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Poisson Distribution Results for ϑ-Spirallike Functions of Order γ

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Abstract. The main objective of this organized paper is to establish the Poisson distribution conditions for the ϑ-spirallike function classes $S_\vartheta(\gamma;\psi)$ and $K_\vartheta(\gamma;\psi)$. We also investigate an integral operator associated with the Poisson distribution.

§1 Introduction

Let E be the open unit disk on the complex plane $\mathbb{C}$. We denote by A, the class of functions f which are analytic in E with $f(0) = f'(0) - 1 = 0$. Such a function has the following Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

If f is univalent, we indicate a subset of A by S.

We now refer to the class of spirallike functions.

Let $\vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. A logarithmic ϑ-spiral is a curve given by

$$\omega = \omega_0 \exp(-e^{-i\vartheta} t) \qquad (-\infty < t < \infty)\,,$$

where ω_0 is a nonzero complex number. Thus $\omega = \omega(t)$ is a logarithmic ϑ-spiral, then

$$\Im\left[e^{i\vartheta} \log \omega(t)\right] = \text{constant} \qquad (-\infty < t < \infty)\,.$$

Observe that 0-spirals are radial half-lines. For each ϑ $\left(|\vartheta| < \frac{\pi}{2}\right)$ there is a unique ϑ-spiral which joins a given point $\omega \neq 0$ to the origin (see [7]).

A domain D containing the origin is named ϑ-spirallike if for each point $\omega \neq 0$ in D the arc of the ϑ-spiral from ω to the origin lies entirely in D. This obviously implies that D is simply connected. An analytic univalent function f is named ϑ-spirallike if its range is ϑ-spirallike. The class of ϑ-spirallike functions in the open unit disk E is indicated by S_ϑ. Analytically, this means that a function $f \in A$ belongs to the class S_ϑ if and only if

$$\Re\left(e^{i\vartheta} \frac{zf'(z)}{f(z)}\right) > 0 \qquad \left(|\vartheta| < \frac{\pi}{2}, z \in E\right).$$

The class S_ϑ of ϑ-spirallike functions was defined by Špaček [16] and he proved that $S_\vartheta \subset S^*$. We note at this point that $S_0 = S^*$. Libera [8] extended this definition to functions ϑ-spirallike

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of order γ $(0 \leq \gamma < 1)$ indicated by $S_\vartheta(\gamma)$ as follows:

$$\Re\left(e^{i\vartheta}\frac{zf'(z)}{f(z)}\right) > \gamma\cos\vartheta \qquad \left(|\vartheta| < \frac{\pi}{2}, z \in E\right).$$

Clearly, $S_\vartheta(\gamma) \subset S_\vartheta$. Later, Murugusundaramoorthy and Magesh [12] (see also [10]) introduce the following subclasses of ϑ-spirallike functions as below:

$$S_\vartheta(\gamma;\psi) = \left\{f \in A : \Re\left(e^{i\vartheta}\frac{zf'(z)}{(1-\psi)f(z)+\psi zf'(z)}\right) > \gamma\cos\vartheta,\ |\vartheta| < \frac{\pi}{2}, z \in E\right\}$$

and

$$K_\vartheta(\gamma;\psi) = \left\{f \in A : \Re\left(e^{i\vartheta}\frac{zf''(z)+f'(z)}{f'(z)+\psi zf''(z)}\right) > \gamma\cos\vartheta,\ |\vartheta| < \frac{\pi}{2}, z \in E\right\},$$

where $0 \leq \psi < 1$ and $0 \leq \gamma < 1$.

In this organized paper, we provide some results for the Poisson distribution conditions of the ϑ-spirallike function classes $S_\vartheta(\gamma;\psi)$ and $K_\vartheta(\gamma;\psi)$. We also investigate an integral operator associated with this condition.

§2 Problem Formulation

We aim to start by stating the Poisson distribution and some basis lemmas for further investigations.

The Poisson distribution is one of the most powerful tools in the examination of many problems of multivariate data research fields. Stated differently, nowadays, the Poisson distribution is generated from univalent functions. In this context, Porwal [14] introduced some basic inequalities. Corresponding inequalities have been obtained for different subclasses of analytic functions (see [1], [5], [13]). Moreover, the well known elementary distributions such as Borel, Logarithmic, Pascal and Poisson have been partially used in the theory of univalent functions (for example, see [2], [3], [4], [6], [9], [11], [15], [17]).

Suppose that X is a non-negative discrete random variable. Then, the probability distribution function is given as below:

$$P(X=k) = \frac{m^k e^{-m}}{k!} \qquad (k = 0, 1, 2, 3, \ldots),$$

where m is called the parameter.

Based upon the above function, let us develop the following power series:

$$P(z) = z + \sum_{k=2}^{\infty}\frac{e^{-m}m^{k-1}}{(k-1)!}z^k.$$

Performing the obvious calculations, we have that the radius of convergence of the above power series is infinity.

For the sake of simplicity, the following notations

$$\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!} = e^m - 1,$$

$$\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-2)!} = me^m$$

and

$$\sum_{k=3}^{\infty}\frac{m^{k-1}}{(k-3)!} = m^2e^m$$

will be used in the proof of theorems.

Lemma 1. *(see [12]) A function f defined by (1) is in the class $S_\vartheta(\gamma;\psi)$ if*

$$\sum_{k=2}^{\infty}\left[(1-\psi)(k-1)\sec\vartheta+(1-\gamma)(1-\psi+k\psi)\right]|a_k|\leq 1-\gamma,$$

where $|\vartheta|<\frac{\pi}{2},\ 0\leq\psi<1$ *and* $0\leq\gamma<1$.

Lemma 2. *A function f defined by (1) is in the class $K_\vartheta(\gamma;\psi)$ if*

$$\sum_{k=2}^{\infty}k\left[(1-\psi)(k-1)\sec\vartheta+(1-\gamma)(1-\psi+k\psi)\right]|a_k|\leq 1-\gamma,$$

where $|\vartheta|<\frac{\pi}{2},\ 0\leq\psi<1$ *and* $0\leq\gamma<1$.

§3 Results

In this part, we offer to get the Poisson distribution conditions for the classes $S_\vartheta(\gamma;\psi)$ and $K_\vartheta(\gamma;\psi)$.

Theorem 3. *Let $m>0$. If the following result holds*

$$\left[(1-\psi)\sec\vartheta+(1-\gamma)\psi\right]me^m\leq 1-\gamma, \tag{2}$$

then $P(z)\in S_\vartheta(\gamma;\psi)$.

Proof. By considering Lemma 1 with the Poisson relation

$$P(z)=z+\sum_{k=2}^{\infty}\frac{e^{-m}m^{k-1}}{(k-1)!}z^k,$$

it is enough to prove that

$$\sum_{k=2}^{\infty}\left[(1-\psi)(k-1)\sec\vartheta+(1-\gamma)(1-\psi+k\psi)\right]\frac{e^{-m}m^{k-1}}{(k-1)!}\leq 1-\gamma.$$

Next, writing $k=(k-1)+1$, after routine computations, we find that

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left[(1-\psi)(k-1)\sec\vartheta+(1-\gamma)(1-\psi+k\psi)\right]\frac{e^{-m}m^{k-1}}{(k-1)!}\\
=\ &\left[(1-\psi)\sec\vartheta+(1-\gamma)\psi\right]\sum_{k=2}^{\infty}(k-1)\frac{e^{-m}m^{k-1}}{(k-1)!}\\
&+(1-\gamma)\sum_{k=2}^{\infty}\frac{e^{-m}m^{k-1}}{(k-1)!}\\
=\ &\left[(1-\psi)\sec\vartheta+(1-\gamma)\psi\right]\sum_{k=2}^{\infty}\frac{e^{-m}m^{k-1}}{(k-2)!}\\
&+(1-\gamma)\sum_{k=2}^{\infty}\frac{e^{-m}m^{k-1}}{(k-1)!}\\
=\ &\left\{\left[(1-\psi)\sec\vartheta+(1-\gamma)\psi\right]me^m+(1-\gamma)(e^m-1)\right\}e^{-m}\\
=\ &\left[(1-\psi)\sec\vartheta+(1-\gamma)\psi\right]m+(1-\gamma)(1-e^{-m}).
\end{aligned}$$

Indeed, the last expression is bounded by $1-\gamma$ if and only if (2) holds.

Thus, the proof is complete. □

Setting $\psi = 0$ in Theorem 3, we get

Corollary 4. *Let $m > 0$. Then $P(z) \in S_\vartheta(\gamma)$, with*

$$me^m \sec\vartheta \leq 1-\gamma.$$

Theorem 5. *Let $m > 0$. If the following result holds*

$$[(1-\psi)\sec\vartheta + (1-\gamma)\psi]\, m^2 e^m + [2(1-\psi)\sec\vartheta + (1-\gamma)(2\psi+1)]\, me^m \leq 1-\gamma, \qquad (3)$$

then $P(z) \in K_\vartheta(\gamma;\psi)$.

Proof. By considering Lemma 2 with the Poisson relation

$$P(z) = z + \sum_{k=2}^{\infty} \frac{e^{-m} m^{k-1}}{(k-1)!} z^k,$$

it is enough to prove that

$$\sum_{k=2}^{\infty} k\,[(1-\psi)(k-1)\sec\vartheta + (1-\gamma)(1-\psi+k\psi)]\, \frac{e^{-m} m^{k-1}}{(k-1)!} \leq 1-\gamma.$$

Further, writing $k^2 = (k-1)(k-2) + 3(k-1) + 1$ and $k = (k-1)+1$, we immediately find that

$$\begin{aligned}
&\sum_{k=2}^{\infty} k\,[(1-\psi)(k-1)\sec\vartheta + (1-\gamma)(1-\psi+k\psi)]\, \frac{e^{-m} m^{k-1}}{(k-1)!} \\
= \;& [(1-\psi)\sec\vartheta + (1-\gamma)\psi] \sum_{k=2}^{\infty} (k-1)(k-2) \frac{e^{-m} m^{k-1}}{(k-1)!} \\
& + [2(1-\psi)\sec\vartheta + (1-\gamma)(2\psi+1)] \sum_{k=2}^{\infty} (k-1) \frac{e^{-m} m^{k-1}}{(k-1)!} + (1-\gamma) \sum_{k=2}^{\infty} \frac{e^{-m} m^{k-1}}{(k-1)!} \\
= \;& [(1-\psi)\sec\vartheta + (1-\gamma)\psi] \sum_{k=3}^{\infty} \frac{e^{-m} m^{k-1}}{(k-3)!} \\
& + [2(1-\psi)\sec\vartheta + (1-\gamma)(2\psi+1)] \sum_{k=2}^{\infty} \frac{e^{-m} m^{k-1}}{(k-2)!} + (1-\gamma) \sum_{k=2}^{\infty} \frac{e^{-m} m^{k-1}}{(k-1)!} \\
= \;& \big\{ [(1-\psi)\sec\vartheta + (1-\gamma)\psi]\, m^2 e^m + [2(1-\psi)\sec\vartheta + (1-\gamma)(2\psi+1)]\, me^m \\
& + (1-\gamma)(e^m - 1) \big\}\, e^{-m} \\
= \;& [(1-\psi)\sec\vartheta + (1-\gamma)\psi]\, m^2 + [2(1-\psi)\sec\vartheta + (1-\gamma)(2\psi+1)]\, m \\
& + (1-\gamma)(1-e^{-m}).
\end{aligned}$$

Indeed, the last expression is bounded previously by $1-\gamma$ if and only if (3) holds. □

Setting $\psi = 0$ in Theorem 5, we get

Corollary 6. *Let $m > 0$. Then $P(z) \in K_\vartheta(\gamma)$, with*

$$m^2 e^m \sec\vartheta + [2\sec\vartheta + (1-\gamma)]\, me^m \leq 1-\gamma.$$

§4 An Integral Operator

In this part, we will prove similar results associated with a special integral operator $I(z)$ as below:

$$I(z) = \int_0^z \frac{P(\xi)}{\xi} d\xi. \tag{4}$$

Theorem 7. *Suppose that $m > 0$. If the following result holds*

$$[(1-\psi)\sec\vartheta + (1-\gamma)\psi]\,(1-e^{-m}) + (1-\psi)(1-\gamma-\sec\vartheta)\frac{(1-e^{-m}-me^{-m})}{m} \leq 1-\gamma,$$

then $I(z) \in S_\vartheta(\gamma;\psi)$.

Proof. From (4), we obtain

$$\begin{aligned} I(z) &= z + \sum_{k=2}^{\infty} \frac{e^{-m}m^{k-1}}{(k-1)!}\frac{z^k}{k} \\ &= z + \sum_{k=2}^{\infty} \frac{e^{-m}m^{k-1}}{k!} z^k. \end{aligned}$$

By considering Lemma 1, it can be written

$$\sum_{k=2}^{\infty} [(1-\psi)(k-1)\sec\vartheta + (1-\gamma)(1-\psi+k\psi)]\,\frac{e^{-m}m^{k-1}}{k!} \leq 1-\gamma.$$

Next, after routine computations, we find that

$$\begin{aligned} &\sum_{k=2}^{\infty} [(1-\psi)(k-1)\sec\vartheta + (1-\gamma)(1-\psi+k\psi)]\,\frac{e^{-m}m^{k-1}}{k!} \\ =\ & [(1-\psi)\sec\vartheta + (1-\gamma)\psi] \sum_{k=2}^{\infty} k\frac{e^{-m}m^{k-1}}{k!} \\ & +(1-\psi)(1-\gamma-\sec\vartheta)\sum_{k=2}^{\infty}\frac{e^{-m}m^{k-1}}{k!} \\ =\ & \left\{[(1-\psi)\sec\vartheta + (1-\gamma)\psi]\,(e^m-1) + (1-\psi)(1-\gamma-\sec\vartheta)\left(\frac{e^m-1-m}{m}\right)\right\} e^{-m} \\ =\ & [(1-\psi)\sec\vartheta + (1-\gamma)\psi]\,(1-e^{-m}) + (1-\psi)(1-\gamma-\sec\vartheta)\left(\frac{1-e^{-m}-me^{-m}}{m}\right). \end{aligned}$$

Here the proof ends. □

Corollary 8. *Let $m > 0$. Then $P(z) \in S_\vartheta(\gamma)$, with*

$$\sec\vartheta(1-e^{-m}) + (1-\gamma-\sec\vartheta)\frac{(1-e^{-m}-me^{-m})}{m} \leq 1-\gamma.$$

Theorem 9. *Let $m > 0$. If the following result holds*

$$[(1-\psi)\sec\vartheta + (1-\gamma)\psi]\, me^m \leq 1-\gamma,$$

then $I(z) \in K_{\vartheta}(\gamma;\psi)$.

Proof. By considering Lemma 1, it can be written

$$\sum_{k=2}^{\infty} k\left[(1-\psi)(k-1)\sec\vartheta + (1-\gamma)(1-\psi+k\psi)\right]\frac{e^{-m}m^{k-1}}{k!} \leq 1-\gamma,$$

or, equivalently

$$\sum_{k=2}^{\infty}\left[(1-\psi)(k-1)\sec\vartheta + (1-\gamma)(1-\psi+k\psi)\right]\frac{e^{-m}m^{k-1}}{(k-1)!} \leq 1-\gamma.$$

The continuing part of the proof is similar to Theorem 3, so the proof ends. □

Corollary 10. *Let* $m > 0$. *Then* $P(z) \in K_{\vartheta}(\gamma)$, *with*

$$me^{m}\sec\vartheta \leq 1-\gamma.$$

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