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Asymptotic analysis of a nonlinear stochastic eco-epidemiological system with feedback control

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Abstract. This paper proposes a new stochastic eco-epidemiological model with nonlinear incidence rate and feedback controls. First, we prove that the stochastic system has a unique global positive solution. Second, by constructing a series of appropriate stochastic Lyapunov functions, the asymptotic behaviors around the equilibria of deterministic model are obtained, and we demonstrate that the stochastic system exists a stationary Markov process. Third, the conditions for persistence in the mean and extinction of the stochastic system are established. Finally, we carry out some numerical simulations with respect to different stochastic parameters to verify our analytical results. The obtained results indicate that the stochastic perturbations and feedback controls have crucial effects on the survivability of system.

§1 Introduction

In the last many decades, epidemic has been a leading cause of death. To prevent outbreak and expansion of infectious diseases, people have implemented some suitable measures, where epidemiological models [4, 13, 24–27, 30, 32, 40] have given important insights to analyze the spreading and control of epidemic. In general, we assume that the population comprises two subgroups: susceptible individuals (S) and the already infected individuals (I). In [6], the SI model is given by:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = S(t)\Big(a - bS(t) - cI(t)\Big), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = I(t)\Big(dS(t) - e - fI(t)\Big), \end{cases} \tag{1}$$

where a, b, c, d, e and f are positive constants. a denotes the recruitment birth rate, b and f stand for the density restriction coefficients of $S(t)$ and $I(t)$, respectively. c is the contact

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rate, d is the rate of transmission, e is the diseased death rate. There are some scholars who have discussed several different SI models, the readers can refer to [16,23,31,41] and references therein.

However, the infectious rate is a crucial factor in SI models, and there are some different communication forms of infectious diseases according to principle of mass action. Bilinear and standard incidences have been extensively applied in disease analysis [2,22]. In 1973, [5] introduced a saturated incidence (see e.g. [23,25])

$$g(I)S = \frac{kI}{1+\alpha I}S$$

into epidemic model, where kI denotes infection force, $\frac{1}{1+\alpha I}$ represents the inhibition action. When $\alpha = 0$, we notice that the saturated incidence turns into a bilinear incidence ($g(I)S = kIS$). According to the crowding effect of $I(t)$ or some protection measures to $S(t)$, the number of effective contacts may saturate at a high level because the saturated incidence rate includes the behavioral change and crowding action of infective individuals.

In the real world, ecosystems are continuously disrupted by unpredictable forces [12]. Most often, the disturbance functions are called as control variables. In the last decades, many scholars prefer to study ecosystem with the help of feedback controls (see [11, 12, 28]). For instance, Gopalsamy and Weng [11] considered a feedback control variables into the logistic model with delay, and obtained the sufficient conditions for the global asymptotic dynamics of solutions, where control variables satisfy properties of certain differential equations. [28] proposed feedback control variables in SI model with the bilinear incidence to explore the global stability of the model. Specifically, Tripathi and Abbas [28] proposed the SI model which is described by:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = S(t)\Big(a - bS(t) - \dfrac{cI(t)}{1+kI(t)} - pu(t)\Big), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = I(t)\Big(-d - eI(t) + \dfrac{cS(t)}{1+kI(t)} - qv(t)\Big), \\ \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = -p_1u(t) + q_1S(t), \\ \dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = -p_2v(t) + q_2I(t), \end{cases} \tag{2}$$

with initial condition

$$S(0) > 0,\ I(0) > 0,\ u(0) > 0 \text{ and } v(0) > 0, \tag{3}$$

where $u(t)$ and $v(t)$ denote feedback control variables. k is inhibitory effect, and p_1, p_2, q_1 and q_2 denote the feedback control coefficients. The else parameters have the similar meanings as for system (1). According to biological considerations, all parameters k, p, q, p_1, p_2, q_1 and q_2 are positive constants. We know that the solution of (2) with (3) is positive when $t \geq 0$ in [21]. In (2), the basic reproduction number $R_0 = \frac{Qkc}{2kd+P+Qc^2}$ controls whether or not the disease persists, here $P = \frac{p_2e+qq_2}{p_2}$ and $Q = \frac{p_1}{bp_1+pq_1}$. If $d = Qac$, $R_0 < 1$ and $(2kd + P + Qc^2)a = kd$, then $E_0(S_0, 0, u_0, 0)$ is globally asymptotically stable. If $R_0 > 1$ and $ckI_* < \min\{2b,\ 2(e + ckS_*)\}$, then $E_*(S_*, I_*, u_*, v_*)$ is globally asymptotically stable.

On the other hand, many scholars only considered the deterministic models that ignore the effects of environmental fluctuations. However, in the real world, environmental noise is one of the vital elements in ecosystems. And almost all practical observations showed that the stochastic fluctuations in biological growth process are obvious, and the probability of random fluctuations is not small. Since the fluctuations exist, a real ecosystem can not maintain at a steady state, and environmental perturbation will break this equilibrium either by directly affecting the density or by indirectly changing the values of parameters. Therefore, in many cases, it is imprecise to use deterministic models to analyze and predict changes in ecosystem behavior. In order to adapt to different practical needs, it is necessary to use the stochastic biological mathematical models to describe the ecosystem, so that the objective reality can be more comprehensive understanding. Inspired by these ideas, many authors (see [1, 3, 7–10, 14, 15, 17–19, 21, 29, 33–39, 42]) showed that it is more precise to reflect how the environmental noise influences the population dynamics. [10] formulated a stochastic eco-epidemiological model with a nonlinear functional response. By making use of the technique of inequalities and Lyapunov methods, the authors showed that the larger environmental interference destroys the persistence of the eco-epidemiology model. [36] proposed stochastic SIR and SEIR models with nonlinear incidence rate, and proved that under some conditions, the solution exists a unique stationary distribution and is ergodic. Hence making use of the model with stochastic fluctuations can more accurately predict the dynamic behavior of the system. In what follows, we suppose that environmental perturbations are directly proportional to $S(t)$ and $I(t)$ and are affected on the $\frac{\mathrm{d}S(t)}{\mathrm{d}t}$ and $\frac{\mathrm{d}I(t)}{\mathrm{d}t}$ in model (2), respectively.

For all we know, there are no investigators to consider the global dynamics of stochastic epidemic models with feedback control yet. Therefore, this paper discusses the global asymptotic behaviors of a stochastic SI model with nonlinear incidence rate and feedback controls and investigates the effect of environmental noise on the survivability of the model. Corresponding to system (2), we take into account the stochastic system as follows:

$$\begin{cases} \mathrm{d}S(t) = S(t)\left(a - bS(t) - \dfrac{cI(t)}{1+kI(t)} - pu(t)\right)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t), \\ \mathrm{d}I(t) = I(t)\left(-d - eI(t) + \dfrac{cS(t)}{1+kI(t)} - qv(t)\right)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}B_2(t), \\ \mathrm{d}u(t) = \left(-p_1 u(t) + q_1 S(t)\right)\mathrm{d}t, \\ \mathrm{d}v(t) = \left(-p_2 v(t) + q_2 I(t)\right)\mathrm{d}t, \end{cases} \tag{4}$$

where σ_1^2 and σ_2^2 denote the noise intensity, $B_1(t)$ and $B_2(t)$ are mutually independent standard Brownian motions defined in the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

Next, we mainly investigate the global dynamics of the new stochastic eco-epidemiological model with nonlinear incidence rate and feedback controls. In view of feedback control, the energy of the stochastic eco-epidemiological model is not conserved, which causes difficulties for the model analysis. One of the main contributions of this paper is to solve the corresponding difficulties.

This study is organized as follows. In the next section, the unique nonnegative solution of (4) is proved. Section 3 discusses the global asymptotic behaviors of (4) around the equilibria of the corresponding deterministic system (2). And we validate that (4) exists a stationary Markov process under certain conditions. In Section 4, the conditions for the persistence in mean and extinction of (4) are obtained, we further present a number of simulations to illustrate the main results and summarize our main results in Section 5.

§2 Existence and uniqueness of the global positive solution

Ecologically, $S(t)$, $I(t)$, $u(t)$ and $v(t)$ in (4) are nonnegative for all $t \geq 0$. Next, we verify that the global positive solution is existent and unique.

Theorem 2.1. *Model (4) with initial data (3) admits a unique positive solution* $(S(t), I(t), u(t), v(t)) \in R_+^4$ *with probability 1 for* $t > 0$.

Proof. Since the coefficients of (4) are local Lipschitz continuous, then there has a unique local solution $(S(t), I(t), u(t), v(t))$ on $t \in [0, \tau_e)$, where τ_e denotes explosion time. Next, we prove $\tau_e = +\infty$ almost surely (a.s.), which shows that $(S(t), I(t), u(t), v(t))$ is global. Set $k_0 \geq 1$ large enough such that $S(0)$, $I(0)$, $u(0)$ and $v(0)$ lie in $[\frac{1}{k_0}, k_0]$. For any $k > k_0$, define the following stopping time:

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \min\{(S(t), I(t), u(t), v(t))\} \leq \frac{1}{k} \text{ or } \max\{(S(t), I(t), u(t), v(t))\} \geq k\right\}.$$

Specifically, we let $\inf \emptyset = \infty$. Clearly τ_k denotes a monotonically increasing function as $k \to +\infty$. Let $\tau_\infty = \lim_{k\to+\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = +\infty$ a.s., so $\tau_e = +\infty$ a.s., for $\forall t \geq 0$. By contradiction, there are constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Thus, for some $k_1 \geq k_0$, it holds

$$P\{\tau_k \leq T\} \geq \varepsilon, \quad \forall k \geq k_1. \tag{5}$$

Next, one constructs a C^2-function V: $R_+^4 \to R_+$:

$$\begin{aligned} V(S, I, u, v) &= q_1q_2(S - 1 - \ln S) + q_1q_2(I - 1 - \ln I) + \frac{pq_2}{2}u^2 + \frac{qq_1}{2}v^2 \\ &\quad + q_2(u - 1 - \ln u) + q_1(v - 1 - \ln v) \\ &:= V_1(S, I, u, v) + V_2(u, v), \end{aligned}$$

here

$$V_1(S, I, u, v) = q_1q_2(S - 1 - \ln S) + q_1q_2(I - 1 - \ln I) + \frac{pq_2}{2}u^2 + \frac{qq_1}{2}v^2,$$
$$V_2(u, v) = q_2(u - 1 - \ln u) + q_1(v - 1 - \ln v).$$

Using Itô's formula yields

$$\mathrm{d}V(S, I, u, v) = LV\mathrm{d}t + q_1q_2(S - 1)\sigma_1\mathrm{d}B_1(t) + q_1q_2(I - 1)\sigma_2\mathrm{d}B_2(t),$$

where $LV = LV_1 + LV_2$:

$$\begin{aligned} LV_1 &= q_1q_2\left((S - 1)(a - bS - \frac{cI}{1 + kI} - pu) + \frac{\sigma_1^2}{2}\right) \\ &\quad + q_1q_2\left((I - 1)(-d - eI + \frac{cS}{1 + kI} - qv) + \frac{\sigma_2^2}{2}\right) \\ &\quad + pq_2(-p_1u^2 + q_1uS) + qq_1(-p_2v^2 + q_2vI) \end{aligned}$$

$$\begin{aligned}
&= q_1q_2\left(aS-bS^2-a+bS+\frac{cI}{1+kI}+pu+\frac{\sigma_1^2}{2}\right)-p_1pq_2u^2\\
&\quad +q_1q_2\left(-dI-eI^2+d+eI-\frac{cS}{1+kI}+qv+\frac{\sigma_2^2}{2}\right)-q_1qp_2v^2\\
&= q_1q_2\left(-bS^2+(a+b-\frac{c}{1+kI})S\right)+q_1q_2\left(-eI^2+(e-d+\frac{c}{1+kI})I\right)\\
&\quad +q_2(-p_1pu^2+q_1pu)+q_1(-qp_2v^2+q_2qv)+q_1q_2\left(d-a+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}\right),
\end{aligned}$$

$$\begin{aligned}
LV_2 &= q_2\left(1-\frac{1}{u}\right)\Big(-p_1u(t)+q_1S(t)\Big)+q_1\left(1-\frac{1}{v}\right)\Big(-p_2v(t)+q_2I(t)\Big)\\
&= q_1q_2S(t)+p_1q_2-p_1q_2u(t)-\frac{q_1q_2S(t)}{u(t)}+q_1q_2I(t)+q_1p_2-q_1p_2v(t)-\frac{q_1q_2I(t)}{v(t)}.
\end{aligned}$$

$$\begin{aligned}
LV &= q_1q_2\left(-bS^2+(a+b-\frac{c}{1+kI})S\right)+q_1q_2\left(-eI^2+(e-d+\frac{c}{1+kI})I\right)\\
&\quad +q_2(-p_1pu^2+q_1pu)+q_1(-qp_2v^2+q_2qv)+q_1q_2\left(d-a+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}\right)\\
&\quad +q_1q_2S(t)+p_1q_2-p_1q_2u(t)-\frac{q_1q_2S(t)}{u(t)}+q_1q_2I(t)+q_1p_2-q_1p_2v(t)-\frac{q_1q_2I(t)}{v(t)}\\
&= q_1q_2\left(-bS^2+(a+b+1-\frac{c}{1+kI})S\right)+q_1q_2\left(-eI^2+(e-d+1+\frac{c}{1+kI})I\right)\\
&\quad +q_2(-p_1pu^2+(q_1p-p_1)u)+q_1(-qp_2v^2+(q_2q-p_2)v)+q_1q_2\left(d-a+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}\right)\\
&\quad +p_1q_2+q_1p_2-\frac{q_1q_2S(t)}{u(t)}-\frac{q_1q_2I(t)}{v(t)}\\
&\le \max\left\{q_1q_2\left(-bS^2+\Big(a+b+1-\frac{c}{1+kI}\Big)S\right)\right\}\\
&\quad +\max\left\{q_1q_2\left[-eI^2+\big(e-d+1+\frac{c}{1+kI}\big)I\right]\right\}+\max\Big\{q_2\big(-p_1pu^2+(q_1p-p_1)u\big)\Big\}\\
&\quad +\max\Big\{q_1\big(-qp_2v^2+(q_2q-p_2)v\big)\Big\}+q_1q_2\left(d-a+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}\right)+p_1q_2+q_1p_2\\
&\le K
\end{aligned}$$

for some constant $K>0$. Therefore

$$\mathrm{d}V(S,I,u,v)\le K\mathrm{d}t+q_1q_2(S-1)\sigma_1\mathrm{d}B_1(t)+q_1q_2(I-1)\sigma_2\mathrm{d}B_2(t). \tag{6}$$

Integrating (6) from 0 to $T\wedge\tau_k=\min\{T,\tau_k\}$, further taking expectation, we have

$$\mathbb{E}V(S(T\wedge\tau_k),I(T\wedge\tau_k),u(T\wedge\tau_k),v(T\wedge\tau_k))\le V(S(0),I(0),u(0),v(0))+K\mathbb{E}(T\wedge\tau_k).$$

Therefore, we yield

$$\mathbb{E}V(S(T\wedge\tau_k),I(T\wedge\tau_k),u(T\wedge\tau_k),v(T\wedge\tau_k))\le V(S(0),I(0),u(0),v(0))+KT. \tag{7}$$

Let $\Omega_k=\{\tau_k\le T\}$ for all $k\ge k_1$, according to the inequality (5), we know that $P(\Omega_k)\ge\varepsilon$. Hence for any $\omega\in\Omega_k$, there exists at least one of $S(\tau_k,\omega)$, $I(\tau_k,\omega)$, $u(\tau_k,\omega)$ or $v(\tau_k,\omega)$ which equals either $\frac{1}{k}$ or k. As a result, we have

$$V(S(\tau_k,\omega),I(\tau_k,\omega),u(\tau_k,\omega),v(\tau_k,\omega)) \geq \min\left\{k-1-\ln k,\ \frac{1}{k}-1+\ln k\right\}.$$

Then

$$V(S(T\wedge\tau_k),I(T\wedge\tau_k),u(T\wedge\tau_k),v(T\wedge\tau_k)) \geq (k-1-\ln k)\wedge\left(\frac{1}{k}-1+\ln k\right). \tag{8}$$

Then combining equations (7) and (8), we yield

$$\begin{aligned} V(S(0),I(0),u(0),v(0))+KT &\geq \mathbb{E}\Big[\chi_{\Omega_k}(\omega)V(S(\tau_k,\omega),I(\tau_k,\omega),u(\tau_k,\omega),v(\tau_k,\omega))\Big] \\ &\geq \varepsilon(k-1-\ln k)\wedge\left(\frac{1}{k}-1+\ln k\right), \end{aligned}$$

where χ_{Ω_k} is the indicator function of Ω_k. When $k\to+\infty$, we get

$$+\infty > V(S(0),I(0),u(0),v(0))+KT = +\infty,$$

which is a contradiction. Thus, we prove that $\tau_\infty=+\infty$ a.s. □

§3 Asymptotic behaviors

In [28], the authors have obtained two equilibria $E_0(S_0,0,u_0,0)$ and $E_*(S_*,I_*,u_*,v_*)$ for model (2), and under certain conditions, they are globally asymptotically stable, respectively. However, for corresponding stochastic system (4), the two equilibrium points are not existent. In the part, we study the asymptotic behaviors of (4) around E_0 and E_* of (2), respectively.

3.1 Asymptotic behaviors around E_0 of (2)

If $d=Qac$, $R_0<1$ and $(2kd+P+Qc^2)a=kd$, then $E_0(S_0,0,u_0,0)$ of (2) is globally asymptotically stable. However (4) does not exist any equilibrium. Hence we investigate the asymptotic behaviors of (4) around $E_0(S_0,0,u_0,0)$.

Theorem 3.1. *Assume that $(S(t),I(t),u(t),v(t))$ is the solution of (4) with the positive initial data* (3). *When $d=Qac$, $R_0<1$ and $(2kd+P+Qc^2)a=kd$, then*

$$\limsup_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t\left[\Big(S(\theta)-S_0\Big)^2+I(\theta)^2+\Big(u(\theta)-u_0\Big)^2+v(\theta)^2\right]\mathrm{d}\theta \leq \frac{S_0\sigma_1^2}{2L},$$

where $L=\min\left\{b,\ e,\ \frac{p_1p}{q_1},\ \frac{p_2q}{q_2}\right\}$.

Proof. Since $E_0(S_0,0,u_0,0)$ is the disease-free equilibrium of (2), then we get

$$bS_0+pu_0=a,\quad -p_1u_0+q_1S_0=0,\quad S_0=\frac{ap_1}{pq_1+bp_1},\quad u_0=\frac{aq_1}{pq_1+bp_1}.$$

Define

$$V(S,I,u,v)=q_1q_2\left(S-S_0-S_0\ln\frac{S}{S_0}\right)+q_1q_2I+pq_2\frac{(u-u_0)^2}{2}+qq_1\frac{v^2}{2}.$$

Then

$$\mathrm{d}V(S,I,u,v)=LV\mathrm{d}t+q_1q_2(S-S_0)\sigma_1\mathrm{d}B_1(t)+q_1q_2I\sigma_2\mathrm{d}B_2(t), \tag{9}$$

where

$$
\begin{aligned}
LV &= q_1q_2\left[(S-S_0)\Big(a-bS-\frac{cI}{1+kI}-pu\Big)+\frac{S_0\sigma_1^2}{2}\right]\\
&\quad +q_1q_2I\Big(-d-eI+\frac{cS}{1+kI}-qv\Big)\\
&\quad +pq_2(u-u_0)(-p_1u+q_1S)+qq_1v(-p_2v+q_2I)\\
&= q_1q_2\left\{(S-S_0)\Big[-b(S-S_0)-p(u-u_0)-\frac{cI}{1+kI}\Big]+\frac{S_0\sigma_1^2}{2}\right\}\\
&\quad +q_1q_2\Big(-dI-eI^2+\frac{c}{1+kI}(S-S_0)I+\frac{cS_0}{1+kI}I-qIv\Big)\\
&\quad +pq_2(u-u_0)\Big(-p_1(u-u_0)-p_1u_0+q_1(S-S_0)+q_1S_0\Big)\\
&\quad +qq_1(-p_2v^2+q_2Iv)\\
&= -q_1q_2b(S-S_0)^2-q_1q_2eI^2-p_1pq_2(u-u_0)^2-p_2qq_1v^2\\
&\quad -q_1q_2dI+\frac{q_1q_2cS_0}{1+kI}I+\frac{q_1q_2S_0}{2}\sigma_1^2\\
&= -q_1q_2b(S-S_0)^2-q_1q_2eI^2-p_1pq_2(u-u_0)^2-p_2qq_1v^2\\
&\quad +\frac{acp_1q_1q_2}{pq_1+bp_1}I(\frac{1}{1+kI}-1)+\frac{q_1q_2S_0}{2}\sigma_1^2\\
&\le -q_1q_2b(S-S_0)^2-q_1q_2eI^2-p_1pq_2(u-u_0)^2-p_2qq_1v^2+\frac{q_1q_2S_0}{2}\sigma_1^2.
\end{aligned}
$$

Integrating (9) from 0 to t, and taking expectation, we have

$$
\begin{aligned}
\mathbb{E}V(t)-\mathbb{E}V(0) &\le -q_1q_2b\mathbb{E}\int_0^t\Big(S(\theta)-S_0\Big)^2\mathrm{d}\theta-q_1q_2e\mathbb{E}\int_0^t I(\theta)^2\mathrm{d}\theta\\
&\quad -p_1pq_2\mathbb{E}\int_0^t\Big(u(\theta)-u_0\Big)^2\mathrm{d}\theta-p_2qq_1\mathbb{E}\int_0^t v(\theta)^2\mathrm{d}\theta\\
&\quad +\frac{q_1q_2S_0}{2}\sigma_1^2t. \qquad (10)
\end{aligned}
$$

Hence, dividing inequality (10) by t and taking the limit superior yield

$$
\limsup_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t\Big[q_1q_2b\big(S(\theta)-S_0\big)^2+q_1q_2eI(\theta)^2+p_1pq_2(u(\theta)-u_0)^2+p_2qq_1v(\theta)^2\Big]\mathrm{d}\theta \le \frac{q_1q_2S_0\sigma_1^2}{2}.
$$

Dividing both sides of the above equation by q_1q_2, we get

$$
\limsup_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t\Big[b(S(\theta)-S_0)^2+eI(\theta)^2+\frac{p_1p(u(\theta)-u_0)^2}{q_1}+\frac{p_2qv(\theta)^2}{q_2}\Big]\mathrm{d}\theta\le\frac{S_0\sigma_1^2}{2},
$$

i.e.

$$
\limsup_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t\left[\Big(S(\theta)-S_0\Big)^2+I(\theta)^2+\Big(u(\theta)-u_0\Big)^2+v(\theta)^2\right]\mathrm{d}\theta\le\frac{S_0\sigma_1^2}{2L},
$$

where $L=\min\left\{b,\ e,\ \frac{p_1p}{q_1},\ \frac{p_2q}{q_2}\right\}$. □

Corollary 3.1. *Taking into account Theorem 3.1, when $\sigma_1=0$, then*

$$
LV\le -q_1q_2b(S-S_0)^2-q_1q_2eI^2-p_1pq_2(u-u_0)^2-p_2qq_1v^2\le 0,
$$

hence, if $d = Qac$, $R_0 < 1$ and $(2kd + P + Qc^2)a = kd$ hold, then $E_0(S_0, 0, u_0, 0)$ of (2) is globally asymptotically stable.

Remark 3.1. To Theorem 3.1, when the environmental fluctuation is small enough, the solution $(S(t), I(t), u(t), v(t))$ of (4) fluctuates around $E_0(S_0, 0, u_0, 0)$, and the noise intensity is dependent on σ_1^2.

3.2 Asymptotic behaviors around E_* of (2)

If $R_0 > 1$, $ckI_* < \min\{2(e + ckS_*), 2b\}$, system (2) has an equilibrium $E_*(S_*, I_*, u_*, v_*)$. However, (4) does not have stead-state. Next, we prove the asymptotic behaviors of (4) around $E_*(S_*, I_*, u_*, v_*)$ under certain conditions.

Theorem 3.2 *Assume that $(S(t), I(t), u(t), v(t))$ is the solution of (4) obtained in Theorem 2. When $R_0 > 1$, $ckI_* < \min\{2b,\ 2(e + ckS_*)\}$, we derive*

$$\limsup_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t \Big[(S(\theta) - S_*)^2 + (I(\theta) - I_*)^2 + (u(\theta) - u_*)^2 + (v(\theta) - v_*)^2\Big]\mathrm{d}\theta \le \frac{W}{m},$$

where

$$m = \min\left\{q_1q_2b,\ \frac{q_1q_2e}{1 + kI_*},\ p_1pq_2,\ \frac{p_2qq_1}{1 + kI_*}\right\},\ W = \frac{q_1q_2S_*}{2}\sigma_1^2 + \frac{q_1q_2I_*}{2(1 + kI_*)}\sigma_2^2,$$

simultaneously (4) exists a stationary Markov process.

Proof. It follows from $E_*(S_*, I_*, u_*, v_*)$ of (2) that

$$\begin{cases} a - bS_* - \dfrac{cI_*}{1 + kI_*} - pu_* = 0, \\ -d - eI_* + \dfrac{cS_*}{1 + kI_*} - qv_* = 0, \\ -p_1u_* + q_1S_* = 0, \\ -p_2v_* + q_2I_* = 0. \end{cases}$$

Define

$$\begin{aligned} V(S, I, u, v) &= q_1q_2\left(S - S_* - S_*\ln\frac{S}{S_*}\right) + \frac{q_1q_2}{1 + kI_*}\left(I - I_* - I_*\ln\frac{I}{I_*}\right) \\ &\quad + pq_2\frac{(u - u_*)^2}{2} + \frac{qq_1}{1 + kI_*}\frac{(v - v_*)^2}{2} \\ &:= q_1q_2V_1 + \frac{q_1q_2}{1 + kI_*}V_2 + pq_2V_3 + \frac{qq_1}{1 + kI_*}V_4. \end{aligned}$$

With help of Itô's formula, we have

$$\mathrm{d}V_1 = LV_1\mathrm{d}t + (S - S_*)\sigma_1\mathrm{d}B_1(t),$$

with

$$\begin{aligned} LV_1 &= (S - S_*)\left[a - bS - \frac{cI}{1 + kI} - pu\right] + \frac{S_*\sigma_1^2}{2} \\ &= (S - S_*)\left[bS_* + \frac{cI_*}{1 + kI_*} + pu_* - bS - \frac{cI}{1 + kI} - pu\right] + \frac{S_*\sigma_1^2}{2} \\ &= (S - S_*)\left[-b(S - S_*) - p(u - u_*) - \frac{c(I - I_*)}{(1 + kI)(1 + kI_*)}\right] + \frac{S_*\sigma_1^2}{2} \end{aligned}$$

$$= -b(S-S_*)^2 - p(S-S_*)(u-u_*) - \frac{c(S-S_*)(I-I_*)}{(1+kI)(1+kI_*)} + \frac{S_*\sigma_1^2}{2}.$$

Similarly, we have

$$\mathrm{d}V_2 = LV_2\mathrm{d}t + (I-I_*)\sigma_2\mathrm{d}B_2(t),$$

$$\begin{aligned}
LV_2 &= (I-I_*)\left[-d-eI+\frac{cS}{1+kI}-qv\right]+\frac{I_*\sigma_2^2}{2}\\
&= (I-I_*)\left[eI_*-\frac{cS_*}{1+kI_*}+qv_*-eI+\frac{cS}{1+kI}-qv\right]+\frac{I_*\sigma_2^2}{2}\\
&= (I-I_*)\left[-e(I-I_*)-q(v-v_*)+\frac{c\Big((S-S_*)-k(S_*I-SI_*)\Big)}{(1+kI)(1+kI_*)}\right]+\frac{I_*\sigma_2^2}{2}\\
&= -e(I-I_*)^2-q(I-I_*)(v-v_*)+\frac{c(S-S_*)(I-I_*)}{1+kI}-\frac{ckS_*(I-I_*)^2}{(1+kI)(1+kI_*)}+\frac{I_*\sigma_2^2}{2}.
\end{aligned}$$

Also, we get

$$\begin{aligned}
\mathrm{d}V_3 &= (u-u_*)(-p_1u+q_1S)\mathrm{d}t\\
&= (u-u_*)\Big[-p_1(u-u_*)-p_1u_*+q_1(S-S_*)+q_1S_*\Big]\mathrm{d}t\\
&= \Big[-p_1(u-u_*)^2+q_1(S-S_*)(u-u_*)\Big]\mathrm{d}t\\
\mathrm{d}V_4 &= (v-v_*)(-p_2v+q_2I)\mathrm{d}t\\
&= (v-v_*)\Big[-p_2(v-v_*)-p_2v_*+q_2(I-I_*)+q_2I_*\Big]\mathrm{d}t\\
&= \Big[-p_2(v-v_*)^2+q_2(I-I_*)(v-v_*)\Big]\mathrm{d}t.
\end{aligned}$$

Thus, we have

$$\mathrm{d}V = LV\mathrm{d}t + q_1q_2(S-S_*)\sigma_1\mathrm{d}B_1(t) + \frac{q_1q_2(I-I_*)\sigma_2}{1+kI_*}\mathrm{d}B_2(t), \tag{11}$$

$$\begin{aligned}
LV &= q_1q_2LV_1+\frac{q_1q_2}{1+kI_*}LV_2+pq_2\mathrm{d}V_3+\frac{qq_1}{1+kI_*}\mathrm{d}V_4\\
&= -q_1q_2b(S-S_*)^2-q_1q_2p(S-S_*)(u-u_*)-\frac{q_1q_2c(S-S_*)(I-I_*)}{(1+kI)(1+kI_*)}\\
&\quad+\frac{q_1q_2S_*\sigma_1^2}{2}-\frac{q_1q_2e(I-I_*)^2}{1+kI_*}-\frac{q_1q_2ckS_*(I-I_*)^2}{(1+kI)(1+kI_*)^2}-\frac{q_1q_2q(I-I_*)(v-v_*)}{1+kI_*}\\
&\quad+\frac{q_1q_2c(S-S_*)(I-I_*)}{(1+kI)(1+kI_*)}+\frac{q_1q_2I_*\sigma_2^2}{2(1+kI_*)}-p_1pq_2(u-u_*)^2\\
&\quad+q_1q_2p(S-S_*)(u-u_*)-\frac{p_2q_1q}{1+kI_*}(v-v_*)^2+\frac{q_1q_2q}{1+kI_*}(I-I_*)(v-v_*)\\
&\le -q_1q_2b(S-S_*)^2-\frac{q_1q_2e}{1+kI_*}(I-I_*)^2-p_1pq_2(u-u_*)^2\\
&\quad-\frac{p_2q_1q}{1+kI_*}(v-v_*)^2+\frac{q_1q_2S_*}{2}\sigma_1^2+\frac{q_1q_2I_*}{2(1+kI_*)}\sigma_2^2.
\end{aligned}$$

Obviously, we note that if

$$\frac{q_1q_2S_*}{2}\sigma_1^2+\frac{q_1q_2I_*}{2(1+kI_*)}\sigma_2^2<\min\left\{q_1q_2bS_*^2,\ \frac{q_1q_2e}{1+kI_*}I_*^2,\ p_1pq_2u_*^2,\ \frac{p_2q_1q}{1+kI_*}v_*^2\right\},$$

then the domain

$$\begin{aligned}-q_1q_2b(S-S_*)^2 &- \frac{q_1q_2e}{1+kI_*}(I-I_*)^2 - p_1pq_2(u-u_*)^2\\ &- \frac{p_2q_1q}{1+kI_*}(v-v_*)^2 + \frac{q_1q_2S_*}{2}\sigma_1^2 + \frac{q_1q_2I_*}{2(1+kI_*)}\sigma_2^2 < 0\end{aligned}$$

lies entirely in R_+^4. With the help of [9, Lemma 3.1], let U be any neighborhood of the domain with $\overline{U} \subseteq R_+^4$, thus for every $(S, I, u, v) \in R_+^4 \setminus U$, we get $LV \leq -C$ (C represents a positive constant). Therefore, the system (4) has a stationary Markov process.

Further, integrating (11) from 0 to t, then taking expectation, we yield

$$\begin{aligned}\mathbb{E}V(t) - \mathbb{E}V(0) \leq\ & -q_1q_2b\mathbb{E}\int_0^t (S(\theta)-S_*)^2 \mathrm{d}\theta - p_1pq_2\mathbb{E}\int_0^t (u(\theta)-u_*)^2 \mathrm{d}\theta\\ & -\frac{q_1q_2e}{1+kI_*}\mathbb{E}\int_0^t (I(\theta)-I_*)^2 \mathrm{d}\theta - \frac{p_2q_1q}{1+kI_*}\mathbb{E}\int_0^t (v(\theta)-v_*)^2 \mathrm{d}\theta\\ & +\frac{q_1q_2S_*}{2}\sigma_1^2 t + \frac{q_1q_2I_*}{2(1+kI_*)}\sigma_2^2 t.\end{aligned}$$

By simple calculation can get

$$\begin{aligned}\limsup_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t & \Big[q_1q_2b(S(\theta)-S_*)^2 + \frac{q_1q_2e(I(\theta)-I_*)^2}{1+kI_*} + p_1pq_2(u(\theta)-u_*)^2\\ & + \frac{p_2q_1q(v(\theta)-v_*)^2}{1+kI_*}\Big]\mathrm{d}\theta \leq \frac{q_1q_2S_*}{2}\sigma_1^2 + \frac{q_1q_2I_*}{2(1+kI_*)}\sigma_2^2,\end{aligned}$$

clearly

$$\limsup_{t\to+\infty} \frac{\mathbb{E}\int_0^t \Big[(S(\theta)-S_*)^2 + (I(\theta)-I_*)^2 + (u(\theta)-u_*)^2 + (v(\theta)-v_*)^2\Big]\mathrm{d}\theta}{t} \leq \frac{W}{m}, \tag{12}$$

where

$$m = \min\left\{q_1q_2b,\ \frac{q_1q_2e}{1+kI_*},\ p_1pq_2,\ \frac{p_2qq_1}{1+kI_*}\right\} \text{ and } W = \frac{q_1q_2S_*}{2}\sigma_1^2 + \frac{q_1q_2I_*}{2(1+kI_*)}\sigma_2^2.$$

□

Corollary 3.2. *Taking into account Theorem 3.2, when $\sigma_1 = \sigma_2 = 0$, we yield*

$$LV \leq -q_1q_2b(S-S_*)^2 - \frac{q_1q_2e}{1+kI_*}(I-I_*)^2 - p_1pq_2(u-u_*)^2 - \frac{p_2q_1q}{1+kI_*}(v-v_*)^2 \leq 0.$$

Therefore, if $R_0 > 1$, $ckI_* < \min\{2(e+ckS_*), 2b\}$, then $E_*(S_*, I_*, u_*, v_*)$ of (2) is globally asymptotically stable.

Remark 3.2. It follows from Theorem 3.2 that if the environmental fluctuation is small enough, then solution of (4) fluctuates around $E_*(S_*, I_*, u_*, v_*)$. In addition, the noies intensity is positively correlated with σ_1^2 and σ_2^2.

§4 Survivability analysis

In random sense, we know that (4) does not exist equilibria, though we have learned the stability of the equilibrium points for (2), it can not illustrate the persistence of (4). Based on this, in the following, we study the persistence in mean and extinction of (4), because these two properties are vary important in a ecosystem.

Definition 4.1. ([23]) Extinction and persistence in mean are defined as follows.

1. We call the population $X(t)$ extinct if
$$\lim_{t\to+\infty} X(t) = 0 \text{ a.s.}$$

2. We call the population $X(t)$ persistent in mean if
$$\liminf_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t X(\theta)\mathrm{d}\theta > 0 \text{ a.s.}$$

4.1 Persistence in mean

Theorem 4.1. *Assume that $(S(t), I(t), u(t), v(t))$ is the solution of (4) with (3). When $R_0 > 1$, $ckI_* < \min\{2b,\ 2(e + ckS_*)\}$ and*
$$\alpha = \max\{\sigma_1, \sigma_2\} < \min\left\{S_*\sqrt{\frac{m}{W_0}},\ I_*\sqrt{\frac{m}{W_0}},\ u_*\sqrt{\frac{m}{W_0}},\ v_*\sqrt{\frac{m}{W_0}}\right\}$$
hold, then we get
$$\begin{cases} \liminf\limits_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t S(\theta)\mathrm{d}\theta > 0, \\ \liminf\limits_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t I(\theta)\mathrm{d}\theta > 0, \\ \liminf\limits_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t u(\theta)\mathrm{d}\theta > 0, \\ \liminf\limits_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t v(\theta)\mathrm{d}\theta > 0, \end{cases}$$
where
$$W_0 = \frac{q_1 q_2 S_*}{2} + \frac{q_1 q_2 I_*}{2(1 + kI_*)},$$
and m is defined in Theorem 3.2.

Proof. Taking into account inequality (12) in Theorem 3.2, we yield
$$\begin{cases} \limsup\limits_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t (S(\theta) - S_*)^2\mathrm{d}\theta \le \frac{W}{m}, \\ \limsup\limits_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t (I(\theta) - I_*)^2\mathrm{d}\theta \le \frac{W}{m}, \\ \limsup\limits_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t (u(\theta) - u_*)^2\mathrm{d}\theta \le \frac{W}{m}, \\ \limsup\limits_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t (v(\theta) - v_*)^2\mathrm{d}\theta \le \frac{W}{m}. \end{cases} \tag{13}$$
When $S(t) \ge 0$ and $S_* > 0$, applying the inequality $2S(t)S_* \ge S_*^2 - (S(t) - S_*)^2$, we get
$$S(t) \ge \frac{S_*}{2} - \frac{(S(t) - S_*)^2}{2S_*}.$$
Moreover
$$W = \frac{q_1 q_2 S_*}{2}\sigma_1^2 + \frac{q_1 q_2 I_*}{2(1 + kI_*)}\sigma_2^2 \le \alpha^2\left[\frac{q_1 q_2 S_*}{2} + \frac{q_1 q_2 I_*}{2(1 + kI_*)}\right] = \alpha^2 W_0.$$

If $\alpha < S_*\sqrt{\frac{m}{W_0}}$, we yield

$$\begin{aligned}\liminf_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t S(\theta)\mathrm{d}\theta &\geq \frac{S_*}{2}-\limsup_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t\frac{(S(\theta)-S_*)^2}{2S_*}\mathrm{d}\theta\\ &\geq \frac{S_*}{2}-\frac{W}{2S_*m}\geq\frac{S_*}{2}-\frac{\alpha^2W_0}{2S_*m}\\ &> 0.\end{aligned}$$

Similarly, if $\alpha < I_*\sqrt{\frac{m}{W_0}}$, we have

$$\begin{aligned}\liminf_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t I(\theta)\mathrm{d}\theta &\geq \frac{I_*}{2}-\limsup_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t\frac{(I(\theta)-I_*)^2}{2I_*}\mathrm{d}\theta\\ &\geq \frac{I_*}{2}-\frac{W}{2I_*m}\geq\frac{I_*}{2}-\frac{\alpha^2W_0}{2I_*m}\\ &> 0.\end{aligned}$$

If $\alpha < u_*\sqrt{\frac{m}{W_0}}$, we have

$$\begin{aligned}\liminf_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t u(\theta)\mathrm{d}\theta &\geq \frac{u_*}{2}-\limsup_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t\frac{(u(\theta)-u_*)^2}{2u_*}\mathrm{d}\theta\\ &\geq \frac{u_*}{2}-\frac{W}{2u_*m}\geq\frac{u_*}{2}-\frac{\alpha^2W_0}{2u_*m}\\ &> 0.\end{aligned}$$

If $\alpha < v_*\sqrt{\frac{m}{W_0}}$, we have

$$\begin{aligned}\liminf_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t v(\theta)\mathrm{d}\theta &\geq \frac{v_*}{2}-\limsup_{t\to+\infty}\frac{1}{t}\mathbb{E}\int_0^t\frac{(v(\theta)-v_*)^2}{2v_*}\mathrm{d}\theta\\ &\geq \frac{v_*}{2}-\frac{W}{2v_*m}\geq\frac{v_*}{2}-\frac{\alpha^2W_0}{2v_*m}\\ &> 0,\end{aligned}$$

completing the proof. □

Remark 4.1. Theorem 4.1 indicates that if $R_0 > 1$, $ckI_* < \min\{2b,\ 2(e+ckS_*)\}$ and the noise is sufficiently small, system (4) will be persistent in mean. And it also illustrates that the population can resist a sufficiently small intensity of random disturbance from environmental to keep persistence.

4.2 Extinction

Extinction is also an important property in biological mathematical system. In the subsection, we give a lemma which is used for the proof of population extinction.

Lemma 4.1. *([20]) Suppose $X(t)\in C[\Omega\times R_+, R_+^0]$, here $R_+^0 := \{a|a>0, a\in R\}$.*

1. *If $\exists\lambda$, β_i and constants $\lambda_0>0$, $T>0$, when $t\geq T$, satisfying*

$$\ln X(t)\leq\lambda t-\lambda_0\int_0^t X(\theta)\mathrm{d}\theta+\sum_{i=1}^n\beta_iB_i(t)\quad a.s.,$$

then

$$\begin{cases} \lim\limits_{t\to+\infty} X(t) = 0 \ a.s., & \text{if } \lambda < 0; \\ \limsup\limits_{t\to+\infty} \dfrac{1}{t}\displaystyle\int_0^t X(\theta)\mathrm{d}\theta \le \dfrac{\lambda}{\lambda_0} \ a.s., & \text{if } \lambda \ge 0. \end{cases}$$

2. If $\exists \beta_i$ and constants $\lambda > 0$, $\lambda_0 > 0$, $T > 0$, when $t \ge T$, satisfying

$$\ln X(t) \ge \lambda t - \lambda_0 \int_0^t X(\theta)\mathrm{d}\theta + \sum_{i=1}^n \beta_i B_i(t) \quad a.s.,$$

then $\liminf\limits_{t\to+\infty} \dfrac{1}{t}\displaystyle\int_0^t X(\theta)\mathrm{d}\theta \ge \dfrac{\lambda}{\lambda_0}$ *a.s.*

Theorem 4.2. *Assume that $(S(t), I(t), u(t), v(t))$ is the solution of (4) with (3). If $\sigma_1^2 > 2a$, then*

$$\lim_{t\to+\infty} S(t) = 0 \ a.s., \ \lim_{t\to+\infty} I(t) = 0 \ a.s., \ \lim_{t\to+\infty} u(t) = 0 \ a.s., \ \lim_{t\to+\infty} v(t) = 0 \ a.s.$$

Proof. By Itô's formula to (4) implies that

$$\begin{aligned} \mathrm{d}\ln S(t) &= \left(a - bS(t) - \frac{cI(t)}{1+kI(t)} - pu(t) - \frac{\sigma_1^2}{2}\right)\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t) \\ &\le \left(a - \frac{\sigma_1^2}{2} - bS(t)\right)\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \end{aligned} \tag{14}$$

then by direct calculations, we yield

$$\frac{1}{t}\ln\frac{S(t)}{S(0)} \le \left(a - \frac{\sigma_1^2}{2}\right) - \frac{b}{t}\int_0^t S(\theta)\mathrm{d}\theta + \frac{1}{t}\int_0^t \sigma_1 \mathrm{d}B_1(\theta). \tag{15}$$

Since $\sigma_1^2 > 2a$, from Lemma 4.2, it holds

$$\lim_{t\to+\infty} S(t) = 0 \text{ a.s.}$$

Similarly, we have

$$\mathrm{d}\ln I(t) = \left(-d - eI(t) + \frac{cS(t)}{1+kI(t)} - qv(t) - \frac{\sigma_2^2}{2}\right)\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t).$$

According to $\lim_{t\to+\infty} S(t) = 0$ a.s., when $t > T$, there exists a constant $\epsilon > 0$ small enough satisfying $S(t) \le \frac{\epsilon}{c}$, therefore

$$\mathrm{d}\ln I(t) \le \left(\epsilon - d - eI(t) - \frac{\sigma_2^2}{2}\right)\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t). \tag{16}$$

then

$$\frac{1}{t}\ln\frac{I(t)}{I(0)} \le \left(\epsilon - d - \frac{\sigma_2^2}{2}\right) - \frac{e}{t}\int_0^t I(\theta)\mathrm{d}\theta + \frac{1}{t}\int_0^t \sigma_2 \mathrm{d}B_2(\theta).$$

Combined Lemma 4.2 with the arbitrariness of ϵ, it yields

$$\lim_{t\to+\infty} I(t) = 0 \text{ a.s.}$$

Thus, from (4), one has

$$\lim_{t\to+\infty} u(t) = 0 \text{ a.s.}, \ \lim_{t\to+\infty} v(t) = 0 \text{ a.s.}$$

□

Remark 4.2. Theorem 4.2 indicates that when environmental noise is large enough, the populations will be extinct.

Theorem 4.3. *Assume that $(S(t), I(t), u(t), v(t))$ is the solution of (4) with (3). When*

$$\begin{cases} b\sigma_2^2 + c\sigma_1^2 > 2(ac - bd), \\ \sigma_1^2 < 2a \quad \text{and} \quad \dfrac{pq_1}{bp_1} < 1 \end{cases}$$

hold, then we obtain

$$\begin{cases} \dfrac{a - \frac{1}{2}\sigma_1^2}{b} \geq \limsup\limits_{t\to+\infty} \dfrac{\int_0^t S(\theta)\mathrm{d}\theta}{t} \geq \liminf\limits_{t\to+\infty} \dfrac{\int_0^t S(\theta)\mathrm{d}\theta}{t} \geq \dfrac{\left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right)}{b} > 0 \ a.s., \\ \lim\limits_{t\to+\infty} I(t) = 0 \ a.s., \\ \dfrac{q_1(a - \frac{1}{2}\sigma_1^2)}{bp_1} \geq \limsup\limits_{t\to+\infty} u(t) \geq \liminf\limits_{t\to+\infty} u(t) \geq \dfrac{q_1\left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right)}{bp_1} > 0 \ a.s., \\ \lim\limits_{t\to+\infty} v(t) = 0 \ a.s. \end{cases}$$

Proof. Since the solution of (4) with (3) is nonnegative, and from (4), it yields

$$\mathrm{d}S(t) \leq S(t)[a - bS(t)]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1. \tag{17}$$

We therefore have (15) by simple calculation (17). When $\sigma_1^2 \leq 2a$, in other words, $a - \frac{\sigma_1^2}{2} \geq 0$, by Lemma 4.2, we find that

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t S(\theta)\mathrm{d}\theta \leq \frac{a - \frac{1}{2}\sigma_1^2}{b} \quad \text{a.s.} \tag{18}$$

Next, we investigate the second equation of (4). Making use of (18), we yield

$$\mathrm{d}I(t) \leq I(t)\left[-d - eI(t) + \frac{c(a - \frac{1}{2}\sigma_1^2)}{b}\right]\mathrm{d}t + \sigma_2 I(t)\mathrm{d}B_2(t). \tag{19}$$

Using Itô's formula to (19), we can obtain

$$\mathrm{d}\ln I(t) \leq \left(-d - eI(t) + \frac{c(a - \frac{1}{2}\sigma_1^2)}{b} - \frac{\sigma_2^2}{2}\right)\mathrm{d}t + \sigma_2\mathrm{d}B_2(t). \tag{20}$$

Simple calculations shows that

$$\frac{1}{t}\ln\frac{I(t)}{I(0)} \leq \left(-d - \frac{\sigma_2^2}{2} + \frac{c(a - \frac{1}{2}\sigma_1^2)}{b}\right) - \frac{e}{t}\int_0^t I(\theta)\mathrm{d}\theta + \frac{1}{t}\int_0^t \sigma_2\mathrm{d}B_2(\theta).$$

If $b\sigma_2^2 + c\sigma_1^2 > 2(ac - bd)$ hold, that is to say, $-d - \frac{\sigma_2^2}{2} + \frac{c(a - \frac{1}{2}\sigma_1^2)}{b} < 0$ hold, making use of Lemma 4.2, we know

$$\lim_{t\to+\infty} I(t) = 0 \ \text{a.s.} \tag{21}$$

By (4), when equation (21) is satisfied, we easily get

$$\lim_{t\to+\infty} v(t) = 0 \ \text{a.s.}$$

Taking account of the third equation of system (4), when equation (18) holds, we get

$$\mathrm{d}u(t) \leq \left(-p_1 u(t) + \frac{q_1(a - \frac{1}{2}\sigma_1^2)}{b}\right)\mathrm{d}t,$$

$$x(t) = \frac{q_1(a - \frac{1}{2}\sigma_1^2)}{bp_1} + \widetilde{c}\exp(-p_1 t)$$

is the solution of the following equation:

$$\mathrm{d}x(t) = \left(-p_1 x(t) + \frac{q_1(a - \frac{1}{2}\sigma_1^2)}{b}\right)\mathrm{d}t.$$

According to comparison theorem $u(t) \leq x(t)$, we yield

$$u(t) \leq \frac{q_1(a - \frac{1}{2}\sigma_1^2)}{bp_1} + \widetilde{c}\exp(-p_1 t),$$

then

$$\limsup_{t\to+\infty} u(t) \leq \frac{q_1(a - \frac{1}{2}\sigma_1^2)}{bp_1} \text{ a.s.,} \tag{22}$$

where $\widetilde{c}$ is some constant.

Next, we consider the first equation of (4) again, when equation (21) and equation (22) hold, we yield

$$\mathrm{d}S(t) \geq S(t)\left[a - bS(t) - \frac{q_1(a - \frac{1}{2}\sigma_1^2)}{bp_1}\right]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1. \tag{23}$$

Similarly, by Itô's formula to (23), we get

$$\frac{1}{t}\ln\frac{S(t)}{S(0)} \geq \left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right) - \frac{b}{t}\int_0^t S(\theta)\mathrm{d}\theta + \frac{1}{t}\int_0^t \sigma_1\mathrm{d}B_1(\theta). \tag{24}$$

If $a - \frac{\sigma_1^2}{2} > 0$ and $\frac{pq_1}{bp_1} < 1$ hold, that is to say, $\left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right) > 0$ holds, by Lemma 4.2, it yields

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t S(\theta)\mathrm{d}\theta \geq \frac{\left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right)}{b} > 0 \text{ a.s.} \tag{25}$$

Finally, we consider the fourth equation of (4) again, when (25) holds, we have

$$\mathrm{d}u(t) \geq \left[-p_1 u(t) + \frac{q_1\left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right)}{b}\right]\mathrm{d}t,$$

by computation, we find that

$$\liminf_{t\to+\infty} u(t) \geq \frac{q_1\left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right)}{bp_1} > 0 \text{ a.s.}$$

Based on the above work, when

$$\begin{cases} b\sigma_2^2 + c\sigma_1^2 > 2(ac - bd), \\ \sigma_1^2 < 2a \quad \text{and} \quad \dfrac{pq_1}{bp_1} < 1 \end{cases}$$

are satisfied, we get

$$\begin{cases} \dfrac{a - \frac{1}{2}\sigma_1^2}{b} \geq \limsup\limits_{t\to+\infty}\dfrac{\int_0^t S(\theta)\mathrm{d}\theta}{t} \geq \liminf\limits_{t\to+\infty}\dfrac{\int_0^t S(\theta)\mathrm{d}\theta}{t} \geq \dfrac{\left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right)}{b} > 0 \text{ a.s.,} \\ \lim\limits_{t\to+\infty} I(t) = 0 \text{ a.s.,} \\ \dfrac{q_1(a - \frac{1}{2}\sigma_1^2)}{bp_1} \geq \limsup\limits_{t\to+\infty} u(t) \geq \liminf\limits_{t\to+\infty} u(t) \geq \dfrac{q_1\left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right)}{bp_1} > 0 \text{ a.s.,} \\ \lim\limits_{t\to+\infty} v(t) = 0 \text{ a.s.} \end{cases}$$

□

Remark 4.3. By Theorem 4.2, we can give these biological significance as follows:

1. The condition $b\sigma_2^2 + c\sigma_1^2 > 2(ac - bd)$ shows that when the intensity σ_1^2 and σ_2^2 of environmental fluctuations are bigger, the infected individuals $I(t)$ is extinct;

2. The conditions $\sigma_1^2 < 2a$ and $\frac{pq_1}{bp_1} < 1$ implies that when σ_1^2 is smaller and the feedback controls have also less harmful to the susceptible individuals, then the susceptible individuals $S(t)$ is persistence in mean.

Therefore, we can obtain that making use of the environmental fluctuations reasonably and decreasing the perniciousness of the feedback controls to the susceptible individuals $S(t)$ which are advantageous to control the development of infectious diseases.

§5 Simulations and Conclusions

Next, we present some numerical simulations. We investigate the discrete equations as follows:

$$\begin{cases} S_{n+1} = S_n + S_n\left(a - bS_n - \dfrac{cI_n}{1+kI_n} - pu_n\right)\Delta t + \sigma_1 S_n \Delta W_{1k}, \\ I_{n+1} = I_n + I_n\left(-d - eI_n + \dfrac{cS_n}{1+kI_n} - qv_n\right)\Delta t + \sigma_2 I_n \Delta W_{2k}, \\ u_{n+1} = u_n + (-p_1 u_n + q_1 S_n)\Delta t, \\ v_{n+1} = v_n + (-p_2 v_n + q_2 I_n)\Delta t, \end{cases}$$

where $\Delta t = 0.01$, $\Delta W_{ik} = W_i(t_{k+1}) - W_i(t_k)$ $(i = 1, 2)$ is the Gaussian distribution $N(0, \Delta t)$.

In Figure 1, we set $S(0) = 0.12$, $I(0) = 0.12$, $u(0) = 0.12$, $v(0) = 0.12$, $a = 0.35$, $b = 2$, $c = 3$, $d = 0.21$, $e = 1$, $p = 1.5$, $q = 1$, $p_1 = 1$, $p_2 = 2$, $q_1 = 2$, $q_2 = 1$, $k = 0.1$, $\sigma_1 = \sigma_2 = 0.02$, and $\Delta t = 0.01$.

By computation, we get that

$$E_0 = (S_0, I_0, u_0, v_0) = (0.07, 0, 0.14, 0), \ R_0 = 0.018 < 1.$$

The result of Figure 1 shows that it satisfies the result of Theorem 3.1.

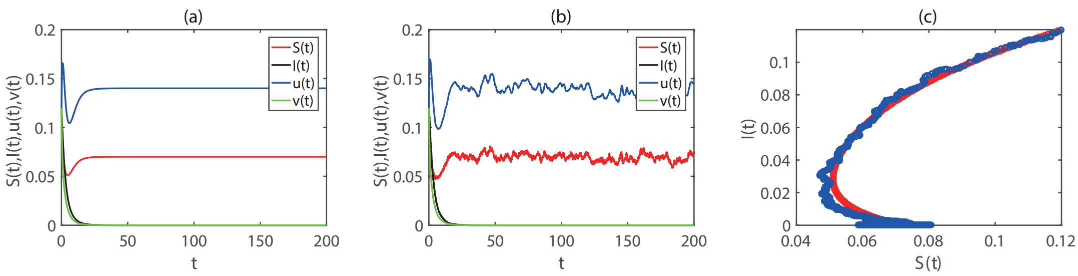

Fig 1. (a) The deterministic model (2) with $\sigma_1 = \sigma_2 = 0$; (b) The stochastic system (4) with $\sigma_1 = \sigma_2 = 0.02$; (c) Phase portrait: the red line describes (2), and the blue orbit expresses (4).

In Figure 2, we set $S(0) = 0.12$, $I(0) = 0.12$, $u(0) = 0.12$, $v(0) = 0.12$, $a = 0.3$, $b = 2$, $c = 3$,

$d = 0.15$, $e = 1$, $p = 0.2$, $q = 0.01$, $p_1 = 1$, $p_2 = 8$, $q_1 = 1.5$, $q_2 = 2$, $k = 8$, $\sigma_1 = \sigma_2 = 0.02$, and $\Delta t = 0.01$.

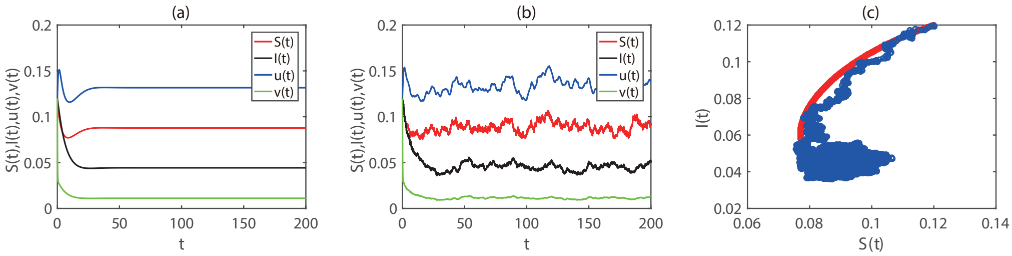

Fig 2. (a) The deterministic model (2) with $\sigma_1 = \sigma_2 = 0$; (b) The stochastic system (4) with $\sigma_1 = \sigma_2 = 0.02$; (c) Phase portrait: the red line describes (2), and the blue orbit expresses (4).

Then, we get that

$$\begin{cases} E_* = (S_*, I_*, u_*, v_*) = (0.0878, 0.0443, 0.1317, 0.011), \\ R_0 = 1.4263 > 1, \ 2b = 4 > ckI_* = 1.0632, \\ 2(e + ckS_*) = 6.2144 > ckI_* = 1.0632. \end{cases}$$

The result of Figure 2 proves that it satisfies our conclusion in Theorem 3.2.

In Figure 3, we choose $\sigma_1 = \sigma_2 = 0.01$, other parameters are all same with Figure 2. Figure 3 indicates that the solution for the stochastic system (4) fluctuates in a sufficiently small domain. Considering Figure 3 (b) and (c), one can know that there exists a stationary Markov process. The result of Figure 3 shows that our conclusions are true.

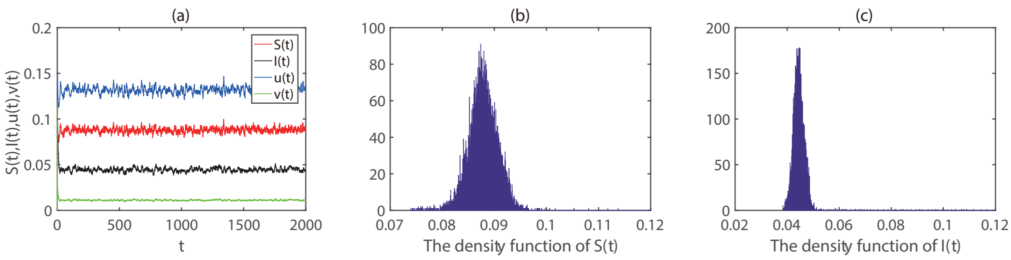

Fig 3. (a) Time series diagram of (4); (b) The density functions of $S(t)$; (c) The density functions of $I(t)$.

In Figure 4, one can choose the parameters of (2) as follows:
$S(0) = 0.12$, $I(0) = 0.12$, $u(0) = 0.12$, $v(0) = 0.12$, $a = 0.2$, $b = 2$, $c = 3$, $d = 0.1$, $e = 1$, $p = 0.5$, $q = 0.1$, $p_1 = 1$, $p_2 = 6$, $q_1 = 1.5$, $q_2 = 2$, $k = 6$, and $\Delta t = 0.01$.

Under this condition, we obtain that

$$\begin{cases} E_* = (S_*, I_*, u_*, v_*) = (0.04856, 0.0255, 0.0728, 0.0085), \\ R_0 = 1.1888 > 1, \ 2b = 4 > ckI_* = 0.459, \\ 2(e + ckS_*) = 3.74816 > ckI_* = 0.459. \end{cases}$$

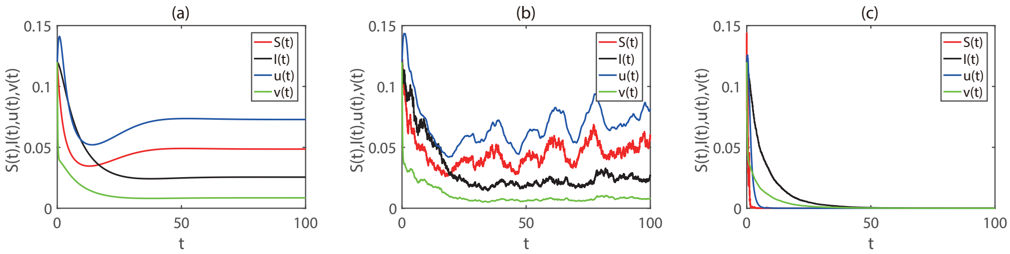

Fig 4. (a) The deterministic model (2) with $\sigma_1 = \sigma_2 = 0$; (b) Persistence in mean of stochastic system (4) with $\sigma_1 = \sigma_2 = 0.05$; (c) The extinction of stochastic system (4) with $\sigma_1 = 0.66$, $\sigma_2 = 0.01$.

In Figure 4 (b), we let $\sigma_1 = \sigma_2 = 0.05$. Then, we have

$$\alpha = \max\{\sigma_1, \sigma_2\} = 0.05 < \min\left\{S_*\sqrt{\frac{m}{W_0}},\ I_*\sqrt{\frac{m}{W_0}},\ u_*\sqrt{\frac{m}{W_0}},\ v_*\sqrt{\frac{m}{W_0}}\right\} = 0.4613.$$

Therefore, Theorem 4.1 holds. Figure 4 (b) illustrates that if the noise perturbations are small enough, the solution is are persistent in mean.

In Figure 4 (c), we set $\sigma_1 = 0.66$, $\sigma_2 = 0.01$. Then we get $\sigma_1^2 = 0.4356 > 2a = 0.4$, which satisfies Theorem 4.2. Figure 4 (c) shows that if the noise perturbation σ_1 is large enough, the solution is extinct.

In Figure 5, we choose the parameters of (2) as follows:
$S(0) = 0.03$, $I(0) = 0.03$, $u(0) = 0.03$, $v(0) = 0.03$, $a = 0.1$, $b = 1$, $c = 0.3$, $d = 0.02$, $e = 0.1$, $p = 0.1$, $q = 0.01$, $p_1 = 1$, $p_2 = 6$, $q_1 = 1.5$, $q_2 = 3$, $k = 6$, and $\Delta t = 0.01$.

Then, we obtain that

$$\begin{cases} E_* = (S_*, I_*, u_*, v_*) = (0.0823, 0.0198, 0.1235, 0.0099), \\ R_0 = 3.698 > 1,\ 2b = 2 > ckI_* = 0.0356, \\ 2(e + ckS_*) = 0.3481 > ckI_* = 0.0356. \end{cases}$$

In Figure 5 (b), we let $\sigma_1 = 0.2$, $\sigma_2 = 0.2$. Then, we have

$$\begin{cases} b\sigma_2^2 + c\sigma_1^2 = 0.052 > 0.02 = 2(ac - bd), \\ \sigma_1^2 = 0.04 < 0.2 = 2a \quad \text{and} \quad \dfrac{pq_1}{bp_1} = 0.15 < 1. \end{cases}$$

Therefore, the all conditions of Theorem 4.2 are satisfied. Figure 5 (b) illustrates that if the feedback controls have less effects on the susceptible individuals, then the healthy individuals are persistent in mean.

In Figure 6, we choose the parameters of (2) as follows:
$S(0) = 0.03$, $I(0) = 0.03$, $u(0) = 0.03$, $v(0) = 0.03$, $a = 0.11$, $b = 1$, $c = 0.5$, $d = 0.026$, $e = 0.1$, $p = 0.5$, $q = 0.01$, $p_1 = 0.99$, $p_2 = 6$, $q_1 = 2$, $q_2 = 3$, $k = 6$, and $\Delta t = 0.01$.

Hence, we obtain that

$$\begin{cases} E_* = (S_*, I_*, u_*, v_*) = (0.05386, 0.00354, 0.10881, 0.00177), \\ R_0 = 2.7568 > 1,\ 2b = 2 > ckI_* = 0.0106, \\ 2(e + ckS_*) = 0.5232 > ckI_* = 0.0106. \end{cases}$$

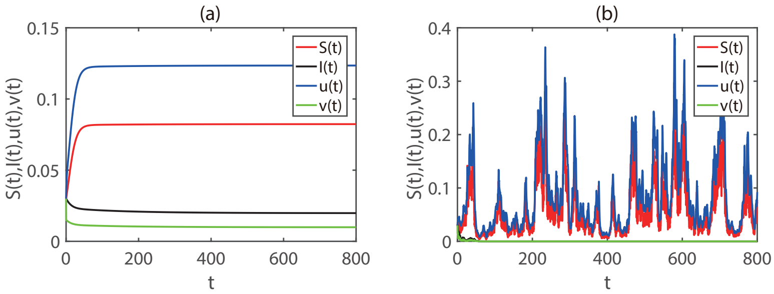

Fig 5. (a) The deterministic model (2) with $\sigma_1 = \sigma_2 = 0$; (b) Persistence in mean of stochastic system (4) with $\sigma_1 = \sigma_2 = 0.2$.

In Figure 6 (b), we let $\sigma_1 = 0.2$, $\sigma_2 = 0.2$. Then, we have

$$\begin{cases} b\sigma_2^2 + c\sigma_1^2 = 0.06 > 0.058 = 2(ac - bd), \\ \sigma_1^2 = 0.04 < 0.22 = 2a \quad \text{and} \quad \dfrac{pq_1}{bp_1} = 1.0101 > 1. \end{cases}$$

It follows from Figure 6 (b) that if the feedback controls have great effects on the susceptible individuals, then the susceptible population goes extinct. Comparing Figure 5 (b) with Figure 6 (b) shows that the feedback controls have significant effect on persistence and extinction of the susceptible population.

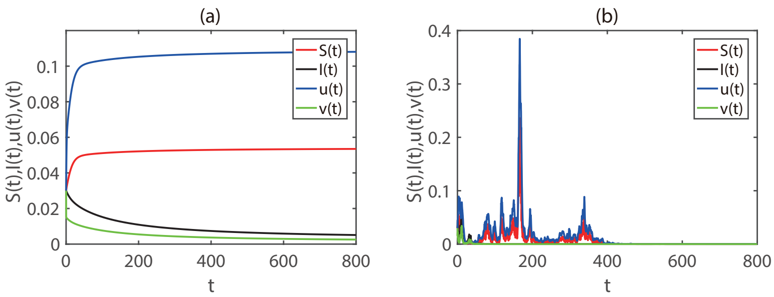

Fig 6. (a) The deterministic model (2) with $\sigma_1 = \sigma_2 = 0$; (b) The extinction of stochastic system (4) with $\sigma_1 = \sigma_2 = 0.2$.

Based on a deterministic SI epidemic model derived by Tripathi and Abbas [28], we propose a stochastic model with saturated incidence rate and feedback controls. Firstly, the existence and uniqueness of the global positive solution of (4) is proved. Then, by stochastic Lyapunov functions with feedback controls $u(t)$ and $v(t)$ and inequality techniques, we obtain the asymptotic dynamics around the equilibria of (2) and prove that the solution of (4) is a stationary Markov process, which implies that the solution of (4) can fluctuate around the equilibria of (2). Moreover, the fluctuation range is dependent on σ_1 and σ_2. Lastly, we investigate the survival of (4). These results show that the stability of the population system can resist on the external noise disturbance is restricted. When the noise perturbation is small enough, the stability of the system has little effect; however, when the perturbation is bigger, it can result in the extinction of population. At the same time, we show that feedback controls are advantageous to the control of the infectious disease. [10] and [36] only consider the effect of environment interference on the epidemiological model. By comparison, the feedback control in this paper has a realistic significance for the extinction of the disease. Therefore, we should take account of the environmental distribution reasonably and decrease the harmfulness of the feedback controls to the susceptible individuals which have important guiding significance for controlling disease. Without stochastic effects, our results are completely consistent with that given in [28]. Consequently, we really extend and develop some results and methods of deterministic models with feedback controls.

Therefore, we summarize the main results as follows:

- Asymptotic behaviors:
  1. When $d = Qac$, $R_0 < 1$ and $(2kd + P + Qc^2)a = kd$, the solution of (4) fluctuates around E_0 of (2). Furthermore, the extent of fluctuation is dependent on σ_1^2;
  2. When $R_0 > 1$ and $ckI_* < \min\{2b,\ 2(e+ckS_*)\}$, the solution of (4) fluctuates around E_* of (2). There is a positive correlation between the extent of fluctuation and the intensity of environmental disturbance σ_1^2 and σ_2^2. Furthermore, when σ_1^2 and σ_2^2 small enough, the solution of (4) has a stationary Markov process.
- Survivability analysis
  1. When $R_0 > 1$, $ckI_* < \min\{2b,\ 2(e + ckS_*)\}$ and
  $$\alpha = \max\{\sigma_1, \sigma_2\} < \min\left\{S_*\sqrt{\frac{m}{W_0}},\ I_*\sqrt{\frac{m}{W_0}},\ u_*\sqrt{\frac{m}{W_0}},\ v_*\sqrt{\frac{m}{W_0}}\right\}$$
  hold, the solution of the stochastic model (4) is persistent in mean;
  2. When $\sigma_1^2 > 2a$, the populations of system (4) is extinct;
  3. When
  $$\begin{cases} b\sigma_2^2 + c\sigma_1^2 > 2(ac - bd), \\ \sigma_1^2 < 2a \ \text{ and } \ \dfrac{pq_1}{bp_1} < 1 \end{cases}$$
  hold, then we get
  $$\begin{cases} \dfrac{a - \frac{1}{2}\sigma_1^2}{b} \geq \limsup\limits_{t\to+\infty} \dfrac{\int_0^t S(\theta)\mathrm{d}\theta}{t} \geq \liminf\limits_{t\to+\infty} \dfrac{\int_0^t S(\theta)\mathrm{d}\theta}{t} \geq \dfrac{(a - \frac{\sigma_1^2}{2})(1 - \frac{pq_1}{bp_1})}{b} > 0 \ a.s., \\ \lim\limits_{t\to+\infty} I(t) = 0 \ a.s., \\ \dfrac{q_1(a - \frac{1}{2}\sigma_1^2)}{bp_1} \geq \limsup\limits_{t\to+\infty} u(t) \geq \liminf\limits_{t\to+\infty} u(t) \geq \dfrac{q_1\left(a - \frac{\sigma_1^2}{2}\right)\left(1 - \frac{pq_1}{bp_1}\right)}{bp_1} > 0 \ a.s., \\ \lim\limits_{t\to+\infty} v(t) = 0 \ a.s., \end{cases}$$
  that is, $I(t)$ and $v(t)$ are extinct, $S(t)$ and $u(t)$ are persistent in mean.
- Stochastic distribution and feedback controls have important effects on the dynamics behaviors of the epidemic model.

Some amusing topics deserve further study. One can consider some realistic and complex issues, for example, a stochastic feedback control model with impulsive effects or a stochastic feedback control model with delays.

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