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Semi entropy of uncertain random variables and its application to portfolio selection

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Abstract. Semi entropy is a measure to characterize the indeterminacy of the uncertain random variable considering the values of the uncertain random variable which are lower than the mean. As important roles of semi entropy in finance, this paper presents the concept of semi entropy for uncertain random variables. In order to compute semi entropy for uncertain random variables, Monte-Carlo approach is provided. As an application of semi entropy, portfolio selection problems are optimized based on mean-semi entropy mode.

§1 Introduction

Semi entropy of an uncertain random variable is a device to measure indeterminacy of the uncertain random variable via considering the values of the uncertain random variable which are lower than the mean. Zhou et al. [23] introduced the concept of semi entropy for LR (Left-Right) fuzzy variables based on credibility measure and applied it to portfolio selection problem involving new markets as fuzzy variables. Furthermore, several authors devoted their works to the case of credibilistic portfolio selection problems via semi entropy, for instance [7, 8, 24]. It is mentioned that return of new market can be modeled as fuzzy variable or uncertain variable. Also, in many situations, we deal with a portfolio including new markets and historical markets which modeled as uncertain random variables. Therefore, we want to propose the concept of semi entropy for uncertain random variable and apply it to portfolio optimization problem. Thus, we first review some topics in the case of entropy and portfolio selection of uncertain returns.

After foundation of uncertainty theory, Liu [10] proposed the concept of entropy for uncertain variables via inception of Shanon entropy for random variables. After that, Dai [5] and Yao and Dai [22] presented the concept of quadratic and sine entropy for uncertain variables, respectively. Furthermore, Chen et al. [4] introduced cross (relative) entropy for uncertain variables for measuring the difference between two uncertainty distributions.

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In many situations, we are faced with some phenomena involving uncertainty and randomness. These phenomena can be modeled by chance theory which was presented by Liu [17]. For characterizing indeterminacy of uncertain random variables, Sheng et al. [20] proposed the concept of uncertain random variables and invoked it to the minimum spanning tree. Also, Jia et al. [9] proposed the concept of cross entropy for uncertain random variables to characterize the distance between two chance distributions. After that, Ahmadzade et al. [3] introduced the concept of partial entropy of uncertain random variables and for optimizing portfolio selection problems, for instance see [1, 2]. Since values of uncertain random variables which greater than mean are interested in finance, we should minimize entropy, variance or other risk measures with respect to the values which lower than mean. Therefore, we introduce the concepts of semi entropy and partial semi entropy for uncertain random variables. Furthermore, we propose Monte Carlo simulation approach for computing the value of partial semi entropy. As an application of semi entropy in finance, we optimize the portfolio selection problem via mean-semi entropy model.

This paper is organized as follows. Section 2 recalls some concepts of uncertainty theory and chance theory as they are needed. The concept of semi entropy and partial semi entropy are proposed for uncertain random variable and their properties are studied in Section 3. The Monte-Carlo approach for calculating partial semi entropy of uncertain random variables is provided in Section 4. In Section 5, portfolio selection problems with the uncertain random returns are solved via mean-semi entropy model. Finally, some brief conclusions are obtained in Section 6.

§2 Preliminaries

In this section, we review some concepts of chance theory, including chance measure, uncertain random variable, chance distribution, operational law, and expected value, and variance, and so on.

2.1 Uncertain Variables

In this subsection, we provide several definitions and elementary concepts of uncertainty theory that will be used in the next sections. For more details, the reader refers to [10, 11].

Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:
(i) (Normality) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
(ii) (Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
(iii) (Subadditivity) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

(iv) (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying $\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$ where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 1. *An uncertain variable ξ is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B.*

Definition 2. *(Liu [10]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$.

Theorem 1. *(Liu [10]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables, and $f_1, f_2, \cdots, f_n$ be measurable functions. Then $f_1(\xi_1), f_2(\xi_2), \cdots, f_n(\xi_n)$ are independent uncertain variables.*

Definition 3. *(Liu [10]) The events $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\Lambda_i^*\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\}$$

such that Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \Gamma\}$, $i = 1, 2, \cdots, n$, respectively, where Γ is sure event.

Definition 4. *(Liu [11]) Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(x)$ is called the inverse uncertainty distribution of ξ.*

Theorem 2. *(Liu [11]) Let $\xi_1, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)).$$

2.2 Uncertain Random Variable

The chance space is refer to the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$, in which $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space and $(\Omega, \mathcal{A}, \Pr)$ is a probability space.

Definition 5. *(Liu [17]) Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event. Then the chance measure of Θ is defined as*

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\} \mathrm{d}r.$$

Liu [17] proved that a chance measure satisfies normality, duality, and monotonicity properties, that is (i) $\mathrm{Ch}\{\Gamma \times \Omega\} = 1$; (ii) $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1$ for any event Θ; (iii) $\mathrm{Ch}\{\Theta_1\} \leq \mathrm{Ch}\{\Theta_2\}$ for any real number set $\Theta_1 \subset \Theta_2$. Besides, Hou [6] proved the subadditivity of chance measure, that is, $\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \leq \sum_{i=1}^{\infty} \mathrm{Ch}\{\Theta_i\}$ for a sequence of events $\Theta_1, \Theta_2, \cdots$.

Definition 6. *(Liu [17])*

An uncertain random variable is a measurable function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers, i.e., $\{\xi \in B\}$ is an event for any Borel set B.

To calculate the chance measure, Liu [18] presented a definition of chance distribution.

Definition 7. *(Liu [18]) Let ξ be an uncertain random variable. Then its chance distribution is defined by*

$$\Phi(x) = \text{Ch}\{\xi \leq x\}$$

for any $x \in \Re$.

Theorem 3. *(Liu [18]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables. Then the uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ has a chance distribution*

$$\Phi(x) = \int_{\Re^m} F(x, y_1, \cdots, y_m) \text{d}\Psi_1(y_1) \cdots \text{d}\Psi_m(y_m)$$

where $F(x, y_1, \cdots, y_m)$ is the uncertainty distribution of uncertain variable $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ for any real numbers $y_1, y_2, \cdots, y_m$.

Definition 8. *(Liu [18]) Let ξ be an uncertain random variable. Then its expected value is defined by*

$$E[\xi] = \int_0^{+\infty} \text{Ch}\{\xi \geq r\}\text{d}r - \int_{-\infty}^0 \text{Ch}\{\xi \leq r\}\text{d}r$$

provided that at least one of the two integrals is finite.

Let Φ denote the chance distribution of ξ. Liu [18] proved a formula to calculate the expected value of uncertain random variable with chance distribution, that is,

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\text{d}x - \int_{-\infty}^0 \Phi(x)\text{d}x.$$

Theorem 4. *(Liu [17]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables (not necessarily independent), then the where $E[f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)]$ is the expected value of the uncertain variable $f(y_1, \cdots,$*
$y_m, \tau_1, \cdots, \tau_n)$ for any real numbers $y_1, \cdots, y_m$.

Theorem 5. *(Liu [17], Linearity of Expected Value Operator) Assume η_1 and η_2 are random variables (not necessarily independent), τ_1 and τ_2 are independent uncertain variables, and f_1 and f_2 are measurable functions. Then*

$$E[f_1(\eta_1, \tau_1) + f_2(\eta_2, \tau_2)] = E[f_1(\eta_1, \tau_1)] + E[f_2(\eta_2, \tau_2)].$$

Theorem 6. *(Liu [17]) Let $f : R^n \to R$ be a measurable function, and $\xi_1, \xi_2, \cdots, \xi_n$ uncertain random variables on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \text{Pr})$. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain random variable determined by*

$$\xi(\gamma, \omega) = f(\xi_1(\gamma, \omega), \xi_2(\gamma, \omega), \cdots, \xi_n(\gamma, \omega))$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

§3 Semi Entropy of Uncertain Random Variables

In order to propose semi entropy for uncertain random variables, we should recall the concept of entropy for uncertain random variables introduced by Sheng et al. [20].

Definition 9. *(Sheng et al. [20]) "Suppose ξ is an uncertain random variable. Then the entropy of ξ is defined by*

$$H[\xi] = \int_{-\infty}^{\infty} T(\Phi(x))dx,$$

where $T(s) = -s\ln s - (1-s)\ln(1-s)$, and $\Phi(x)$ is chance distribution of ξ."

Since, investors benefit from the values of uncertain random variables which are greater than mean. Thus, we should consider entropy for the values that lower than mean. Furthermore, we should minimize this quantity in portfolio selection. Therefor, we introduce the concept of semi entropy as follows:

Definition 10. *Suppose that ξ is an uncertain random variable with expected value μ and chance distribution $\Phi(x)$. Semi entropy of uncertain random variable ξ is*

$$SH[\xi] = \int_{\mathbb{R}} T\big(\Phi(x)^-\big)\,dx,$$

$$\Phi(x)^- = \begin{cases} \Phi(x), & if \quad x \le \mu, \\ \\ 0, & if \quad x > \mu. \end{cases}$$

and $T(s) = -s\ln s - (1-s)\ln(1-s)$.

In many situations, we want to measure the indeterminacy of uncertain random variables corresponds to uncertain variables. For this purpose, by inception of Ahmadzade et al. [3], we propose the concept of partial semi entropy for uncertain random variables.

Definition 11. *Suppose that $\eta_1, \eta_2, \cdots, \eta_m$ are random variables and $\tau_1, \tau_2, \cdots, \tau_m$ are uncertain variables, also $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_m)$ is an uncertain random variable with expected value μ. Partial semi entropy of uncertain random variable ξ is defined as following*

$$PSH[\xi] = \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} T(F(x, y_1, \cdots, y_m)^-)\,dx\,d\Psi(y_1, \cdots, y_m)$$

where

$$F(x, y_1, \cdots, y_m)^- = \begin{cases} F(x, y_1, \cdots, y_m), & if \quad x \le \mu, \\ \\ 0, & if \quad x > \mu, \end{cases}$$

and $T(s) = -s\ln s - (1-s)\ln(1-s)$ and $F(x, y_1, \cdots, y_m)$ is the uncertainty distribution of uncertain variable $f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_m)$ for any real numbers $y_1, \cdots, y_m$. Also, $\Psi(y_1, \cdots, y_m)$ is the joint probability distribution of $\eta_1, \eta_2, \cdots, \eta_m$ for any $y_1, \cdots, y_m$.

Remark 1. *If uncertain random variables reduce to uncertain ones, we can use above definition for uncertain variables.*

Theorem 7. *Let $\eta_1, \eta_2, \cdots, \eta_n$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_n$, and $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_m$, respectively, and let f be a measurable function. Then*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_n, \tau_1, \tau_2, \cdots, \tau_m)$$

has partial semi entropy

$$PSH[\xi]=\begin{cases}\displaystyle\int_{\mathbb{R}^n}\int_0^{F(\mu,y_1,\cdots,y_n)}(F^{-1}(\beta,y_1,\cdots,y_n)-\mu)\ln\frac{\beta}{1-\beta}d\beta d\Psi_1(y_1)\cdots\Psi_n(y_n), & \\ \hfill if \quad \mu<0, & \\ \displaystyle\int_{\mathbb{R}^n}\int_0^{F(\mu,y_1,\cdots,y_n)}F^{-1}(\beta,y_1,\cdots,y_n)\ln\frac{\beta}{1-\beta}d\beta d\Psi_1(y_1)\cdots\Psi_n(y_n) & \\ \displaystyle+\mu\int_{F(\mu,y_1,\cdots,y_m)}^1\ln\frac{\beta}{1-\beta}d\beta d\Psi_1(y_1)\cdots\Psi_n(y_n), \qquad if \quad \mu>0, & \end{cases}$$

Proof. By using Definition 11 and Fubini's theorem, for $\mu<0$, we have

$$\begin{aligned}PSH[\xi]&=\int_{\mathbb{R}^n}\int_{-\infty}^{\mu}T\big(F(x,y_1,\cdots,y_n)\big)\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n)\\&=\int_{\mathbb{R}^n}\int_{-\infty}^{\mu}\int_0^{F(x,y_1,\cdots,y_n)}T'(\beta)\mathrm{d}\beta\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n)\\&=\int_{\mathbb{R}^n}\int_0^{F(\mu,y_1,\cdots,y_n)}\int_{F^{-1}(\beta,y_1,\cdots,y_n)}^{\mu}T'(\beta)\mathrm{d}x\mathrm{d}\beta\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n)\\&=\int_{\mathbb{R}^n}\int_0^{F(\mu,y_1,\cdots,y_m)}(F^{-1}(\beta,y_1,\cdots,y_n)-\mu)\ln\frac{\beta}{1-\beta}\mathrm{d}\beta\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n).\end{aligned}$$

Similarly, for $\mu>0$, we have

$$\begin{aligned}PSH[\xi]&=\int_{\mathbb{R}^n}\int_{-\infty}^{\mu}T\big(F(x,y_1,\cdots,y_m)\big)\mathrm{d}x\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_n(y_n)\\&=\int_{\mathbb{R}^n}\int_{-\infty}^{0}\int_0^{F(x,y_1,\cdots,y_n)}T'(\beta)\mathrm{d}\beta\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n)\\&\quad+\int_{\mathbb{R}^n}\int_0^{\mu}\int_{F(x,y_1,\cdots,y_n)}-T'(\beta)\mathrm{d}\beta\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n)\\&=\int_{\mathbb{R}^n}\int_0^{F(0,y_1,\cdots,y_n)}\int_{F^{-1}(\beta,y_1,\cdots,y_n)}^{0}T'(\beta)\mathrm{d}x\mathrm{d}\beta\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n)\\&\quad+\int_{\mathbb{R}^n}\int_{F(0,y_1,\cdots,y_n)}^{F(\mu,y_1,\cdots,y_n)}\int_0^{F^{-1}(\beta,y_1,\cdots,y_n)}-T'(\beta)\mathrm{d}x\mathrm{d}\beta\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n)\\&\quad+\int_{\mathbb{R}^n}\int_{F(\mu,y_1,\cdots,y_n)}\int_0^{\mu}-T'(\beta)\mathrm{d}x\mathrm{d}\beta\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n)\\&=\int_{\mathbb{R}^n}\int_0^{F(\mu,y_1,\cdots,y_n)}F^{-1}(\beta,y_1,\cdots,y_n)\ln\frac{\beta}{1-\beta}\mathrm{d}\beta\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n)\\&\quad+\mu\int_{\mathbb{R}^n}\int_{F(\mu,y_1,\cdots,y_n)}^1\ln\frac{\beta}{1-\beta}\mathrm{d}\beta\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_n(y_n).\end{aligned}$$

□

§4 Monte Carlo Simulation for Partial Semi Entropy

By invoking Theorem 7, we can write partial semi entropy of an uncertain random variable via expectation of a function of random variables as follows:

$$PSH[\xi]=\begin{cases} E\Big[\big(F^{-1}(U,\eta_1,\cdots,\eta_n)-\mu\big)\ln\big(\dfrac{U}{1-U}\big)I_{(0,F(\mu,\eta_1,\cdots,\eta_n)}(U)\Big], & if \quad \mu<0,\\ E\Big[F^{-1}(U,\eta_1,\cdots,\eta_n)\ln\big(\dfrac{U}{1-U}\big)I_{(0,F(\mu,\eta_1,\cdots,\eta_n))}(U)\Big] & \\ +\mu E\Big[\ln\big(\dfrac{U}{1-U}\big)I_{(F(\mu,\eta_1,\cdots,\eta_n),1)}(U)\Big], & if \quad \mu>0, \end{cases}$$

where, $U,\eta_1,\cdots,\eta_m$ are random variables with standard uniform distribution, $\Psi_1(y_1),\cdots,$ $\Psi_m(y_m)$, respectively. Thus, we can use Monte Carlo simulation for partial semi entropy via the following steps. Consider $\xi=f(\eta,\tau)$ as an uncertain random variable.
Step 1. Randomly generate $u_1,u_2,\cdots,u_N$ and $y_1,y_2,\cdots,y_M$ from standard uniform distribution and probability distribution $\Psi(y)$ corresponding to the random variable η, respectively.
Step 2. If $\mu<0$, compute $\big(F^{-1}(u_i,y_j)-\mu\big)\ln\big(\frac{u_i}{1-u_i}\big)I_{(0,F(\mu,y_j))}(u_i)$ for $i=1,2,\cdots,N$ and $j=1,2,\cdots,M$.

otherwise, compute $\Big[F^{-1}(u_i,y_j)\ln\big(\frac{u_i}{1-u_i}\big)I_{(0,F(\mu,y_j))}(u_i)+\mu\ln\big(\frac{u_i}{1-u_i}\big)I_{(F(\mu,y_j),1)}(u_i)\Big]$
Step 3. Consider

$$\begin{cases} \dfrac{1}{NM}\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{M}\big(F^{-1}(u_i,y_j)-\mu\big)\ln\big(\dfrac{u_i}{1-u_i}\big)I_{(0,F(\mu,y_j))}(u_i), & if \quad \mu<0,\\ \dfrac{1}{NM}\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{M}\Big[F^{-1}(u_i,y_j)\ln\big(\dfrac{u_i}{1-u_i}\big)I_{(0,F(\mu,y_j))}(u_i)+\mu\ln\big(\dfrac{u_i}{1-u_i}\big)I_{(F(\mu,y_j),1)}(u_i)\Big] & if \quad \mu>0, \end{cases}$$

as an approximation for partial-semi-entropy PSH.

Example 1. *Suppose η is a random variable such that $\eta\sim Exp(\frac{1}{5})$. Also, let τ is an uncertain variable with uncertainty distribution $\mathcal{N}(3,5)$. Consider $\xi=\eta+\tau$ as an uncertain random variable. We want to calculate partial semi entropy of ξ. Since $\mu=E[\xi]=E[\tau]+E[\eta]=5+3=8$ is greater than zero, we use the following formula for computing the partial semi entropy:*

$$PSH[\xi]=E\Big[F^{-1}(U,\eta)\ln\big(\frac{U}{1-U}\big)I_{(0,F(\mu,\eta)}(U)\Big] + \mu E\Big[\ln\big(\frac{U}{1-U}\big)I_{(F(\mu,\eta),1)}(U)\Big],$$

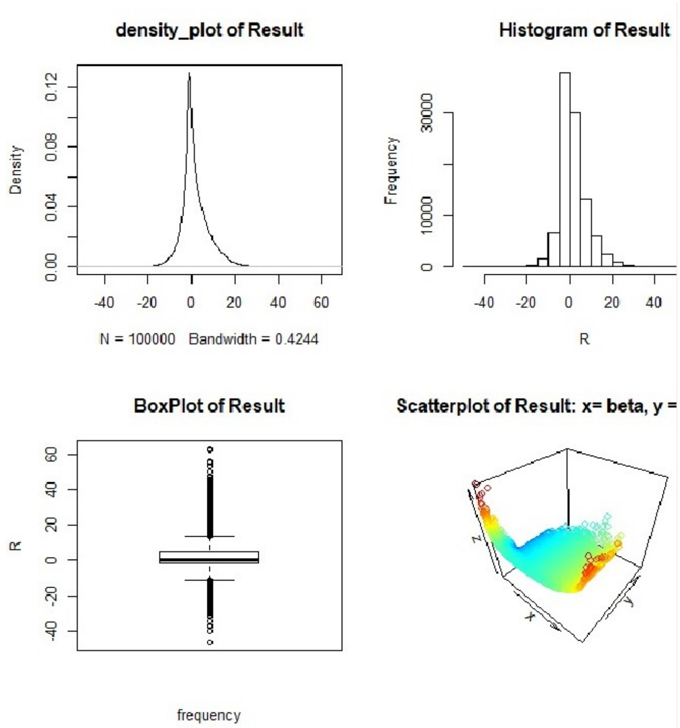

Figure 1. Statistical Plots of The Random Sample in Example 1.

in fact, partial semi entropy of ξ is

$$PSH[\xi] = \int_0^\infty \int_0^{\left(1+\exp(\frac{\pi(y-5)}{5\sqrt{3}})\right)^{-1}} \left(y+3+\frac{5\sqrt{3}}{\pi}\ln\frac{\beta}{1-\beta}\right)\ln\frac{\beta}{1-\beta}\frac{1}{5}\exp(-\frac{y}{5})d\beta dy$$

$$+8\int_0^\infty \int_{\left(1+\exp(\frac{\pi(y-5)}{5\sqrt{3}})\right)^{-1}}^1 \ln\frac{\beta}{1-\beta}\frac{1}{5}\exp(-\frac{y}{5})d\beta dy = 1.646667.$$

The value of above integral is obtained via Monte Carlo simulation, i.e. the integral is mean of the random sample. Other statistical properties of the random sample such as density function, box plot, histogram and scatter plot are displayed in Figure 1.

Example 2. *Let η be a random variable such that $\eta \sim \mathcal{N}(5,4)$. Also, let τ is an uncertain variable with uncertainty distribution $\mathcal{N}(3,5)$. Consider $\xi = \eta + \tau$ as an uncertain random variable. We want to calculate partial semi entropy of ξ. Since $\mu = E[\xi] = E[\tau] + E[\eta] = 5 + 3 = 8$ is greater than zero, we use the following formula for computing the partial semi entropy:*

$$PSH[\xi] = E\Big[F^{-1}(U,\eta)\ln(\frac{U}{1-U})I_{(0,F(\mu,\eta))}(U)\Big]$$

$$+\mu E\Big[\ln(\frac{U}{1-U})I_{(F(\mu,\eta),1)}(U)\Big],$$

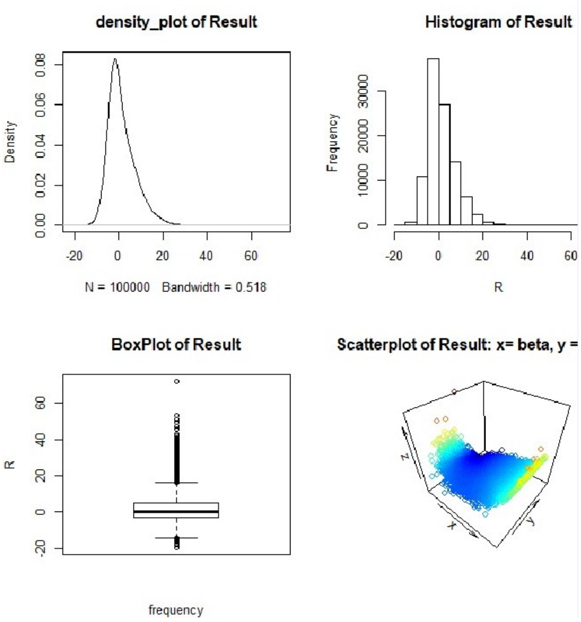

Figure 2. Statistical Plots of The Random Sample in Example 2.

in fact, partial semi entropy of ξ *is*

$$PSH[\xi] = \int_0^\infty \int_0^{\left(1+\exp(\frac{\pi(y-5)}{5\sqrt{3}})\right)^{-1}} \left(y+3+\frac{5\sqrt{3}}{\pi}\ln\frac{\beta}{1-\beta}\right)\ln\frac{\beta}{1-\beta}d\beta d\Psi(y)$$
$$+8\int_0^\infty \int_{\left(1+\exp(\frac{\pi(y-5)}{5\sqrt{3}})\right)^{-1}}^1 \ln\frac{\beta}{1-\beta}d\beta d\Psi(y) = 1.436056,$$

where, $d\Psi(y) = \frac{1}{\sqrt{2\pi\times 4}}\exp\left(-\frac{(y-5)^2}{2\times 4}\right)dy$. *The value of above integral is obtained via Monte Carlo simulation, i.e. the integral is mean of the random sample. Other statistical properties of the random sample such as density function, box plot, histogram and scatter plot are displayed in Figure 2.*

§5 Portfolio Optimization of Uncertain Random Returns

In many situations, we deal with several securities involving historical and new markets. We can consider historical and new markets as random and uncertain variables, respectively. In the case of new markets, we have not enough data to predict probability distributions. Thus, we invite experts to derive belief degree or uncertain distribution of return for new markets. Besides, the values of uncertain random variables are greater than mean, conclude interest in finance, we should minimize entropy for values which lower than mean. Therefore, in order to solve the portfolio selection problems with uncertain random returns, we propose two mean-semi-entropy models via partial semi entropy.

Assume we have n securities with uncertain random returns $\xi_1, \xi_2, \cdots, \xi_n$, respectively. Also, consider x_i's as investment proportions in security $i, i = 1, 2, ..., n$. In portfolio selection, we want to obtain a large return. In fact, we want to derive a large amount of $x_1\xi_1+\cdots+x_n\xi_n$. Since, $\xi_1,\cdots,\xi_n$ are uncertain random variables, it is reasonable to obtain a large amount of $E[x_1\xi_1+\cdots+x_n\xi_n]$.

Based on the investor's view, we introduce the following portfolio selection models. When upper bound of partial semi entropy of returns is known, the investor will prefer a portfolio with large expectation.

$$\begin{cases} \max\limits_{x_i} E[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n] \\ \\ subject \ \ to: \\ \\ PSH[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]\leq\lambda \\ \\ x_1+x_2+...+x_n=1, \quad x_i\geq 0, \quad i=1,2,...,n, \end{cases}$$

where, λ is predetermined parameters.

When upper bound of expectation of returns is known, the investor will prefer a portfolio with small partial semi entropy.

$$\begin{cases} \min\limits_{x_i} PSH[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n] \\ \\ subject \ \ to: \\ \\ E[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]\geq\delta \\ \\ x_1+x_2+...+x_n=1, \quad x_i\geq 0, \quad i=1,2,...,n, \end{cases}$$

where, δ is predetermined parameters.

Sometimes the investor want to maximize expectation and minimize partial semi entropy of returns. This major can be modeled as follows.

$$\begin{cases} \max\limits_{x_i} E[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n] \\ \\ \min\limits_{x_i} PSH[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n] \\ \\ subject \ \ to: \\ \\ x_1+x_2+...+x_n=1, \quad x_i\geq 0, \quad i=1,2,...,n, \end{cases}$$

Example 3. Consider we have four securities with uncertain random returns shown in Table 1, with $\xi_i=\tau_i+\eta_i, \quad i=1,2,\cdots,n$.

Table 1.

No	Uncertain Term	Random Term
1	$\tau_1\sim\mathcal{N}(0.042,0.18)$	$\eta_1\sim\mathcal{N}(0.02,0.09)$
2	$\tau_2\sim\mathcal{N}(0.039,0.21)$	$\eta_2\sim\mathcal{N}(0.01,0.16)$
3	$\tau_3\sim\mathcal{N}(0.031,0.16)$	$\eta_3\sim\mathcal{N}(0.01,0.05)$
4	$\tau_4\sim\mathcal{N}(0.02,0.26)$	$\eta_4\sim\mathcal{N}(0.02,0.06)$

We want to maximize the mean of total returns with constrained semi-entropy.

$$\begin{cases} \max\limits_{x_i} 0.062x_1 + 0.049x_2 + 0.041x_3 + 0.04x_4 \\ \\ subject\ \ to: \\ \\ PSH[x_1\xi_1 + x_2\xi_2 + x_3\xi_3 + x_4\xi_4] < 0.01 \\ \\ x_1 + x_2 + x_3 + x_4 = 1, \quad x_i \geq 0, \quad i = 1,2,3,4. \end{cases}$$

$$\begin{aligned} & PSH[x_1\xi_1 + x_2\xi_2 + x_3\xi_3 + x_4\xi_4] \\ & = \int_{\mathbb{R}^4} \int_0^{\left(1+\exp(\frac{\pi(x_1y_1+x_2y_2+x_3y_3+x_4y_4-0.02x_1-0.01x_2-0.01x_3-0.02x_4)}{(0.18x_1+0.21x_2+0.16x_3+0.26x_4)\sqrt{3}})\right)^{-1}} \\ & \qquad \left[\left(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + 0.042x_1 + 0.039x_2 + 0.031x_3 + 0.02x_4\right)\right. \\ & \qquad \left. + \left(\frac{(0.18x_1 + 0.21x_2 + 0.16x_3 + 0.26x_4)\sqrt{3}}{\pi} \ln \frac{\beta}{1-\beta}\right) \ln \frac{\beta}{1-\beta}\right] \\ & \qquad\quad \mathrm{d}\beta \mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\mathrm{d}\Psi_3(y_3)\mathrm{d}\Psi_4(y_4) \\ & \quad + (0.042x_1 + 0.039x_2 + 0.031x_3 + 0.02x_4) \\ & \qquad \int_{\mathbb{R}^4} \int_{\left(1+\exp(\frac{\pi(x_1y_1+x_2y_2+x_3y_3+x_4y_4-0.02x_1-0.01x_2-0.01x_3-0.02x_4)}{(0.18x_1+0.21x_2+0.16x_3+0.26x_4)\sqrt{3}})\right)^{-1}}^1 \ln \frac{\beta}{1-\beta} \\ & \qquad\quad \mathrm{d}\beta \mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\mathrm{d}\Psi_3(y_3)\mathrm{d}\Psi_4(y_4), \end{aligned}$$

where Ψ_1, Ψ_2, Ψ_3 and Ψ_4 are probability distribution functions of $\mathcal{N}(0.02, 0.09)$, $\mathcal{N}(0.01, 0.16)$, $\mathcal{N}(0.01, 0.05)$ and $\mathcal{N}(0.02, 0.06)$, respectively. Now, by solving the crisp optimization problem, we obtain the optimal solutions as Table 2. Also, the expected value of the total returns is 0.046200.

Table 2.

No	1	2	3	4
Proportion of Portfolio	0.24	0.02	0.74	0

Now, We want to minimize the semi-entropy of total returns with constrained mean.

$$\begin{cases} \min\limits_{x_i} PSH[x_1\xi_1 + x_2\xi_2 + x_3\xi_3 + x_4\xi_4] \\ \\ subject\ \ to: \\ \\ 0.062x_1 + 0.049x_2 + 0.041x_3 + 0.04x_4 > 0.055 \\ \\ x_1 + x_2 + x_3 + x_4 = 1, \quad x_i \geq 0, \quad i = 1,2,3,4. \end{cases}$$

Now, by solving the crisp optimization problem, we obtain the optimal solutions as Table 3. Also, the semi-entropy of the total returns is 0.056852.

Table 3.

No	1	2	3	4
Proportion of Portfolio	0.68	0	0.32	0

Now, we want to optimize the portfolio selection problem via multi-objective model.

$$\begin{cases} \max\limits_{x_i} 0.062x_1 + 0.049x_2 + 0.041x_3 + 0.04x_4 \\ \min\limits_{x_i} PSH[x_1\xi_1 + x_2\xi_2 + x_3\xi_3 + x_4\xi_4] \\ subject\ \ to: \\ x_1 + x_2 + x_3 + x_4 = 1, \quad x_i \geq 0, \quad i = 1, 2, 3, 4. \end{cases}$$

Now, by solving the crisp optimization problem, we obtain the optimal solutions as Table 4. Also, the mean and semi-entropy of the total returns are 0.043320 and 0.000234, respectively.

Table 4.

No	1	2	3	4
Proportion of Portfolio	0.08	0.08	0.84	0

§6 Conclusions

This paper presented the concept of partial semi entropy of uncertain random variables as a risk measure. Also, some properties of this concept are studied. Furthermore, for computing partial semi entropy for uncertain random variables, Monte-Carlo approach was provided. As an application in finance, portfolio selection problems were optimized by mean-semi entropy. For better understanding of main results, some examples and figures were obtained.

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