Appl. Math. J. Chinese Univ.
2022, 37(3): 404-414

Modified integral equation combined with the decomposition method for time fractional differential equations with variable coefficients

Muhammad Amin Sadiq Murad

Abstract. In this paper, the modified integral equation, namely, Elzaki transformation coupled with the Adomian decomposition method called Elzaki Adomian decomposition method (EADM) is used to investigate the solution of time-fractional fourth-order parabolic partial differential equations (PDEs) with variable coefficients. The introduced method is used to solve two models of the proposed problem, the analytical and approximate solutions of the models are obtained. The outcomes illustrate that the proposed technique is a highly accurate, and facilitates the process of solving differential equations by comparing it, with the exact solution and those obtained by the variation iteration method (VIM) and Laplace homotopy perturbation method (LHPM).

§1 Introduction

Fractional calculus has been applied to many applications in defferent branches of science such as economy, biology, physics, engineering, and many more. Because of its ability in describing the characteristics of various linear and nonlinear phenomena in different fields, numbers of researchers and mathematicians have begun to apply the fractional calculus to their problems. The analytical and exact solutions to the differential equations of fractional order are quite difficult to achieve, therefore the numerical methods are used to find their approximate solutions.

The fourth-order parabolic PDEs played an enormous role in applied sciences in different fields, such as engineering, mechanics, and physics. Many well-known physical problems are described by fourth-order parabolic PDEs with variable coefficients. In [1] deformation of a viscoelastic beam and mathematical modeling of its plate deflection were described by fourth-order parabolic PDEs. Many researchers tried to improve the numerical techniques to investigate the solution of the fourth-order parabolic PDEs with variable coefficients of integer and fractional orders. The variation iteration method to solve the present problem was employed

Received: 2020-05-27. Revised: 2020-07-29.
MR Subject Classification: 65M25, 65R10.
Keywords: Elzaki transformation, Adomian decomposition method, time-fractional fourth-order parabolic,Variation iteration method, Laplace homotopy perturbation.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-022-4159-5.

by Biazar and Ghazvini [2], who modified Laplace VIM for solving the fourth-order PDEs with variable coefficients [3], Adomian decomposition method (ADM) [4], homotopy perturbation method(HPM) [5], variation iteration method and Adomian decomposition method [6], a comparison between He's homotopy perturbation method and variation iteration method [7], and tension spline method [8]. Pandey and Mishra solved the time-fractional fourth-order PDEs using the homotopy analysis Sumudu transform technique [9]. Recently, Javidi and Ahmad used the Laplace homotopy perturbation method to find the solution for the proposed problem [10].

Elzaki integral transform is a modification of the Laplace and Sumudu transforms which were invented by Tariq [11]. Elzaki transformation is an efficient and powerful technique that has found the exact solutions to several differential equations that cannot be solved by Sumudu transform [12]. In the last decade, the Elzaki transform has been used to study the exact and approximate solutions for many differential equations with integer and fractional orders [13–16]. In [17] HPM coupled with the Elzaki transform is used to solve the fourth-order parabolic PDEs, also Elzaki transform and HPM are combined to investigate the solution of time-fractional Navier-Stokes equation and system of non-linear PDEs see [18,19]. Tariq and Biazar used Elzaki transform combined with the Adomian decomposition method to solve a class of time-fractional PDEs [20]. The well-known sin-gordon equation was solved by Elzaki transform and Adomian polynomial technique [21], also Loyinmi and Akinfe used Elzaki transformation coupled with HPM to solve Fisher's reaction-diffusion [22]. Naveed and Muhammad [23], applied Elzaki transform combined with the variation iteration method to solve nonlinear oscillators.

The purpose of the present paper is to expand the applications of the decomposition method combined with the Elzaki transformation and show the computational efficiency of the EADM technique in solving fractional differential equations. Consider time-fractional fourth-order parabolic PDE with variable coefficients:

$$D_t^{\alpha} v\left(x,y,t\right) + \beta\left(x,y\right)\frac{\partial^4 v}{\partial x^4} + \gamma\left(x,y\right)\frac{\partial^4 v}{\partial y^4} = 0, \quad t>0,\ a<x,y<b, \tag{1}$$

where $\beta\left(x,y\right), \gamma\left(x,y\right) > 0$ and $1 < \alpha \leq 2$
subjected to the initial conditions

$$v\left(x,y,0\right) = h_0\left(x,\ y\right), \quad v_t\left(x,y,0\right) = h_1\left(x,\ y\right),$$

and the boundary conditions

$$\begin{aligned} v(a,y,t) &= f_0(y,t), & v(b,y,t) &= f_1(y,t), \\ v(x,a,t) &= f_2(x,t), & v(x,b,t) &= f_3(x,t), \\ \frac{\partial^2 v}{\partial x^2}(a,y,t) &= g_0(y,t), & \frac{\partial^2 v}{\partial x^2}(b,y,t) &= g_1(y,t), \\ \frac{\partial^2 v}{\partial y^2}(x,a,t) &= g_2(x,t), & \frac{\partial^2 v}{\partial y^2}(x,b,t) &= g_3(x,t). \end{aligned}$$

where h_0, h_1, f_i, and $g_i, i = 0,1,2,3$ are continuous functions. The existence and uniqueness of the proposed problem are studied in [24][25].

§2 Periliminary

In this section, we introduce some definitions and properties of fractional calculus and Elzaki transform which are used in this article.

Definition 2.1. *[24] A real valued function $g(y), y > 0$is belong to the space C_σ, $\sigma \in \mathbb{R}$ if there exists at least a real number $d > \sigma$, such that $g(y) = y^d g_1(y)$ where $g_1(y) \in C(0, \infty)$, and it is said to be in the space C_σ^n if $g^n \in \mathbb{R}_\sigma$, $n \in \mathbb{N}$.*

Definition 2.2. *The function $f(u)$ is called Riemann-Liouville fractional integral of order $\alpha \geq 0$, if it defines as:*

$$J^\alpha f(u) = \frac{1}{\Gamma(\alpha)} \int_0^u (u-t)^{\alpha-1} f(t)\, dt, \ \alpha > 0, \ t > 0. \tag{2}$$

In particular $J^0 f(u) = f(u)$.

For $\theta \geq 0$ and $\vartheta \geq -1$ some properties of the operator J^α

1. $J^\alpha J^\theta f(u) = J^{\alpha+\theta} f(u)$
2. $J^\alpha J^\theta f(u) = J^\theta J^\alpha f(u)$
3. $J^\alpha x^\vartheta = \frac{\Gamma(\vartheta+1)}{\Gamma(\alpha+\vartheta+1)} x^{\alpha+\vartheta}$.

Definition 2.3. *The function $f \in C_{-1}^n$, $n \in \mathbb{N}$ is called Caputo fractional derivative if it defines as*

$$D^\alpha f(u) = \frac{1}{\Gamma(n-\alpha)} \int_0^u (u-t)^{n-\alpha-1} f^{(n)}(t)\, dt, \quad n-1 < \alpha \leq n. \tag{3}$$

Definition 2.4. *[11] The Elzaki transform of the function $f(u)$ is defined as:*

$$E[f(u)] = T(v) = v \int_0^\infty f(u)\, e^{\frac{-u}{v}} du \qquad u > 0. \tag{4}$$

In partial derivative case:

$$E\left[\frac{\partial f(x,u)}{\partial u}\right] = v \int_0^\infty \frac{\partial f}{\partial u} e^{\frac{-u}{v}} du.$$

Using integration by parts, we obtain

$$E\left[\frac{\partial f(x,u)}{\partial u}\right] = \frac{T(x,v)}{v} - v f(x,0)$$

Suppose that f is piecewise continuous, then we can calculate $E\left[\frac{\partial f}{\partial x}\right]$ as follows:

$$E\left[\frac{\partial f(x,u)}{\partial x}\right] = \int_0^\infty v e^{\frac{-u}{v}} \ \frac{\partial f(x,u)}{\partial x} du = \frac{\partial}{\partial x} \int_0^\infty v e^{\frac{-u}{v}} \ f(x,u)\, du = \frac{\partial}{\partial x} T(x,v),$$

Similarly, we can have:

$$E\left[\frac{\partial^2 f(x,u)}{\partial x^2}\right] = \frac{d^2 T(x,v)}{dx^2}$$

Assume that $\frac{\partial f}{\partial u} = h$, then we have:

$$E\left[\frac{\partial^2 f(x,u)}{\partial u^2}\right] = E\left[\frac{\partial h(x,u)}{\partial u}\right] = \frac{1}{v} E[h(x,u)] - v h(x,0)$$

$$E\left[\frac{\partial^2 f(x,u)}{\partial u^2}\right] = \frac{T(x,v)}{v^2} - f(x,0) - v \frac{\partial f}{\partial u}(x,u)$$

Table 1. Elzaki table of transform for some functions.

$f(t)$	$E[f(t)] = T(v)$
1	v^2
t	v^3
t^n	$n!v^{n+2}$
e^{at}	$\frac{v^2}{1-av}$
$\sin at$	$\frac{av^3}{1+a^2v^2}$
$\cos at$	$\frac{v^2}{1+a^2v^2}$
$\frac{t^{a-1}}{\Gamma(a)},\ a=0$	v^{a+1}

By mathematical induction one can extend this result to the n^{th} partial derivative as:

$$E\left[\frac{\partial^n f(x,u)}{\partial u^n}\right] = \frac{T(x,v)}{v^n} - \sum_{i=0}^{n-1} v^{2-n+i}\frac{\partial^i f(x,0)}{\partial u^i}. \tag{5}$$

§3 Elzaki Adomian decomposition method (EADM)

Consider the following time- fractional fourth-order PDE:

$$D_t^\alpha v(x,y,t) + r(x,y)\frac{\partial^4 v(x,y,t)}{\partial x^4} + l(x,y)\frac{\partial^4 v(x,y,t)}{\partial y^4} = g(x,y,t), \qquad 1<\alpha\le 2 \tag{6}$$

Subject to the initial conditions

$$v(x,y,0) = f(x,y), \qquad v_t(x,y,0) = h(x,y),$$

where $g(x,y,t)$be a source term,r and l are known functions, $f(x,y)$ and $h(x,y)$ are algebraic functions, also $D_t^\alpha v(x,y,t)$ is the Caputo fractional derivative of $v(x,t)$which is introduced as:

$$D_t^\alpha v(x,y,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{v^{(n)}(x,y,\varphi)}{(t-\varphi)^{\alpha+1-n}}d\varphi, \qquad n-1<\alpha\le n,$$

Elzaki form of the Caputo operators

$$E[D_t^\alpha v(x,y,t)] = \frac{1}{s^\alpha}E[v(x,y,t)] - \sum_{k=0}^{n-1} v^{(k)}(x,y,0)\,s^{2-\alpha+k} \tag{7}$$

Applying the Elzaki transform on (6), we obtain

$$E[D_t^\alpha v(x,y,t)] + E\left[r(x,y)\frac{\partial^4 v(x,y,t)}{\partial x^4} + l(x,y)\frac{\partial^4 v(x,y,t)}{\partial y^4}\right] = E[g(x,y,t)]$$

Taking Caputo operator of Elzaki transform, we obtain

$$E[v(x,y,t)] = s^2 f(x,y) + s^3 h(x,y) - s^\alpha E\left[r(x,y)\frac{\partial^4 v(x,y,t)}{\partial x^4} + l(x,y)\frac{\partial^4 v(x,y,t)}{\partial y^4}\right] + s^\alpha E[g(x,y,t)]$$

Taking the Elzaki inverse for both sides, gives

$$v(x,y,t) = H(x,y,t) - E^{-1}\left[s^\alpha E\left[r(x,y)\frac{\partial^4 v(x,y,t)}{\partial x^4} + l(x,y)\frac{\partial^4 v(x,y,t)}{\partial y^4}\right]\right],$$

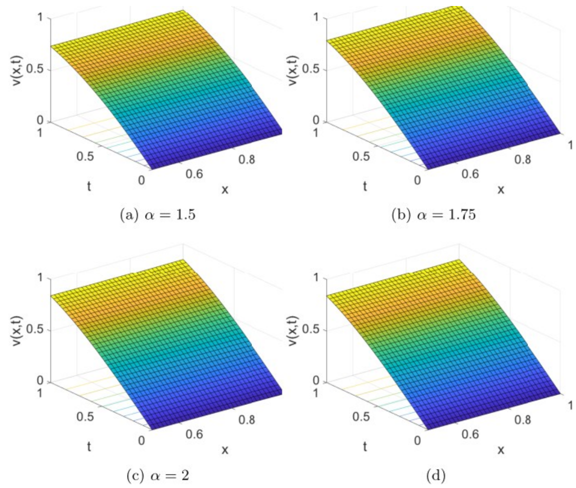

Figure 1. The EADM solutions of $v\,(x,t)$ of (9) at (a) $\alpha = 1.5$, (b) $\alpha = 1.75$, (c) $\alpha = 2$, and (d) exact solution.

where $H\,(x,y,t)$ represents the initial conditions and the source term arises. Applying Adomian decomposition method to get out the nonlinear terms (if any), then the approximate solution of the proposed problem is the infinite series:

$$v\,(x,y,t) = \sum_{n=0}^{\infty} v_n\,(x,y,t)\,. \tag{8}$$

§4 Numerical results

We will apply the decomposition method coupled with the modified integral transform (Elzaki transform) on two models of time-fractional fourth-order parabolic PDEs with variable coefficients. These applications show that the proposed technique converges very fast and gives the effective results.

In Figure 1 we have four graphs; (a) and (b) consist of the solutions of (9) at $\alpha = 1.5$ and $\alpha = 1.75$ respectively, using EADM. Figure 1 (a) and (b) show that fractional order approaches to integer order solution surfaces of fractional order are convergent to the integer order surface. In Figure 1 the graphs (c) and (d) represent EADM solution of $v\,(x,t)$ at $\alpha = 2$ and the exact solution of (9) respectively. Figure 1 (c) and (d) show that the proposed method is in good agreement with the exact solution. Consequently, any surface can be modeled as desired by a physical phenomenon happening in nature.

Example 4.1. *Consider one dimension time-fractional fourth-order parabolic PDEs with variable coefficients:*

$$D_t^{\alpha} v\,(x,t) + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 v\,(x,t)}{\partial x^4} = 0, \quad \frac{1}{2} < x < 1, \quad t > 0,\ 1 < \alpha \le 2, \tag{9}$$

with the initial conditions $v(x,0)=0,\ \frac{\partial v}{\partial t}(x,0)=1+\frac{x^5}{120}$,
and the boundary conditions

$$v\left(\frac{1}{2},t\right)=\left(1+\frac{\left(\frac{1}{2}\right)^5}{120}\right)\sin(t,\alpha), \qquad v(1,t)=\left(\frac{121}{120}\right)\sin(t,\alpha),$$

$$\frac{\partial^2 v}{\partial x^2}\left(\frac{1}{2},t\right)=\frac{1}{6}\left(\frac{1}{2}\right)^3\sin(t,\alpha), \qquad \frac{\partial^2 v}{\partial x^2}(1,t)=\frac{1}{6}\sin(t,\alpha),$$

where $\sin(t,\alpha)=\sum_{i=0}^{\infty}\frac{(-1)^i t^{i\alpha+1}}{\Gamma(i\alpha+2)}$

Applying the Elzaki transform on (9), we obtain

$$E\left[D_t^\alpha v(x,t)+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 v}{\partial x^4}\right]=0.$$

Using the formula (7), we obtain

$$E[v(x,t)]=s^3\left(1+\frac{x^5}{120}\right)-s^\alpha E\left[\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 v}{\partial x^4}\right].$$

Applying the inverse Elzaki transform

$$v(x,t)=E^{-1}\left[s^3\left(1+\frac{x^5}{120}\right)\right]-E^{-1}\left[s^\alpha E\left[\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 v}{\partial x^4}\right]\right],$$

$$v(x,t)=\left(1+\frac{x^5}{120}\right)t-E^{-1}\left[s^\alpha E\left[\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 v}{\partial x^4}\right]\right].$$

Using the ADM, we obtain

$$v_0(x,t)=\left(1+\frac{x^5}{120}\right)t$$

$$v_{n+1}(x,t)=-E^{-1}\left[s^\alpha E\left[\left(\frac{1}{x}+\frac{x^4}{120}\right)\sum_{n=0}^{\infty}\frac{\partial^4 v_n}{\partial x^4}\right]\right],\ n=0,\ 1,\ 2,\ \ldots.$$

$$v_1(x,t)=-E^{-1}\left[s^\alpha E\left[\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 v_0}{\partial x^4}\right]\right]$$

$$v_1(x,t)=-E^{-1}\left[s^\alpha E\left[\left(1+\frac{x^5}{120}\right)t\right]\right]=-E^{-1}\left[s^{\alpha+3}\left(1+\frac{x^5}{120}\right)\right]$$

$$v_1(x,t)=-\left(1+\frac{x^5}{120}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$v_2(x,t)=E^{-1}\left[s^\alpha E\left[\left(1+\frac{x^5}{120}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right]=E^{-1}\left[s^{2\alpha+3}\left(1+\frac{x^5}{120}\right)\right]$$

$$v_2(x,t)=\left(1+\frac{x^5}{120}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}.$$

In the same way, one can have

$$v_3(x,t)=-\left(1+\frac{x^5}{120}\right)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)},$$

$$v_4(x,t)=\left(1+\frac{x^5}{120}\right)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}.$$

According to the ADM the result can be expressed as:

$$v(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+\ldots.$$

Table 2. Exact solution and numerical evaluation when $t = 0.2, 0.4, 0.6$ for (9).

x	$\alpha = 1.5$			$\alpha = 1.75$			$\alpha = 2$	
	v_{VIM}	v_{LHPM}	v_{EADM}	v_{VIM}	v_{LHPM}	v_{EADM}	v_{EADM}	v_{Exact}
0.5	0.196914	0.194734	0.194734	0.197687	0.197359	0.19736	0.198721	0.198721
0.6	0.196991	0.194809	0.194809	0.197763	0.197437	0.197437	0.198798	0.198798
1.0	0.198504	0.196306	0.196306	0.199282	0.198952	0.198953	0.200324	0.200324
0.5	0.377682	0.370692	0.370692	0.383217	0.382210	0.382211	0.389519	0.389519
0.6	0.377828	0.370835	0.370836	0.383366	0.382359	0.382359	0.389670	0.389670
1.0	0.380730	0.373683	0.373683	0.383631	0.385296	0.385296	0.392663	0.392663
0.5	0.531411	0.521418	0.521419	0.547792	0.546533	0.546537	0.564789	0.564789
0.6	0.531617	0.521621	0.521620	0.548005	0.546747	0.546748	0.565008	0.565008
1.0	0.53570	0.525626	0.525627	0.552214	0.550946	0.550947	0.569347	0.569347

$$
\begin{aligned}
v(x,t) = &\left(1+\frac{x^5}{120}\right)t - \left(1+\frac{x^5}{120}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \left(1+\frac{x^5}{120}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\
&- \left(1+\frac{x^5}{120}\right)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \left(1+\frac{x^5}{120}\right)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\cdots \\
v(x,t) = &\left(1+\frac{x^5}{120}\right)\left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\cdots\right).
\end{aligned}
\tag{10}
$$

Assume that $\alpha = 2$, then the equation (10) becomes

$$
\begin{aligned}
v(x,t) &= \left(1+\frac{x^5}{120}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!}\cdots\right) \\
v(x,t) &= \left(1+\frac{x^5}{120}\right)\sin(t).
\end{aligned}
$$

Thus, the exact solution $v(x,t) = \left(1+\frac{x^5}{120}\right)\sin t$ of (9) is obtained where $\alpha = 2$.

Table 2: illustrates the exact and approximate solutions of (9) using EADM for different values of x, t, and α, and compares the obtained results with VIM, LHPM [10], and the exact solution by calculating only the third-order terms of the series. Therefore, the results of EADM are in good agreement with those obtained by VIM and LHPM.

Example 4.2. *Consider the problem of two dimensional time-fractional fourth-order parabolic PDE:*

$$
D_t^\alpha v(x,y,t) + 2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 v}{\partial x^4} + 2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 v}{\partial y^4} = 0, \quad 0 < x, y < 1, \quad t > 0, \tag{11}
$$

with the initial conditions

$$
v(x,y,0) = 0, \ \frac{\partial v}{\partial t}(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}
$$

and the boundary conditions

$$v\left(\frac{1}{2},y,t\right)=\left(2+\frac{\left(\frac{1}{2}\right)^6}{6!}+\frac{y^6}{6!}\right)\sin(t),\ v(1,y,t)=\left(2+\frac{1}{6!}+\frac{y^6}{6!}\right)\sin(t),$$

$$\frac{\partial^2 v}{\partial x^2}\left(\frac{1}{2},y,t\right)=\frac{\left(\frac{1}{2}\right)^4}{6!}\sin(t),\quad \frac{\partial^2 v}{\partial x^2}(1,y,t)=\frac{1}{6!}\sin(t),\ t>1,$$

$$\frac{\partial^2 v}{\partial y^2}\left(x,\frac{1}{2},t\right)=\frac{\left(\frac{1}{2}\right)^4}{6!}\sin(t),\quad \frac{\partial^2 v}{\partial y^2}(x,1,t)=\frac{1}{6!}\sin(t),\ t>1.$$

Applying the Elzaki transform on (11), we get

$$E\left[D_t^\alpha v(x,y,t)+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 v}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 v}{\partial y^4}\right]=0.$$

Using formula (7), we get

$$E[v(x,y,t)]=s^3\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)-s^\alpha E\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 v}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 v}{\partial y^4}\right].$$

Applying the inverse Elzaki transform, we have

$$v(x,y,t)=E^{-1}\left[s^3\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\right]-E^{-1}\left[s^\alpha E\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 v}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 v}{\partial y^4}\right]\right],$$

$$v(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t-E^{-1}\left[s^\alpha E\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 v}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 v}{\partial y^4}\right]\right].$$

Using the ADM, we obtain

$$v_0(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t$$

$$v_{n+1}(x,y,t)=-E^{-1}\left[s^\alpha E\left[\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sum_{n=0}^{\infty}\frac{\partial^4 v_n}{\partial x^4}\right]\right],\ n=0,\ 1,\ 2,\ \ldots.$$

$$v_1(x,y,t)=-E^{-1}\left[s^\alpha E\left[\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{\partial^4 v_0}{\partial x^4}\right]\right]$$

$$v_1(x,y,t)=-E^{-1}\left[s^\alpha E\left[\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t\right]\right]=-E^{-1}\left[s^{\alpha+3}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\right]$$

$$v_1(x,y,t)=-\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

In the same way, one can have

$$v_2(x,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

$$v_3(x,y,t)=-\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)},$$

$$v_4(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}.$$

According to the ADM the result can be expressed in the series:

$$v(x,y,t)=v_0(x,y,t)+v_1(x,y,t)+v_2(x,y,t)+v_3(x,y,t)+\ldots.$$

Table 3. Exact solution and numerical evaluation when $t = 0.2$ for (11).

x	$\alpha = 1.5$		$\alpha = 1.75$		$\alpha = 2$		
	v_{VIM}	v_{EADM}	v_{VIM}	v_{EADM}	v_{VIM}	v_{EADM}	v_{Exact}
0.1	0.389267	0.389368	0.391901	0.394618	0.394672	0.397339	0.397339
0.2	0.389927	0.389368	0.391901	0.394618	0.394672	0.397339	0.397339
0.3	0.389268	0.389368	0.391902	0.394618	0.394672	0.397339	0.397339
0.4	0.389270	0.389370	0.391904	0.394627	0.394674	0.397339	0.397339
0.5	0.389276	0.389376	0.391910	0.394644	0.394681	0.397341	0.397341
0.6	0.389293	0.389393	0.391927	0.394682	0.394698	0.397347	0.397347
0.7	0.389332	0.389432	0.391965	0.394762	0.394736	0.397344	0.397344
0.8	0.389409	0.389511	0.392044	0.394991	0.394816	0.397483	0.397483
0.9	0.389555	0.389655	0.392192	0.394991	0.394963	0.397632	0.397632

Thus, we have

$$\begin{aligned} v(x,y,t) = {} & \left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t-\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ & -\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\cdots, \end{aligned}$$

$$v(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(t-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\cdots\right). \tag{12}$$

Assume that $\alpha = 2$, then the equation (12) becomes

$$v(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\frac{t^9}{9!}\cdots\right)$$

$$v(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sin t.$$

Thus, the exact solution $v(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sin t$ of (11) is obtained where $\alpha = 2$. Table 3: gives the exact and approximate solutions of (11) for different values of x and α using EADM, and compares the obtained results with those obtained by VIM [7] and the exact solution by calculating only the third-order terms of the series. Therefore, EADM converges faster than the variation iteration method.

§5 Conclusion

The objective of this work is to investigate the solution of time-fractional fourth-order PDEs with variable coefficients, and show that the Elzaki Adomian decomposition method is an efficient method and facilitate the process of solving the time-fractional differential equations. The proposed technique applied to two models with different dimensions; the exact and approximate solutions are obtained for each model. In the first model, our obtained results using EADM are in good agreement with those developed by VIM and LHPM, in the second model, it seems that the approximate solution using EADM converge faster than VIM. Finally, according to the obtained results, the Elzaki Adomian decomposition method is better than VIM and LHPM, also the facilitation of solving time-fractional differential equations can take as an advantage over the other methods.

References

[1] A Q M Khaliq, E H Twizell. *A family of second order methods for variable coefficient fourth order parabolic partial differential equations*, Int J Comput Math, 1987, 23(1): 63-76.

[2] J Biazar, H Ghazvini. *He's variational iteration method for fourth-order parabolic equations*, Comput Math with Appl, 2007, 54(7-8): 1047-1054.

[3] M Nadeem, F Li, H Ahmad. *Modified Laplace variational iteration method for solving fourth-order parabolic partial differential equation with variable coefficients*, Comput Math with Appl, 2019, 78(6): 2052-2062.

[4] B Boukary, J Loufouilou-Mouyedo, J Bonazebi-Yindoula, G Bissanga. *Application of the Adomian Decomposition Method (ADM) for Solving the Singular Fourth-Order Parabolic Partial Differential Equation*, J Appl Math Phys, 2018, 6(7): 1476-1480.

[5] N A Khan, N U Khan, M Ayaz, A Mahmood, N Fatima. *Numerical study of time-fractional fourth-order differential equations with variable coefficients*, J King Saud Univ, 2011, 23(1): 91-98.

[6] S T Mohyauddin, A Yildirim, M M Hosseini, Y Khan. *Numerical study of time-fractional fourth-order differential equations with variable coefficients*, World Appl Sci J, 2010, 10(2): 1321-1327.

[7] A A Elbeleze, A Kilicman, B M Taib. *Numerical methods for fractional order singular partial differential equations with variable coefficients*, Math Probl Eng, 2014, 2014(1): 1-8.

[8] T Sultana, P Khandelwal. *Tension spline technique for the solution of fourth-order parabolic partial differential equation*, Malaya J Mat, 2018, 6(3): 514-520.

[9] R K Pandey, H K Mishra. *Numerical Simulation of Time-fractional Fourth Order Differential Equations via Homotopy Analysis Fractional Sumudu Transform Method*, ajna, 2015, 3(3): 52-64.

[10] M Javidi, B Ahmad. *Numerical solution of fourth-order time-fractional partial differential equations with variable coefficients*, J Appl Anal Comput, 2015, 5(1): 52-63.

[11] T M Elzaki. *The New Integral Transform*, Global J Pure Appl Math, 2011, 7(1): 57-64.

[12] E M Hilal. *Elzaki and Sumudu transforms for solving some differential equations*, Global J Pure Appl Math, 2012, 8(2): 167-173.

[13] U Laorach, W Ketpratum, P Pue-on. *Applications of the Elzaki transform method for solving quadratic Riccati differential equations Elzaki transformation*, Int J Math Comput Sci, 2020, 15(2): 291-300.

[14] D Malo, M S Murad, R Masiha, S Abdulazeez. *A New Computational Method Based on Integral Transform for Solving Linear and Nonlinear Fractional Systems*, Jurnal Matematika MANTIK, 2021, 7(1): 9-19.

[15] T M Elzaki, S M Elzaki, E A Elnour. *ELzaki transform fundamental properties investigations and applications*, Glob J Math Sci Theory Pract, 2012, 4(1): 1-13.

[16] M Duz, T M Elzaki. *Solution of constant coeffients partial derivative equations with elzaki transform method*, J App Eng Math, 2019, 9(3): 563-570.

[17] P K Bhadane, V H Pradhan. *Elzaki transform homotopy perturbation method for solving fourth order parabolic PDE with variable coefficients*, Int J Eng R App, 2013, 3(6): 2170-2173.

[18] R M Jena, S Chakraverty. *Solving time-fractional Navier-Stokes equations using homotopy perturbation Elzaki transform*, SN Appl Sci, 2019, 1(1): 1-13.

[19] T M Elzaki, J Biazar. *Homotopy perturbation method and Elzaki transform for solving system of nonlinear partial differential equations*, World Appl Sci J, 2013, 24(7): 944-948.

[20] O E Ige, R A Oderinu, T M Elzaki. *Adomian polynomial and Elzaki transform method for solving sine-gordon equations*, Int J Appl Math, 2019, 49(3): 1-7.

[21] A C Loyinmi, T K Akinfe. *Exact solutions to the family of Fisher's reaction-diffusion equation using Elzaki homotopy transformation perturbation method*, Eng Reports, 2020, 2(2): 1-32.

[22] N Anjum, M Suleman, D Lu, J H He, M Ramzan. *Numerical iteration for nonlinear oscillators by Elzaki transform*, J Low Freq Noise Vib Act Control, 2019, 39(4): 1-6.

[23] R V Slonevskii, R R Stolyarchuk, D Lu, J H He, M Ramzan. *Rational-fractional methods for solving stiff systems of differential equations*, J Math. Sci, 2008, 150(5): 2434-2438.

[24] B Liang, X Peng, C Qu. *Existence of Solutions to a Nonlinear Parabolic Equation of Fourth-Order in Variable Exponent Spaces*, Entropy, 2016, 18(11): 413.

[25] Y Cao, C Liu. *Global existence and non-extinction of solutions to a fourth-order parabolic equation*, Appl Math Lett, 2016, 61(1): 20-25.

Department of Mathematics, College of Science, University of Duhok, Duhok, Iraq.
Email:Muhammad.math@uod.ac