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A generalized Liouville's formula

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Abstract. A generalized Liouville's formula is established for linear matrix differential equations involving left and right multiplications. Its special cases are used to determine the localness of characteristics of symmetries and solutions to Riemann-Hilbert problems in soltion theory.

§1 Introduction

Lax pairs play a key role in solving soliton equations [1]. A Lax pair consists of two matrix spectral problems

$$\Phi_x = U(u,\lambda)\Phi,\ \ \Phi_t = V(u,\lambda)\Phi,$$

where U and V are two square matrices depending on the potential u and the spectral parameter λ [2]. The compatibility condition of the Lax pair is the zero curvature equation

$$U_t - V_x + [U,V] = 0,$$

which presents a so-called soliton equation.

The modern Riemann-Hilbert method in soliton theory [3] uses an equivalent Lax pair [4]

$$\Psi_x = i\lambda[\Lambda,\Psi] + Q(u,\lambda)\Psi,\ \ \Psi_t = i\lambda^m[\Lambda,\Psi] + P(u,\lambda)\Psi,$$

where i is the imaginary unit, m is a natural number, Λ is a constant diagonal matrix, and $\mathrm{tr}(Q) = \mathrm{tr}(P) = 0$. A property that $\det\Psi$ is independent of x and t is needed in solving a related Riemann-Hilbert problem, which determines N-soliton solutions to the resulting soliton equation [3].

In this letter, we would like to explore a more general result on the derivative of $\det\Psi$, which constitutes a generalized Liouville's formula. One specific case can be used to prove the localness of characteristics of symmetries of soliton hierarchies, and another can be used in determining solutions to the associated Riemann-Hilbert problems.

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§2 Generalized Liouville's formula

We present a general result on the derivative of the determinant of a matrix which solves a linear matrix differential equation involving left and right multiplications.

Theorem 2.1. *Let μ, A and B be square matrices depending on x. If μ satisfies a linear matrix differential equation*

$$\mu_x = A\mu + \mu B, \tag{2.1}$$

then we have

$$(\det \mu)_x = [tr(A) + tr(B)] \det \mu, \tag{2.2}$$

and thus

$$\det \mu(x) = e^{\int_{x_0}^{x}[tr(A(x'))+tr(B(x'))]\,dx'} \det \mu(x_0), \tag{2.3}$$

where $x_0 \in \mathbb{R}$ is a given initial point.

Proof: Assume that

$$\mu = (\mu_{jk})_{n\times n} = (\mu_1, \mu_2, \cdots, \mu_n),\ B = (b_{jk})_{n\times n} = (b_1, b_2, \cdots, b_n),$$

where n is a natural number, and μ_j and b_j $(1 \le j \le n)$ are the j-th columns of μ and B, respectively. Let us denote the (j,k) cofactor of μ by M_{jk}, where $1 \le j,k \le n$. Then, we have

$$\sum_{j=1}^{n} \mu_{jk} M_{jl} = \sum_{j=1}^{n} \mu_{kj} M_{lj} = \begin{cases} 0, & \text{if } 1 \le k \ne l \le n, \\ \det \mu, & \text{if } 1 \le k = l \le n. \end{cases} \tag{2.4}$$

On one hand, from $\mu_x = A\mu + \mu B$, we get

$$\mu_{j,x} = A\mu_j + \mu b_j,\ 1 \le j \le n.$$

Thus, we can compute that

$$\begin{aligned}(\det \mu)_x &= \sum_{j=1}^{n} |\mu_1, \cdots, \mu_{j-1}, \mu_{j,x}, \mu_{j+1} \cdots, \mu_n| \\ &= \sum_{j=1}^{n} |\mu_1, \cdots, \mu_{j-1}, A\mu_j + \mu b_j, \mu_{j+1}, \cdots, \mu_n| \\ &= \sum_{j=1}^{n} |\mu_1, \cdots, \mu_{j-1}, A\mu_j, \mu_{j+1}, \cdots, \mu_n| \\ &\quad + \sum_{j=1}^{n} |\mu_1, \cdots, \mu_{j-1}, \mu b_j, \mu_{j+1}, \cdots, \mu_n|. \end{aligned} \tag{2.5}$$

On the other hand, noting

$$A\mu_j = (\sum_{k=1}^{n} a_{1k}\mu_{kj}, \cdots, \sum_{k=1}^{n} a_{nk}\mu_{kj})^T,\ 1 \le j \le n,$$

$$\mu b_j = (\sum_{k=1}^{n} \mu_{1k} b_{kj}, \cdots, \sum_{k=1}^{n} \mu_{nk} b_{kj})^T,\ 1 \le j \le n,$$

where $A = (a_{jk})_{n\times n}$ is assumed, we can compute by the Laplace expansion of a determinant along a column that

$$\sum_{j=1}^{n}|\mu_1,\cdots,\mu_{j-1},A\mu_j,\mu_{j+1},\cdots,\mu_n| = \sum_{j=1}^{n}\sum_{l=1}^{n}\Big(\sum_{k=1}^{n}a_{lk}\mu_{kj}\Big)M_{lj}$$
$$= \sum_{j,k,l=1}^{n}a_{lk}\mu_{kj}M_{lj} = \sum_{k,l=1}^{n}a_{lk}\sum_{j=1}^{n}\mu_{kj}M_{lj}$$
$$= \sum_{k=1}^{n}a_{kk}\det\mu = \mathrm{tr}(A)\det\mu, \tag{2.6}$$

and

$$\sum_{j=1}^{n}|\mu_1,\cdots,\mu_{j-1},\mu b_j,\mu_{j+1},\cdots,\mu_n| = \sum_{j=1}^{n}\sum_{l=1}^{n}\Big(\sum_{k=1}^{n}\mu_{lk}b_{kj}\Big)M_{lj}$$
$$= \sum_{j,k,l=1}^{n}\mu_{lk}b_{kj}M_{lj} = \sum_{j,k=1}^{n}b_{kj}\sum_{l=1}^{n}\mu_{lk}M_{lj}$$
$$= \sum_{j=1}^{n}b_{jj}\det\mu = \mathrm{tr}(B)\det\mu, \tag{2.7}$$

where (2.4) has been used twice. Now the results in the theorem follow immediately from (2.5), (2.6) and (2.7). The proof is finished. □

We remark that alternatively, the derivative formula for a determinant, (2.2), can be proved through using Jacobi's formula (see, e.g., [5]):

$$(\det\mu)_x = \mathrm{tr}(\mathrm{adj}(\mu)\mu_x), \tag{2.8}$$

where $\mathrm{adj}(\mu)$ is the adjugate of μ (i.e., the transpose of its cofactor matrix).

It is also interesting to note that solutions to the linear matrix differential equation (2.1) can be decomposed as follows:

$$\mu = \mu_a\nu\mu_b, \tag{2.9}$$

where ν is an arbitrary constant square matrix, and μ_a and μ_b solve the two special linear matrix differential equations

$$\mu_{a,x} = A\mu_a,\ \ \mu_{b,x} = \mu_b B, \tag{2.10}$$

respectively. Actually, a solution $\mu(x)$ to (2.1) with an initial condition $\mu(x_0) = \mu_0$ can be written as $\mu_a(x)\mu_0\mu_b(x)$, if one takes $\mu_a(x_0)$ and $\mu_b(x_0)$ to be the identity matrix.

§3 Special cases

We present a few applications of Theorem 2.1 to special cases.

Firstly, a direct result follows, when one of the two matrices A and B in Theorem 2.1 is zero.

Proposition 3.1. *Let μ and A be square matrices depending on x. If $\mu_x = A\mu$ or $\mu_x = \mu A$, then $(\det\mu)_x = tr(A)\det\mu$.*

The first half in the above proposition yields Liouville's formula (see, e.g., [6]):

$$\det \mu(x) = \mathrm{e}^{\int_{x_0}^{x} \mathrm{tr}(A(x'))\,dx'} \det \mu(x_0),\ \ x_0 \in \mathbb{R}. \tag{3.1}$$

That is why we call the formula (2.3) a generalized Liouville's formula.

A natural consequence that follows from Liouville's formula is Abel's identity. Let y_1 and y_2 solve an ordinary differential equation of second order:

$$y_{xx} + py_x + qy = 0,$$

where p and q are two functions of x. Then we have Abel's identity

$$W(x) = W(x_0)\mathrm{e}^{-\int_{x_0}^{x} p(x')\,dx'},\ \ x_0 \in \mathbb{R}, \tag{3.2}$$

where $W(x)$ is the Wronskian of y_1 and y_2:

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_{1,x} & y_{2,x} \end{vmatrix}.$$

This is because

$$\begin{bmatrix} y_1 & y_2 \\ y_{1,x} & y_{2,x} \end{bmatrix}_x = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ y_{1,x} & y_{2,x} \end{bmatrix}.$$

Secondly, we prove the following result originated in soliton theory.

Proposition 3.2. *Let μ, A and B be square matrices depending on x. If $\mu_x = [A,\mu] + B\mu$, then $(\det \mu)_x = tr(B) \det \mu$.*

Proof: Since we have

$$\mu_x = [A,\mu] + B\mu = (A + B)\mu - \mu A,$$

it follows from Theorem 2.1 that

$$(\det \mu)_x = [\mathrm{tr}(A + B) - \mathrm{tr}(A)] \det \mu = \mathrm{tr}(B) \det \mu,$$

which completes the proof. □

In particular, if $B = 0$, then we have $(\det \mu)_x = 0$, which is widely used to prove localness of characteristics of symmetries in a soliton hierarchy (see, e.g., [7]- [10]).

Also, if $\mathrm{tr}(B) = 0$, then we still have $(\det \mu)_x = 0$, which is crucially used in determining solutions to the associated Riemann-Hilbert problems in soliton theory (see, e.g., [3]).

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