.1) with the mixed conditions (1.2) will be discussed.

4.1 Function approximation

Any function $u:[0,b]\to\mathbb{R}$ can be expand in the Bernstein basis as

$$u_N(x)=B_N(u(x))=\sum_{r=0}^{N} u\left(\frac{br}{N}\right)B_{r,N}(x-c), \tag{4.1}$$

where, N is chosen as any positive integer such that $N\geq m$, $u\left(\frac{br}{N}\right)$ are unknown Bernstein coefficients, $0\leq c\leq b$ and $B_{r,N}(x)$ are the Bernstein basis polynomials of degree N described in Section 3. Let $a_r=u\left(\frac{br}{N}\right)$, then, Eq. (4.1) can be written in the following matrix form

$$\mathbf{B}_N(x-c)\mathbf{A}=\sum_{r=0}^{N} a_r B_{r,N}(x-c), \quad x\in[0,b], \tag{4.2}$$

where $\mathbf{B}_N(x-c)=[B_{0,N}(x-c)\ B_{1,N}(x-c)...\ B_{N,N}(x-c)]$ and $\mathbf{A}=[a_0\ a_1...\ a_N]^T$. Substituting (2.3) in (4.2), one obtains

$$u_N(x)=\mathbf{B}_N(x-c)\mathbf{A}=\mathbf{X}(x-c)\mathbf{D}^T\mathbf{A}, \tag{4.3}$$

where $\mathbf{X}(x-c) = [1 \;\; x-c \;\; (x-c)^2 \; ... \; (x-c)^N]$ and $\mathbf{D} = (d_{ij})_{(N+1)\times(N+1)}$, in which

$$d_{ij} = \begin{cases} \frac{(-1)^{j-i}}{b^j}\binom{N}{i}\binom{N-i}{j-i} & \text{for } i \leq j, \\ 0 & \text{for } i > j. \end{cases}$$

The k-order derivative $X(x-c)$ is given by

$$X^{(k)}(x-c) = \mathbf{X}(x-c)(\mathbf{B}^T)^k, \tag{4.4}$$

where

$$\mathbf{B}^T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

for $k = 0, 1, 2, ..., m$. Substituting Eq. (4.4) in k-order derivative of Eq. (4.3), we get

$$u_N^{(k)}(x) = \mathbf{X}(x-c)(\mathbf{B}^T)^k\mathbf{D}^T\mathbf{A}, \tag{4.5}$$

Substituting $x - \tau$ instead of x in Eqs. (4.3) and (4.5), results in

$$u_N(x-\tau) = \mathbf{X}(x-c)\mathbf{G}(\tau)\mathbf{D}^T\mathbf{A},$$
$$u_N^{(k)}(x-\tau) = \mathbf{X}(x-c)\mathbf{G}(\tau)(\mathbf{B}^T)^k\mathbf{D}^T\mathbf{A}, \tag{4.6}$$

where

$$\mathbf{G}(\tau) = \begin{bmatrix} \binom{0}{0}(\tau)^0 & -\binom{1}{0}(\tau)^1 & \dots & (-1)^N\binom{N}{0}(\tau)^N \\ 0 & \binom{1}{1}(\tau)^0 & \dots & (-1)^{N-1}\binom{N}{1}(\tau)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \binom{N}{N}(\tau)^0 \end{bmatrix}.$$

Collocating Eqs. (4.5) and (4.6) in the following points

$$x_i = a + \frac{b-a}{N}i, \quad i = 0, 1, ..., N, \tag{4.7}$$

we get

$$u_N^{(k)}(x_i) = \mathbf{X}(x_i - c)(\mathbf{B}^T)^k\mathbf{D}^T\mathbf{A}, \tag{4.8}$$

$$u_N(x_i - \tau) = \mathbf{X}(x_i - c)\mathbf{G}(\tau)\mathbf{D}^T\mathbf{A}, \tag{4.9}$$

$$u_N^{(j)}(x_i - \tau) = \mathbf{X}(x_i - c)\mathbf{G}(\tau)(\mathbf{B}^T)^j\mathbf{D}^T\mathbf{A}, \tag{4.10}$$

for $k = 0, 1, 2, ..., m$ and $j = 0, 1, 2, ..., n$, which can be written in the following matrix forms:

$$\mathbf{U}^{(k)} = \mathbf{X}(\mathbf{B}^T)^k\mathbf{D}^T\mathbf{A}, \tag{4.11}$$

$$\mathbf{U}_1 = \mathbf{X}\mathbf{G}(\tau)\mathbf{D}^T\mathbf{A}, \tag{4.12}$$

$$\mathbf{U}_1^{(j)} = \mathbf{X}\mathbf{G}(\tau)(\mathbf{B}^T)^j\mathbf{D}^T\mathbf{A}, \tag{4.13}$$

where

$$\mathbf{X} = [\mathbf{X}(x_0 - c) \; \mathbf{X}(x_1 - c) \; ... \; \mathbf{X}(x_N - c)]^T,$$

$$\mathbf{U}^{(k)}=\left[u_N^{(k)}(x_0)\ u_N^{(k)}(x_1)\ ...\ u_N^{(k)}(x_N)\right]^T,$$
$$\mathbf{U}_1=\left[u_N(x_0-\tau)\ u_N(x_1-\tau)\ ...\ u_N(x_N-\tau)\right]^T,$$
$$\mathbf{U}_1^{(j)}=\left[u_N^{(j)}(x_0-\tau)\ u_N^{(j)}(x_1-\tau)\ ...\ u_N^{(j)}(x_N-\tau)\right]^T.$$

4.2 Method of solution

Let us consider Eq. (1.1) as follows:

$$A(x)+B(x)=f(x)+C(x), \tag{4.14}$$

where $A(x)=\sum_{k=0}^{m}\beta_k(x)u^{(k)}(x)$, $B(x)=\sum_{j=0}^{n}\gamma_j(x)u^{(j)}(x-\tau)$, and $C(x)=\int_a^b K(x,t)u(t-\tau)dt$. Since $K(x,t)$ is a continuous function, so, $a_i(t-\tau)^iK(x,t)$ is Riemann integrable with respect to t. Hence, using (4.6), $C(x)$ can be written as

$$C(x)=\int_a^b K(x,t)u(t-\tau)dt\simeq\int_a^b K(x,t)\mathbf{X}(x-c)\mathbf{G}(\tau)\mathbf{D}^T\mathbf{A}dt=\mathbf{K}(x)\mathbf{G}(\tau)\mathbf{D}^T\mathbf{A}, \tag{4.15}$$

where $\mathbf{K}(x)=\int_a^b K(x,t)\mathbf{X}(x-c)dt=[k_1(x)\ k_2(x)\ ...\ k_N(x)]$. Collocating points (4.7) in Eq. (4.15), result in

$$\mathbf{C}=\begin{bmatrix}C(x_0)\\C(x_1)\\\vdots\\C(x_N)\end{bmatrix}=\mathbf{K}\mathbf{G}(\tau)\mathbf{D}^T\mathbf{A},\quad \mathbf{K}=\begin{bmatrix}\mathbf{K}(x_0)\\\mathbf{K}(x_1)\\\vdots\\\mathbf{K}(x_N)\end{bmatrix}. \tag{4.16}$$

Setting the collocation points (4.7) in Eq. (4.14), one obtains

$$A(x_i)+B(x_i)=f(x_i)+C(x_i). \tag{4.17}$$

So, we can write system (4.17) in the following matrix form

$$\sum_{k=0}^{m}\mathbf{H}\mathbf{U}^{(k)}+\sum_{j=0}^{n}\mathbf{L}\mathbf{U}_1^{(j)}=\mathbf{F}_1+\mathbf{C}, \tag{4.18}$$

where $\mathbf{C}$ is introduced in (4.16) and

$$\mathbf{H}=\begin{bmatrix}\beta(x_0)&0&\dots&0\\0&\beta(x_1)&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&\beta(x_N)\end{bmatrix},\ \mathbf{L}=\begin{bmatrix}\gamma(x_0)&0&\dots&0\\0&\gamma(x_1)&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&\gamma(x_N)\end{bmatrix},$$

$$\mathbf{U}_1^{(j)}=\begin{bmatrix}U^{(j)}(x_0-\tau)\\U^{(j)}(x_1-\tau)\\\vdots\\U^{(j)}(x_N-\tau)\end{bmatrix},\ \mathbf{U}^{(k)}=\begin{bmatrix}U^{(k)}(x_0)\\U^{(k)}(x_1)\\\vdots\\U^{(k)}(x_N)\end{bmatrix},\ \mathbf{F}_1=\begin{bmatrix}f(x_0)\\f(x_1)\\\vdots\\f(x_N)\end{bmatrix},$$

Substituting (4.11), (4.12),(4.13) and (4.16) in (4.18), results in

$$\sum_{k=0}^{m}\mathbf{H}\mathbf{X}(\mathbf{B}^T)^k\mathbf{D}^T\mathbf{A}+\sum_{j=0}^{n}\mathbf{L}\mathbf{X}\mathbf{G}(\tau)(\mathbf{B}^T)^j\mathbf{D}^T\mathbf{A}-\mathbf{K}\mathbf{G}(\tau)\mathbf{D}^T\mathbf{A}=\mathbf{F}_1, \tag{4.19}$$

or

$$\left[\sum_{k=0}^{m}\mathbf{HX}(\mathbf{B}^T)^k\mathbf{D}^T+\sum_{j=0}^{n}\mathbf{LXG}(\tau)(\mathbf{B}^T)^j\mathbf{D}^T-\mathbf{KG}(\tau)\mathbf{D}^T\right]\mathbf{A}=\mathbf{F}_1. \tag{4.20}$$

Let

$$[\mathbf{M}_1;\mathbf{F}_1] \ or \ \mathbf{M}_1\mathbf{A}=\mathbf{F}_1, \tag{4.21}$$

where $\mathbf{M}_1=\sum_{k=0}^{m}\mathbf{HX}(\mathbf{B}^T)^k\mathbf{D}^T+\sum_{j=0}^{n}\mathbf{LXG}(\tau)(\mathbf{B}^T)^j\mathbf{D}^T-\mathbf{KG}(\tau)\mathbf{D}^T$. Eq. (4.20) can be writen as

$$[\mathbf{P}_1;\mathbf{F}_1]=\begin{bmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,N}; & f(x_0)\\ p_{1,0} & p_{1,1} & \cdots & p_{1,N}; & f(x_1)\\ p_{2,0} & p_{2,1} & \cdots & p_{2,N}; & f(x_2)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ p_{N-m,0} & p_{N-m,1} & \cdots & p_{N-m,N}; & f(x_{N-m})\end{bmatrix},$$

which corresponds to a system of $(N-m+1)$ linear algebraic equations with the unknown coefficients a_i, $i=0,1,2,...,N-m+1$. Using the multipoint boundary conditions (1.2), the following matrix is resulted

$$\left\{\sum_{k=0}^{m-1}\left[a_{k,l}\mathbf{X}(a)+b_{k,l}\mathbf{X}(b)+c_{k,l}\mathbf{X}(c)\right](\mathbf{B}^T)^k\mathbf{D}^T\right\}\mathbf{A}=d_l, \tag{4.22}$$

or

$$\mathbf{Q}_l\mathbf{A}=d_l, \quad l=0,1,2,...,m-1, \tag{4.23}$$

where $\mathbf{Q}_l=\sum_{k=0}^{m-1}\left[a_{k,l}\mathbf{X}(a)+b_{k,l}\mathbf{X}(b)+c_{k,l}\mathbf{X}(c)\right](\mathbf{B}^T)^k\mathbf{D}^T=[q_{l,0},q_{l,1},...,q_{l,N}]$. Eventually, replacing the row matrices (4.23) by the m rows of (4.21), we get

$$[\mathbf{M};\mathbf{F}] \ or \ \mathbf{MA}=\mathbf{F},$$

which corresponds to a system of $(N+1)$ linear algebraic equations. Then, this system can be written as follows:

$$[\mathbf{M};\mathbf{F}]=\begin{bmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,N} & ; & f(x_0)\\ p_{1,0} & p_{1,1} & \cdots & p_{1,N} & ; & f(x_1)\\ p_{2,0} & p_{2,1} & \cdots & p_{2,N} & ; & f(x_2)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ p_{N-m,0} & p_{N-m,1} & \cdots & p_{N-m,N} & ; & f(x_{N-m})\\ q_{0,0} & q_{0,1} & \cdots & q_{0,N} & ; & d_0\\ q_{1,0} & q_{1,1} & \cdots & q_{1,N} & ; & d_1\\ q_{2,0} & q_{2,1} & \cdots & q_{2,N} & ; & d_2\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ q_{m-1,0} & q_{m-1,1} & \cdots & q_{m-1,N} & ; & d_{m-1}\end{bmatrix}. \tag{4.24}$$

Solving (4.24), the unknown values of $\hat{a}_k$ will be obtained, where, $\hat{a}_k=u_N\left(\frac{bk}{N}\right)$, are the approximate values of a_k for $k=0,1,2,...,N$. Therefore, the approximate solution of Eq. (1.1) will be obtained in the following form:

$$B_N(u_N(x)) = \sum_{k=0}^{N} \hat{a}_k B_{k,N}(x-c), \quad 0 \le c \le b. \tag{4.25}$$

§5 Convergence analysis

In this section, we will obtain an estimation of error bound for our numerical method. To do this, we suppose that the infinity norm of a matrix $\mathbf{A}$ is defined by $\|\mathbf{A}\|_I = \max_i \sum_{j=1}^n |a_{ij}|$. The sums of $B_{k,N}$ can be estimated, in an elementary way, by means of the following expressions [7]:

$$T_{N,s} := \sum_{k=0}^{N} (k - Nx)^s B_{k,N}(x), \ x \in [0,1], \tag{5.1}$$

for $N = 1, 2, ...$ and $s = 0, 1, ...$ This relation can be written on the interval $[a, b]$ as follows:

$$T_{N,s} := \sum_{k=0}^{N} ((b-a)k - (a-x)N)^s B_{k,N}(x). \tag{5.2}$$

Lemma 5.1. *[7] If $h(y) > 0$ is bounded on $[0,1]$ and converges to zero with h, then for any $r = 0, 1, ...,$ uniformly in x, we have $N^r \sum_{k=0}^{N} h\left(\left|\frac{k}{N} - x\right|\right)\left(\frac{k}{N} - x\right)^{2r} B_{k,N}(x) \longrightarrow 0$.*

Therefore, the following result for $h(y) > 0$, $a < y < b$ is concluded

$$N^r \sum_{k=0}^{N} h\left(\left|\frac{aN + (b-a)k}{N} - x\right|\right)\left(\frac{aN + (b-a)k}{N} - x\right)^{2r} B_{k,N}(x) \to 0. \tag{5.3}$$

At first, we perform a bound on $\sup_{x\in[a,b]} |u(x) - B_N(u(x))|$.

Theorem 5.2. *Let $u(x)$ is bounded on $[a, b]$, differentiable in some neighborhood of x and has the second derivative $u''(x)$ for some $x \in [a, b]$. If $u(x)$ be the exact solution of problem (1.1) with the mixed conditions (1.2) and $B_N(u(x))$ be the Bernstein approximation of $u(x)$, then*

$$\sup_{x\in[a,b]} |u(x) - B_N(u(x))| \le \frac{(b-a)^2}{8N} \|u''\|_\infty. \tag{5.4}$$

Proof. We can write

$$\begin{aligned} u(\frac{aN + (b-a)k}{N} - x) =& u(x) + \left(\frac{aN + (b-a)k}{N} - x\right) u'(x) \\ &+ \left(\frac{aN + (b-a)k}{N} - x\right)^2 \left[\frac{1}{2} u''(x) + h(\frac{aN + (b-a)k}{N} - x)\right], \end{aligned} \tag{5.5}$$

where $h(y) := h_x(y)$ is bounded for all y and converges to zero with y. This yields

$$\begin{aligned} B_N(u(x)) =& \sum_{k=0}^{N} u\left(a + \frac{(b-a)k}{N}\right) B_{k,N}(x-c) \\ =& u(x) + \frac{1}{2} u''(x) T_{N,2}(x) + N^{-2} \sum_{k=0}^{N} ((b-a)k - (a-x)N)^2 \\ &\times h\left(\frac{aN + (b-a)k}{N} - x\right) B_{k,N}(x-c). \end{aligned} \tag{5.6}$$

Using (5.3), the last term does not exceed $O(\frac{1}{N})$, and then

$$\begin{aligned} |u(x) - B_N(u(x))| &\leq \frac{1}{2}u''(x)T_{N,2}(x) \leq (b-a)\frac{(x-a)(b-x)}{2N(b-a)}\|u''\|_\infty \\ &= \frac{(x-a)(b-x)}{2N}\|u''\|_\infty \leq \frac{(b-a)^2}{8N}\|u''\|_\infty, \end{aligned} \tag{5.7}$$

thus, $\sup_{x\in[a,b]}|u(x) - B_N(u(x))| \leq \frac{(b-a)^2}{8N}\|u''\|_\infty$. □

Theorem 5.3. *Consider the Fredholm integro-differential-difference equation* (1.1) *with variable coefficients and mixed conditions* (1.2). *Let* $B_N(u(x))$ *be the Bernstein approximation of* $u(x)$ *and* $B_N(u_N(x))$ *be the approximate solution of (1.1). Then,*

$$\sup_{x\in[a,b]}|B_N(u(x)) - B_N(u_N(x))| \leq k(\boldsymbol{M})\left\|\widehat{\mathbf{A}}\right\|_I O(\varepsilon), \tag{5.8}$$

where $k(\boldsymbol{M})$ *is condition number of* $\boldsymbol{M}$ *and* $\widehat{\mathbf{A}} = [\hat{a}_0, \hat{a}_1, ..., \hat{a}_N]$ *be the solution of* (4.24) *computed through our presented method.*

Proof. We can write

$$|B_N(u(x)) - B_N(u_N(x))| = \left|\sum_{k=0}^{N}(a_k - \hat{a}_k)\,B_{k,N}(x-c)\right| \leq \sup_{0\leq k\leq N}|a_k - \hat{a}_k|\,\sum_{k=0}^{N}B_{k,N}(x-c), \tag{5.9}$$

where $\sum_{k=0}^{N}B_{k,N}(x-c) = 1$. This implies that

$$\sup_{x\in[a,b]}|B_N(u(x)) - B_N(u_N(x))| \leq \sup_{0\leq k\leq N}|a_k - \hat{a}_k|. \tag{5.10}$$

Note that $\mathbf{A} = [a_0\ a_1 ...\ a_N]^T$ and $\widehat{\mathbf{A}} = [\hat{a}_0\ \hat{a}_1 ...\ \hat{a}_N]^T$ are the exact and approximate solutions of $\mathbf{MA} = \mathbf{F}$, respectively. Also, suppose that the perturbation matrix $\mathbf{E}$ is such that $(\mathbf{M}+\mathbf{E})\,\widehat{\mathbf{A}} = \mathbf{F}$. It can be shown that $\left\|\mathbf{A} - \widehat{\mathbf{A}}\right\|_I \leq k(\mathbf{M})\frac{\|\mathbf{E}\|_I}{\|\mathbf{M}\|_I}\left\|\widehat{\mathbf{A}}\right\|_I$. It is known that the Gaussian elimination with partial pivoting is almost numerically stable. Therefore, if we use this method for solving the linear system $\mathbf{MA} = \mathbf{F}$, then $\frac{\|\mathbf{E}\|_I}{\|\mathbf{M}\|_I}$ is close to the machine precision ε. This implies that

$$\left\|\mathbf{A} - \hat{\mathbf{A}}\right\|_I \leq k(\mathbf{M})\left\|\hat{\mathbf{A}}\right\|_I O(\varepsilon). \tag{5.11}$$

Using relations (4.12) and (5.11), one obtains

$$\sup_{x\in[a,b]}|B_N(u(x)) - B_N(u_N(x))| \leq k(\mathbf{M})\left\|\widehat{\mathbf{A}}\right\|_I O(\varepsilon), \tag{5.12}$$

and the proof is completed. □

Theorem 5.4. *Consider the assumptions of Theorems 5.2 and 5.3. Then, we have the following error estimation*

$$\sup_{x\in[a,b]}|u(x) - B_N(u_N(x))| \leq \frac{(b-a)^2}{8N}\|u''\|_\infty + k(\boldsymbol{M})\left\|\widehat{\mathbf{A}}\right\|_I O(\varepsilon). \tag{5.13}$$

Proof. The triangle inequality implies that

$$\sup_{x\in[a,b]}|u(x) - B_N(u_N(x))| \leq \sup_{x\in[a,b]}|u(x) - B_N(u(x))| + \sup_{x\in[a,b]}|B_N(u(x)) - B_N(u_N(x))|. \tag{5.14}$$

According to Theorems 5.2 and 5.3, the purpose is achieved. □

§6 Numerical examples

In this section, we will apply the Bernstein polynomials method (BPM) on some examples and compare the quality of the computed solutions with the solutions obtained by some other efficient methods. In order to show the error, we introduce the notations $\alpha_N = \frac{(b-a)^2}{8N}$, $e_N(x) = |u(x) - B_N(u_N(x))|$, $\|e_N\|_2 = \left(\int_a^b e_N^2(x)dx\right)^{\frac{1}{2}}$ and $\|\eta_N\| = \sup_{x\in[a,b]} |u(x) - B_N(u_N(x))|$, where $u(x)$ and $B_N(u_N(x))$ are the exact solution and the solution obtained by presented method, respectively. Also, we use the notation $k(\mathbf{M})$ which was defined in Section 5. In all examples, the computing times (CPUs) in seconds to obtain the numerical solutions $B_N(u_N(x))$ are also given.

Example 1. [21] Consider the following integro-differential-difference equation with variable coefficients

$$u'''(x) - xu'(x) + u''(x-1) - xu(x-1) = f(x) + \int_{-1}^{1} u(t-1)dt, \tag{6.1}$$

subject to the initial conditions $u(0) = 0$, $u'(0) = 1$ and $u''(0) = 0$, where $f(x) = -(x+1)(\sin(x-1) + \cos(x)) - \cos(2) + 1$. The exact solution of this equation is $u(x) = sin(x)$. We implement the suggested method with different values of N and approximate the solution of (4.25) for $c = 0$. Tables 1, 2 and Figure 1 show the numerical results for this example. Table 1 compares the approximate solution by BPM with the Legendre spectral collocation method (LSCM) [21] and the Fibonacci collocation method (FCM) [16]. The outcomes reveal that the results by our method are very promising and superior to LSCM and FCM. From Table 2, we conclude that Theorem 5.4 can be applied to this example. Figure 1 depicts the absolute errors of our method with $N = 6$ and 7. One can see that, as N is increased, the error is decreased.

Table 1. Comparison of the approximate solutions of $u(x)$ for Example 1.

x	Exact solution	BPM $N=6$	BPM $N=7$	LSCM [21] $M=6$	LSCM [21] $M=7$	FCM [16] $N=8$	FCM [16] $N=9$
-1	-0.841471	-0.841319	-0.841427	-0.866814	-0.83644	-1.114125	-3.078521
-0.8	-0.717356	-0.717330	-0.717352	-0.729305	-0.71498	-0.869866	-1.875847
-0.6	-0.564642	-0.564551	-0.564637	-0.569211	-0.563732	-0.633677	-1.054038
-0.4	-0.389418	-0.389372	-0.389402	-0.390626	-0.389177	-0.4110374	-0.5333391
-0.2	-0.198569	-0.198455	-0.198567	-0.198802	-0.198643	-0.2014897	-0.2163871
0	0	0	0	0	0	0	0
0.2	0.198569	0.198455	0.198567	0.198769	0.19865	0.2016525	0.2155338
0.4	0.389418	0.389372	0.389402	0.390107	0.389294	0.4137037	0.5200945
0.6	0.564642	0.564551	0.564637	0.566704	0.564314	0.6476974	0.9903835
0.8	0.717356	0.717330	0.717352	0.721914	0.716785	0.9164897	1.689322
1	0.841471	0.841319	0.841427	0.850444	0.840739	1.235210	2.667579
$\|\eta_N\|$	0	5.439E−4	4.703E−5	2.53434E−2	5.03053E−3	3.937393E−1	2.23705E−0

Table 2. Numerical results for Example 1.

N	α_N	$k(A)$	$\|e_N\|_2$	$O(\varepsilon)$	CPUs
6	0.0833	53.729	4.911E−4	6.2189E−6	3.2743
7	0.0714	73.103	1.649E−5	1.3928E−8	5.1047
8	0.0625	96.922	1.203E−6	6.9103E−9	6.2448
9	0.0555	138.27	9.769E−8	5.2018E−10	8.0830

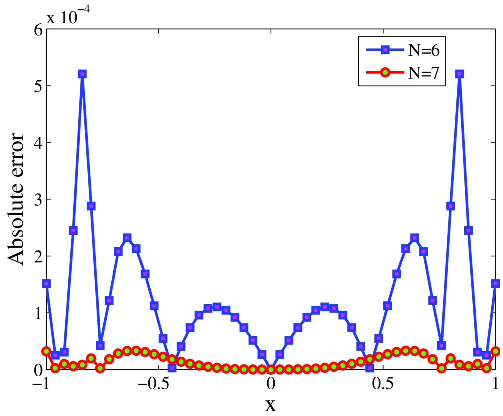

Figure 1. Plot of the absolute errors by presented method with $N = 6, 7$ in Example 1.

Example 2. [27] Consider the following integro-differential-difference equation with variable coefficients

$$(x+4)^2u''(x) - (x+4)u'(x) + u(x-1) - u'(x-1) = f(x) + \int_{-1}^{1} u(t)dt, \tag{6.2}$$

subject to the initial conditions $u(0) = \ln(4)$ and $u'(0) = \frac{1}{4}$, where $f(x) = \ln(x+3) - \frac{1}{x+3} + 3\ln(3) - 5\ln(5)$. The exact solution of this equation is $u(x) = \ln(x+4)$. The numerical results for this example are displayed in Tables 3, 4 and Figure 2. A comparison between the approximate solutions of BPM, LSCM [21] and Boubaker polynomial method (BOPM) [27] are presented in Table 3. From Table 4, we conclude that Theorem 5.4 can be applied to this example. The absolute errors of this method is depicted in Figure 2 with $N = 6$ and 7. It is seen that, as N is increased, the error is decreased and the accuracy increases as well. Therefore, our method for solving this problem is very effective and more accurate with respect to LSCM and BOPM.

Example 3. [8] Consider the following integro-differential-difference equation with variable coefficients

$$u''(x) - xu'(x) + xu(x) - u'(x-1) + u(x-1) = f(x) + \int_{-1}^{1} (3t-2x)u(t)dt, \tag{6.3}$$

subject to the initial conditions $u(0) = 1$ and $u'(0) = 0$, where $f(x) = x(sin(x) + cos(x)) - cos(x) + sin(x-1) + cos(x-1) + 4xsin(1)$. The exact solution of this equation is $u(x) = \cos(x)$. The numerical results for this example are displayed in Table 5, 6 and Figure 3. Table 5 exhibits the approximate solutions for BPM, LSCM [21] and Laguerre collocation method (LCM) [8]. From Table 6, we conclude that Theorem 5.4 can be applied to this example. The absolute

Table 3. Comparison of the approximate solutions of $u(x)$ for Example 2.

x	Exact solution	BPM		LSCM [21]		BOPM [27]	
		$N=6$	$N=7$	$M=6$	$M=7$	$N=6$	$N=7$
-1	1.09861	1.09861	1.09861	1.09861	1.09861	1.098596	1.098657
-0.9	1.1314	1.1314	1.1314	1.1314	1.1314	1.131387	1.131431
-0.8	1.16315	1.16315	1.16315	1.16315	1.16315	1.163138	1.163173
-0.7	1.19392	1.19392	1.19392	1.19392	1.19392	1.193911	1.193937
-0.6	1.22378	1.22378	1.22378	1.22378	1.22378	1.223766	1.223787
-0.5	1.25276	1.25276	1.25276	1.25276	1.25276	1.252755	1.252771
-0.4	1.28093	1.28093	1.28093	1.28093	1.28093	1.280928	1.280939
-0.3	1.30833	1.30833	1.30833	1.30833	1.30833	1.308329	1.308336
-0.2	1.335	1.335	1.335	1.335	1.335	1.335	1.335001
-0.1	1.36098	1.36098	1.36098	1.36098	1.36098	1.360976	1.360977
0	1.38629	1.38629	1.38629	1.38629	1.38629	1.386294	1.386294
$\|\eta_N\|$	0	3.1525E−7	4.2922E−8	5.86528E−7	1.77093E−6	1.66E−5	4.50E−5

Table 4. Numerical results for Example 2.

N	α_N	$k(A)$	$\|e_N\|_2$	$O(\varepsilon)$	CPUs
5	0.1	21.139	4.2015E−8	2.1760E−6	2.1649
6	0.0833	48.046	8.1038E−9	3.3605E−7	4.0137
7	0.0714	63.274	6.3391E−10	1.8214E−9	6.2703
8	0.0625	85.291	7.7344E−11	9.2264E−10	7.7394

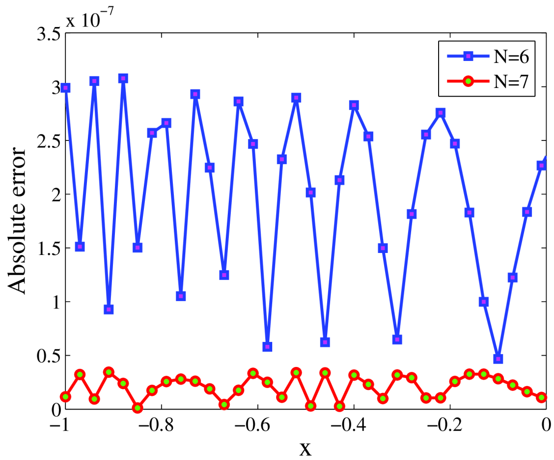

Figure 2. Plot of the absolute errors by presented method with $N=6,7$ in Example 2.

errors of this method is depicted in Figure 3 with $N=6$ and 7. From these results, it is evident that the presented method provides a good approximate solution in comparison with LSCM and LCM.

Table 5. Comparison of the approximate solutions of $u(x)$ for Example 3.

x	Exact solution	BPM $N=6$	BPM $N=7$	LSCM [21] $M=6$	LSCM [21] $M=7$	LCM [8] $N=12$	LCM [8] $N=12, M=15$
0	1	1	1	1	1	0.999999	1
0.1	0.995004	0.995004	0.995004	0.995004	0.995004	0.996593	0.996600
0.2	0.980067	0.980067	0.980067	0.980066	0.980066	0.986641	0.986365
0.3	0.955336	0.955336	0.955336	0.955333	0.955334	0.969544	0.969444
0.4	0.921061	0.921061	0.921061	0.921052	0.921057	0.946139	0.946036
0.5	0.877583	0.877583	0.877583	0.877566	0.877577	0.916382	0.916381
0.6	0.825336	0.825336	0.825336	0.825314	0.825328	0.880504	0.880768
0.7	0.764842	0.764842	0.764842	0.764827	0.76483	0.838776	0.839522
0.8	0.696707	0.696707	0.696707	0.696723	0.696685	0.791509	0.793010
0.9	0.62161	0.62161	0.62161	0.621705	0.62161	0.739046	0.741629
1	0.540302	0.540302	0.540302	0.540552	0.540205	0.681764	0.685810
$\|\eta_N\|$	0	5.2805E−7	9.1044E−8	2.50083E−4	9.68491E−5	1.4146E−2	2.023575E−3

Table 6. Numerical results for Example 3.

N	α_N	$k(A)$	$\|e_N\|_2$	$O(\varepsilon)$	CPUs
6	0.0833	39.576	1.0385E−8	6.2047E−9	3.7728
7	0.0714	57.422	3.6041E−8	1.0225E−10	5.8407
10	0.05	154.02	6.7714E−11	3.3197E−14	8.4226
12	0.0416	183.69	2.1437E−13	1.5581E−15	10.493

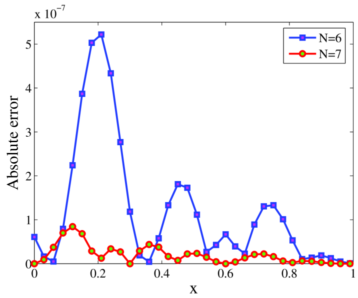

Figure 3. Plot of the absolute errors by presented method with $N = 6, 7$ in Example 3.

Example 4. [15] Finally, consider the following integro-differential-difference equation with variable coefficients

$$u''(x) + xu'(x) + xu(x) + u'(x-1) + u(x-1) = f(x) + \int_{-1}^{0} tu(t-1)dt, \tag{6.4}$$

with conditions

$$u(0) = 1, \quad u'(0) = -1,$$

where $f(x) = e^{-x} + e$. The exact solution of this equation is $u(x) = e^{-x}$. Tables 7, 8, 9 and Figure 4 display the numerical results for this example. Tables 7 and 8 show the approximate solutions and absolute errors of BPM, LSCM [21] and the homotopy analysis method (HAM) [15], respectively. From Table 9, we conclude that Theorem 5.4 can be applied to this example. Figure 4 depicts the absolute errors of our method with $N = 9$ and 10. So, not only because of its performance in providing more efficient results, but also because of its efficiency and accuracy, our method is preferable.

Table 7. Comparison of the approximate solutions of $u(x)$ for Example 4.

x	Exact solution	BPM $N=9$	BPM $N=10$	LSCM [21] $M=10$	LSCM [21] $M=12$	HAM [15] $m=10$	HAM [15] $m=15$
-1	2.71828	2.71828	2.71828	2.71874	2.71832	2.71636	2.71821
-0.8	2.22554	2.22554	2.22554	2.22572	2.22556	1.82165	1.82210
-0.6	1.82212	1.82212	1.82212	1.82216	1.82212	1.82165	1.82210
-0.4	1.49182	1.49182	1.49182	1.49182	1.49182	1.49170	1.49182
-0.2	1.2214	1.2214	1.2214	1.2214	1.2214	1.22140	1.22140
0	1	1	1	1	1	1	1

Table 8. Comparison of the absolute errors for Example 4.

x	BPM $N=9$	BPM $N=10$	LSCM [21] $M=10$	LSCM [21] $M=12$	HAM [15] $m=10$	HAM [15] $m=15$
-1	5.7163E−6	1.3246E−6	4.58316E−4	3.88267E−5	1.92193E−3	7.42184E−5
-0.8	4.4159E−6	1.0190E−7	1.80503E−4	1.52916E−5	1.06345E−3	4.07102E−5
-0.6	1.1415E−7	9.1437E−8	4.38150E−5	3.71185E−6	4.71496E−4	1.54921E−5
-0.4	3.9057E−7	7.4491E−8	3.97513E−6	3.36791E−7	1.2820E−4	4.40566E−7
-0.2	4.2957E−7	1.3824E−8	6.31221E−6	5.34763E−7	3.25198E−4	2.57853E−6
0	0	0	0	0	0	0

Table 9. Numerical results for Example 4.

N	α_N	$k(A)$	$\|e_N\|_2$	$\|\eta_N\|$	$O(\varepsilon)$	CPUs
9	0.0138	132.81	6.3810E−7	5.7163E−6	7.0338E−11	7.2053
10	0.0125	158.30	1.1143E−7	1.3246E−6	3.2905E−13	8.5270
12	0.0104	194.58	6.7013E−10	8.2033E−9	2.9461E−15	10.143
15	0.0083	244.69	9.5582E−13	7.7233E−11	4.4296E−17	12.079

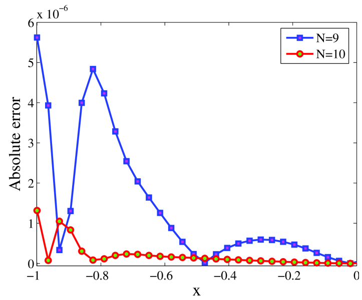

Figure 4. Plot of the absolute errors by presented method with $N = 9, 10$ in Example 4.

§7 Conclusion

In this paper, the Bernstein polynomials method was applied to obtain the numerical solutions of Fredholm integro-differential-difference equation with variable coefficients and mixed conditions. The properties of Bernstein polynomials were used to convert the equation into a system of algebraic equations which could be solved more easily. Some theorems were performed to show the existence, uniqueness and the convergence analysis of this method. The obtained results showed that the Bernstein polynomials method for solving Fredholm integro-differential-difference equation with variable coefficients and mixed conditions was very effective and simple with a high accuracy with respect to some other well-known methods such as Fibonacci collocation method, Legendre spectral collocation method, Boubaker polynomial bases method, Laguerre collocation method and homotopy analysis method.

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