Appl. Math. J. Chinese Univ.
2022, 37(2): 306-316

Convergence analysis for delay Volterra integral equation

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Abstract. In this article we use Chebyshev spectral collocation method to deal with the Volterra integral equation which has two kinds of delay items. We use linear transformation to make the interval into a fixed interval $[-1, 1]$. Then we use the Gauss quadrature formula to approximate the solution. With the help of lemmas, we get the result that the numerical error decay exponentially in the infinity norm and the Chebyshev weighted Hilbert space norm. Some numerical experiments are given to confirm our theoretical prediction.

§1 Introduction

As we know, integral equations have been widely applied in many aspects of natural science, especially with the development of electronic computer technology, see, e.g., Corduneanu [1] and references therein. Volterra integral equation is a type of integral equation that appears in the question of physics or others, such as the problem of celestial particle motion, potential theory, Dirichlet question, electrostatic question, the problem of thermal radiation shielding, and so on. Therefore, it is of great significance to study its numerical solution. There are many methods for solving Volterra integral equations. Detailed introduction can be found in [2]. Here we just list some latest methods: Legendre spectral-collocation method [3, 4], Jacobi spectral-collocation method [5, 6], spectral Galerkin method [7, 8], Chebyshev spectral-collocation method [2, 9] and so on. In this paper, as in [2, 3], we focus on Volterra integral equation. The same as in [2], we use Chebyshev spectral-collocation method to deal with the Volterra integral equation, but in [2] there is no delay. In [3], there is vanishing delay in the Volterra integral equation, but the authors use Legendre spectral-method while in this article we use Chebyshev spectral-collocation method. Another difference is not only the integral interval with vanishing delay, but also the integrand.

The Volterra integral equation we consider in this paper is as follows:

$$y(\tau) = f(\tau) + \int_0^{\tau} R_1(\tau, \xi) y(\xi) d\xi + \int_0^{\phi(\tau)} R_2(\tau, \eta) y(q\eta) d\eta, \qquad \tau \in [0, T], \tag{1}$$

Received: 2020-07-13. Revised: 2020-11-08.
MR Subject Classification: 65M70, 45D05.
Keywords: Chebyshev spectral collocation method, delay, Gauss quadrature formula, convergence analysis.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-022-3563-1.
Supported by Guangdong Provincial Education Projects (2021KTSCX071, HSGDJG21356-372) and Project of Hanshan Normal University (521036).

where $y(\tau)$ is an unknown function on $[0,T]$, $T<\infty$, and $\phi(\tau)$ satisfies the following conditions:

$$\begin{aligned}&\phi(0)=0,\\&0<\phi(\tau)\le\tau,\ 0<\tau\le T,\\&0\le\phi'(\tau),\ 0\le\tau\le T,\\&\phi\in C^m([0,T]),\ m\ge 1.\end{aligned}$$

Other given functions satisfy

$$f(\tau)\in C^m([0,T]),R_1(\tau,\xi)\in C^m(\Omega_1),R_2(\tau,\eta)\in C^m(\Omega_2),m\ge 1,$$

where

$$\Omega_1:=\{(\tau,\xi):0\le\xi\le\tau\le T\},\Omega_2:=\{(\tau,\eta):0\le\eta\le\phi(\tau)\le\tau\le T\}.$$

There are two parts of delay items in the above function. One is $y(q\eta)$, called pantograph delay, where q is a given constant and $0<q<1$. Another one is the generalized delay function $\phi(\tau)$.

This paper is organized as follows. In section 2, we introduce the spectral approach for the equation. Some useful lemmas are provided in section 3, which are important for the convergence analysis. In section 4, we provide the convergence analysis in both L^∞ and $L^2_{\omega^c}$ spaces. At last, numerical experiments are carried out to confirm the theoretical results.

Throughout the paper, C denotes a generic positive constant that is independent of N, but depends on T and the given data.

§2 Chebyshev spectral collocation method

For ease of analysis, equation (1) can be transformed to the fixed interval $[-1,1]$ with the following variable substitution:

$$\begin{aligned}\tau&=\frac{T}{2}(1+x),\ x=\frac{2}{T}\tau-1,\\\xi&=\frac{T}{2}(1+s),\ s=\frac{2}{T}\xi-1,\\\eta&=\frac{T}{2}(1+t),\ t=\frac{2}{T}\eta-1.\end{aligned}$$

Then equation (1) can be rewritten as

$$\begin{aligned}y\Big(\frac{T}{2}(1+x)\Big)=&f\Big(\frac{T}{2}(1+x)\Big)+\int_{-1}^{x}\frac{T}{2}R_1\Big(\frac{T}{2}(1+x),\frac{T}{2}(1+s)\Big)y\Big(\frac{T}{2}(1+s)\Big)ds\\&+\int_{-1}^{\frac{2}{T}\phi\left(\frac{T}{2}(1+x)\right)-1}\frac{T}{2}R_2\Big(\frac{T}{2}(1+x),\frac{T}{2}(1+t)\Big)y\Big(q\frac{T}{2}(1+t)\Big)dt.\end{aligned}\tag{2}$$

Let

$$\begin{aligned}u(x)&:=y\Big(\frac{T}{2}(1+x)\Big),\ g(x):=f\Big(\frac{T}{2}(1+x)\Big),\\\widetilde{K_1}(x,s)&:=\frac{T}{2}R_1\Big(\frac{T}{2}(1+x),\frac{T}{2}(1+s)\Big),\\\widetilde{K_2}(x,t)&:=\frac{T}{2}R_2\Big(\frac{T}{2}(1+x),\frac{T}{2}(1+t)\Big),\\\varphi(x)&:=\frac{2}{T}\phi\Big(\frac{T}{2}(1+x)\Big)-1,\end{aligned}$$

and we have

$$y\Big(q\frac{T}{2}(1+t)\Big)=y\Big(\frac{T}{2}(q+qt)\Big)=y\Big(\frac{T}{2}(1+qt+q-1)\Big)=u(qt+q-1).$$

Now we can rewrite (2) as follows:

$$u(x) = g(x) + \int_{-1}^{x} \widetilde{K}_1(x,s)u(s)ds + \int_{-1}^{\varphi(x)} \widetilde{K}_2(x,t)u(qt+q-1)dt, \qquad x \in [-1,1]. \tag{3}$$

Note that $\{x_i\}_{i=0}^N$ are the $N+1$-point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points (see, e.g., [10]) and equation (3) holds at x_i:

$$u(x_i) = g(x_i) + \int_{-1}^{x_i} \widetilde{K}_1(x_i,s)u(s)ds + \int_{-1}^{\varphi(x_i)} \widetilde{K}_2(x_i,t)u(qt+q-1)dt, \ i = 0,1,2,...,N. \tag{4}$$

In order to get higher accuracy for equation (4), we will transform the interval $[-1, x_i]$, $[-1, \varphi(x_i)]$ into a fixed interval $[-1,1]$ by two simple linear transformation:

$$s_x(z) = \frac{x+1}{2}z + \frac{x-1}{2}, \ z \in [-1,1],$$
$$t_x(v) = \frac{\varphi(x)+1}{2}v + \frac{\varphi(x)-1}{2}, \ v \in [-1,1].$$

Equation (4) is equivalent to

$$u(x_i) = g(x_i) + \int_{-1}^{1} K_1(x_i,z)u(s_{x_i}(z))dz + \int_{-1}^{1} K_2(x_i,v)u(qt_{x_i}(v)+q-1)dv, \tag{5}$$

where

$$K_1(x,z) := \frac{x+1}{2}\widetilde{K_1}(x, s_x(z)), \ K_2(x,v) := \frac{\varphi(x)+1}{2}\widetilde{K_2}(x, t_x(v)).$$

With $N+1$- point of Gauss quadrature formula, we have

$$u(x_i) \approx g(x_i) + \sum_{k=0}^{N} K_1(x_i,z_k)u(s_{x_i}(z_k))\omega_k + \sum_{k=0}^{N} K_2(x_i,v_k)u(qt_{x_i}(v_k)+q-1)\omega_k,$$

$i = 0,1,2,...,N$, where $z_k = v_k$ are the $N+1$-Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto points, the corresponding weight function $\omega_k, k = 0,1,...,N$. We use u_i to approximate $u(x_i)$ and use

$$u^N(x) := \sum_{j=0}^{N} u_j F_j(x)$$

to approximate $u(x)$. $F_j(x)$ is the $j-th$ Lagrange basic function with $\{x_i\}_{i=0}^N$, so if $x = x_i$, then $u^N(x_i) = u_i$. Let

$$u^N(x) = g(x) + \sum_{k=0}^{N} K_1(x,z_k)u^N(s_x(z_k))\omega_k + \sum_{k=0}^{N} K_2(x,v_k)u^N(qt_x(v_k)+q-1)\omega_k.$$

The Chebyshev spectral-collocation method is to seek $u^N(x)$ such that $\{u_i\}_{i=0}^N$ satisfy the following equations for $i = 0,1,2,...,N$,

$$u_i = g(x_i) + \sum_{k=0}^{N} K_1(x_i,z_k)u^N(s_{x_i}(z_k))\omega_k + \sum_{k=0}^{N} K_2(x_i,v_k)u^N(qt_{x_i}(v_k)+q-1)\omega_k. \tag{6}$$

§3 Some useful lemmas

In this section, we will provide some fundamental lemmas, which are important for the following error estimation. In order to discuss clearly, we first introduce some spaces.

Let $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ be a weight function in the usual sense, for $\alpha, \beta > -1$. The set of Jacobi polynomials $\{J_n^{\alpha,\beta}(x)\}_{n=0}^\infty$ forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$ orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1) := \{v\colon v \text{ is measurable and } \|v\|_{\omega^{\alpha,\beta}} < \infty\},$$

equipped with the norm $\|v\|_{\omega^{\alpha,\beta}} = \left(\int_{-1}^{1} |v(x)|^2 \omega^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}}$, and the inner product

$$(u,v)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} u(x)v(x)\omega^{\alpha,\beta}(x)dx,\ \forall u,v \in L^2_{\omega^{\alpha,\beta}}(-1,1).$$

We denote $\left(\frac{\partial^k v}{\partial x^k}\right)(x)$ by $\partial_x^k v, 0 \le k \le m$. For non-negative integer m, define

$$H^m_{\omega^{\alpha,\beta}}(-1,1) := \{v : \partial_x^k v \in L^2_{\omega^{\alpha,\beta}}(-1,1), 0 \leqslant k \leqslant m\},$$

equipped with the norm

$$\|v\|_{H^m_{\omega^{\alpha,\beta}}(-1,1)} := \Big(\sum_{k=0}^{m} \|\partial_x^k v\|^2_{L^2_{\omega^{\alpha,\beta}}(-1,1)}\Big)^{\frac{1}{2}}.$$

For a nonnegative integer N, the semi-norm is defined as:

$$|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(-1,1)} := \Big(\sum_{k=min(m,N+1)}^{m} \|\partial_x^k v\|^2_{L^2_{\omega^{\alpha,\beta}}(-1,1)}\Big)^{\frac{1}{2}}.$$

When $\alpha = \beta = 0$, we denote $H^{m;N}_{\omega^{0,0}}(-1,1)$ by $H^{m;N}(-1,1)$. When $\alpha = \beta = -\frac{1}{2}$, we denote $\omega^{-\frac{1}{2},-\frac{1}{2}}$ by ω^c.

The space $L^\infty(-1,1)$ is the Banach space of the measurable functions $u : (-1,1) \to R$ that are bounded outside a set of measure zero, equipped with the norm

$$\|u\|_{L^\infty(-1,1)} := ess \sup_{x\in(-1,1)} |u(x)|.$$

Lemma 3.1 [10, 11] Assume that $u \in H^m_{\omega^c}(-1,1)$, $m \geqslant 1$. Then the following estimates hold

$$\|u - I_N u\|_{L^2_{\omega^c}(-1,1)} \leqslant CN^{-m} |u|_{H^{m;N}_{\omega^c}(-1,1)}, \tag{7}$$

$$\|u - I_N u\|_{L^\infty(-1,1)} \le CN^{\frac{1}{2}-m} |u|_{H^{m;N}_{\omega^c}(-1,1)}, \tag{8}$$

where I_N is the interpolation operator associated with the $N+1$-point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points $\{x_j\}_{j=0}^N$, promptly

$$I_N v(x) := \sum_{j=0}^{N} v(x_j)F_j(x),\ v \in C([-1,1]).$$

Lemma 3.2 [10, 11] Suppose $u \in H^m_{\omega^c}(-1,1)$, $v \in H^m(-1,1)$ for some $m \geqslant 1$ and $\psi \in P_N$, which denotes the space of all polynomials of degree not exceeding N. Then there exists a constant C independent of N such that

$$\Big|\int_{-1}^{1} u(x)\psi(x)\omega^c(x)dx - \sum_{j=0}^{N} u(x_j)\psi(x_j)\omega_j^c\Big| \leqslant CN^{-m}|u|_{H^{m;N}_{\omega^c}(-1,1)}\|\psi\|_{L^2_{\omega^c}(-1,1)},$$

and

$$\Big|\int_{-1}^{1} v(x)\psi(x)dx - \sum_{j=0}^{N} v(z_j)\psi(z_j)\omega_j\Big| \leqslant CN^{-m}|v|_{H^{m;N}(-1,1)}\|\psi\|_{L^2(-1,1)},$$

where x_j is the $N+1$-point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto point, corresponding weight $\omega_j^c, j = 0,1,...N$ and z_j is the $N+1$-point Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto point, corresponding weight $\omega_j, j = 0,1,...N$.

Lemma 3.3 [6, 12] Let $F_j(x), j = 0,1,...N$ be the $j-th$ Lagrange interpolation polynomials associated with $N+1$-point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points $\{x_j\}_{j=0}^{N}$. Then

$$\|I_N\|_{L^\infty(-1,1)} := \max_{x\in[-1,1]} \sum_{j=0}^{N} |F_j(x)| = O(logN).$$

Lemma 3.4 (Gronwall inequality) [4, 13] Assume that $u(x)$ is a nonnegative, locally integrable function defined on $[-1,1]$ satisfying

$$u(x) \leqslant v(x) + L\int_{-1}^{x} u(\tau)d\tau,$$

where $L \geq 0$ is a constant, $v(x)$ is an integrable function. Then there exists a constant C such that

$$u(x) \leqslant v(x) + C\int_{-1}^{x} v(\tau)d\tau,$$

and

$$\|u(x)\|_{L^\infty(-1,1)} \leqslant C\|v(x)\|_{L^\infty(-1,1)}.$$

Lemma 3.5 Suppose $0 \leqslant M_1, M_2 \leqslant +\infty$. If a nonnegative integrable function $e(x)$ satisfies

$$e(x) \leqslant v(x) + M_1\int_{-1}^{x} e(t)dt + M_2\int_{-1}^{\varphi(x)} e(q\theta+q-1)d\theta,$$

where $v(x)$ is also a nonnegative integrable function, then

$$e(x) \leqslant v(x) + C\int_{-1}^{x} v(\tau)d\tau,$$

and

$$\|e(x)\|_{L^\infty(-1,1)} \leqslant C\|v(x)\|_{L^\infty(-1,1)},$$

where C is a constant.

Proof: With

$$\varphi(x) := \frac{2}{T}\phi\Big(\frac{2}{T}(1+x)\Big) - 1 \leq \frac{2}{T}\Big(\frac{T}{2}\Big)(1+x) - 1 = x,$$

we get

$$e(x) \leq v(x) + M_1\int_{-1}^{x} e(t)dt + M_2\int_{-1}^{x} e(q\theta+q-1)d\theta.$$

Let

$$\rho = q\theta + q - 1,$$

then

$$\theta = \frac{\rho}{q} + \frac{1-q}{q},$$

and

$$\int_{-1}^{x} e(q\theta+q-1)d\theta = \frac{1}{q}\int_{-1}^{qx+q-1} e(\rho)d\rho < \frac{1}{q}\int_{-1}^{x} e(\theta)d\theta.$$

for $0<q<1, qx+q-1 = q(x+1)-1 < x+1-1 = x, x\in[0,1]$.

So

$$e(x) \leq v(x) + (M_1+M_2)\int_{-1}^{x} e(\theta)d\theta.$$

According to lemma 3.4, we have

$$e(x) \leq v(x) + C\int_{-1}^{x} v(\tau)d\tau,$$

and

$$\|e(x)\|_{L^\infty(-1,1)} \leq C\|v(x)\|_{L^\infty(-1,1)}.$$

Lemma 3.6 [14] For all measurable function $f \geq 0$, the following general Hardy inequality

$$\Big(\int_a^b |(Tf)(x)|^q \omega_1(x)dx\Big)^{\frac{1}{q}} \leq C\Big(\int_a^b |f(x)|^p \omega_2(x)dx\Big)^{\frac{1}{p}}$$

holds if and only if

$$\sup_{a<x<b}\Big(\int_x^b \omega_1(t)dt\Big)^{\frac{1}{q}}\Big(\int_a^x \omega_2(t)^{1-p'}dt\Big)^{\frac{1}{p'}} < \infty,\ p' = \frac{p}{p-1},$$

for the case of $1 < p \leq q < \infty$, where T is the form of the following function

$$(Tf)(x) = \int_a^x k(x,t)f(t)dt,$$

with the condition that the function $k(x,t)$ is a given kernel function, ω_1, ω_2 is the weight functions and $-\infty \leq a < b \leq +\infty$.

Lemma 3.7 [6, 15] For all bounded function $v(x)$, there exists a constant C independent of v such that

$$\sup_N \|I_N v\|_{L^2_{\omega^c}(-1,1)} \leqslant C\|v\|_{L^\infty(-1,1)}.$$

§4 Convergence analysis

This section is devoted to providing convergence analysis for the Volterra integral equation. Our goal is to show that the rate of convergence is exponential and the spectral accuracy can be obtained in L^∞ and $L^2_{\omega^c}$ spaces by using Chebyshev spectral-collocation method.

Theorem 4.1 Assume that $u(x)$ is the exact solution of equation (3) and $u^N(x)$ is the approximate solution of the spectrum collocation method for equation (6). For N that is large enough, there is

$$\|u(x) - u^N(x)\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m}\Big(|u|_{H^{m;N}_{\omega^c}(-1,1)} + (K_1^* + K_2^*)\|u\|_{L^\infty(-1,1)}\Big), \tag{9}$$
$$K_i^* := \max_{-1\leq x\leq 1} |K_i(x,z)|_{H^{m;N}(-1,1)}, i = 1, 2.$$

Proof: Note that $e(x) = u(x) - u^N(x)$, subtract (5) from (6) and we have

$$e(x_i) = \int_{-1}^1 K_1(x_i,z)e(s_{x_i}(z))dz + \int_{-1}^1 K_2(x_i,v)e(qt_{x_i}(v)+q-1)dv + J_1(x_i) + J_2(x_i), \tag{10}$$
$$i = 0, 1..., N,$$

where

$$J_1(x) := \int_{-1}^1 K_1(x,z)u^N(s_x(z))dz - \sum_{k=0}^N K_1(x,z_k)u^N(s_x(z_k))\omega_k,$$
$$J_2(x) := \int_{-1}^1 K_2(x,v)u^N(qt_x(v)+q-1)dv - \sum_{k=0}^N K_2(x,v_k)u^N(qt_x(v_k)+q-1)\omega_k.$$

Using lemma 3.2, we have

$$\begin{aligned} |J_1(x)| &\leq CN^{-m}|K_1(x,\cdot)|_{H^{m;N}(-1,1)}\|u^N(s_x(z))\|_{L^2(-1,1)}, \\ |J_2(x)| &\leq CN^{-m}|K_2(x,\cdot)|_{H^{m;N}(-1,1)}\|u^N(qt_x(v)+q-1)\|_{L^2(-1,1)}. \end{aligned} \tag{11}$$

Timing $F_i(x)$ on both sides of the equation (10) and summing up from $i = 0$ to N get

$$\begin{aligned} &I_N u(x) - u^N(x) \\ &= I_N\int_{-1}^1 K_1(x,z)e(s_x(z))dz + I_N\int_{-1}^1 K_2(x,v)e(qt_x(v)+q-1)dv + I_N J_1(x) + I_N J_2(x). \end{aligned}$$

So,

$$e(x) \tag{12}$$
$$= \int_{-1}^{1} K_1(x,z)e(s_x(z))dz + \int_{-1}^{1} K_2(x,v)e(qt_x(v)+q-1)dv + \sum_{j=0,3,4} J_j(x) + \sum_{j=1}^{2} I_N J_j(x),$$

where

$$J_0(x) := u(x) - I_N u(x),$$
$$J_3(x) := I_N \int_{-1}^{1} K_1(x,z)e(s_x(z))dz - \int_{-1}^{1} K_1(x,z)e(s_x(z))dz,$$
$$J_4(x) := I_N \int_{-1}^{1} K_2(x,v)e(qt_x(v)+q-1)dv - \int_{-1}^{1} K_2(x,v)e(qt_x(v)+q-1)dv.$$

We rewrite equation (12) as

$$e(x) \tag{13}$$
$$= \int_{-1}^{x} \widetilde{K_1}(x,s)e(s)ds + \int_{-1}^{\varphi(x)} \widetilde{K_2}(x,t)e(qt+q-1)dt + \sum_{j=0,3,4} J_j(x) + \sum_{j=1}^{2} I_N J_j(x).$$

According to lemma 3.5,

$$\|e\|_{L^\infty(-1,1)} \tag{14}$$
$$\le C(\|J_0\|_{L^\infty(-1,1)} + \|I_N J_1\|_{L^\infty(-1,1)} + \|I_N J_2\|_{L^\infty(-1,1)} + \|J_3\|_{L^\infty(-1,1)} + \|J_4\|_{L^\infty(-1,1)}).$$

Now we estimate the above inequality term by term. Applying the inequality (8) to $J_0(x)$, we get

$$\|J_0(x)\|_{L^\infty(-1,1)} = \|u - I_N u\|_{L^\infty(-1,1)} \le CN^{\frac{1}{2}-m}|u|_{H^{m;N}_{\omega^c}(-1,1)}. \tag{15}$$

Next, using the inequality (11) for $J_1(x)$, we have

$$\max_{-1\le x\le 1} |J_1(x)| \tag{16}$$
$$\le CN^{-m}K_1^* \max_{-1\le x\le 1} \|u^N(s_x(z))\|_{L^2(-1,1)}$$
$$\le CN^{-m}K_1^* \|u^N\|_{L^\infty(-1,1)}$$
$$\le CN^{-m}K_1^* \Big(\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}\Big).$$

According to lemma 3.3,

$$\|I_N J_1(x)\|_{L^\infty(-1,1)} \tag{17}$$
$$\le \|I_N\|_{L^\infty(-1,1)} \|J_1(x)\|_{L^\infty(-1,1)}$$
$$\le CN^{-m}(logN)K_1^* \Big(\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}\Big).$$

Similarly,

$$\|I_N J_2(x)\|_{L^\infty(-1,1)} \le CN^{-m}(logN)K_2^* \Big(\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}\Big). \tag{18}$$

Now we estimate $\|J_3\|_{L^\infty(-1,1)}$ and $\|J_4\|_{L^\infty(-1,1)}$. Use the inequality (8) and let $m=1$,

$$\begin{aligned}&\|J_3\|_{L^\infty(-1,1)} \qquad (19)\\ \le\ & CN^{-\frac{1}{2}}\Big|\int_{-1}^{1}K_1(x,z)e(s_x(z))dz\Big|_{H^{1;N}_{\omega^c}(-1,1)}\\ =\ & CN^{-\frac{1}{2}}\Big|\int_{-1}^{x}\widetilde{K_1}(x,s)e(s)ds\Big|_{H^{1;N}_{\omega^c}(-1,1)}\\ =\ & CN^{-\frac{1}{2}}\Big\|\widetilde{K_1}(x,x)e(x)+\int_{-1}^{x}e(x)\frac{\partial}{\partial x}\widetilde{K_1}(x,s)ds\Big\|_{L^2_{\omega^c}(-1,1)}\\ \le\ & CN^{-\frac{1}{2}}\|e\|_{L^\infty(-1,1)}.\end{aligned}$$

In the same way, there is

$$\begin{aligned}&\|J_4\|_{L^\infty(-1,1)} \qquad (20)\\ =\ & CN^{-\frac{1}{2}}\Big|\int_{-1}^{\varphi(x)}\widetilde{K_2}(x,t)e(qt+q-1)dt\Big|_{H^{1;N}_{\omega^c}(-1,1)}\\ =\ & CN^{-\frac{1}{2}}\Big|\int_{-1}^{x}\widetilde{K_2}(x,\varphi(t))e(q\varphi(t)+q-1)\varphi^{'}(t)dt\Big|_{H^{1;N}_{\omega^c}(-1,1)}\\ =\ & CN^{-\frac{1}{2}}\Big|\widetilde{K_2}(x,\varphi(x))e(q\varphi(x)+q-1)\varphi^{'}(x)|_{L^2_{\omega^c}(-1,1)}\\ &+|\int_{-1}^{x}e(q\varphi(t)+q-1)\varphi^{'}(t)\frac{\partial}{\partial x}\widetilde{K_2}(x,\varphi(t))dt\Big|_{L^2_{\omega^c}(-1,1)}\\ \le\ & CN^{-\frac{1}{2}}\|e\|_{L^\infty(-1,1)}.\end{aligned}$$

Jointing $(14),(15),(17)-(20)$, we have

$$\|e\|_{L^\infty(-1,1)}\le CN^{\frac{1}{2}-m}|u|_{H^{m;N}_{\omega^c}(-1,1)}+CN^{-m}(logN)(K_1^*+K_2^*)\|u\|_{L^\infty(-1,1)}.$$

And because

$$\lim_{N\to\infty}\frac{N^{-m}logN}{N^{\frac{1}{2}-m}}=\lim_{N\to\infty}\frac{logN}{N^{\frac{1}{2}}}=0,$$

when N is large enough,

$$N^{-m}logN<N^{\frac{1}{2}-m}.$$

So we finish the proof of the conclusion:

$$\|e\|_{L^\infty(-1,1)}\le CN^{\frac{1}{2}-m}\Big(|u|_{H^{m,N}_{\omega^c}(-1,1)}+(K_1^*+K_2^*)\|u\|_{L^\infty(-1,1)}\Big).$$

Theorem 4.2 The exact solution of equation (3) is $u(x)$ and $u^N(x)$ is the approximate solution of the spectrum collocation method for equation (6). Let us assume that N is very large, then

$$\|u(x)-u^N(x)\|_{L^2_{\omega^c}(-1,1)}\le CN^{-m}\Big(K_1^*+K_2^*+1\Big)\Big(|u|_{H^{m;N}_{\omega^c}(-1,1)}+\|u^{'}\|_{L^2(-1,1)}+\|u\|_{L^\infty(-1,1)}\Big).$$

Proof: As the same procedure in the deduction from (10) to (13) in theorem 4.1, and with the help of the first conclusion from lemma 3.5 we can derive (13) to the following inequality

$$|e(x)|\le C\int_{-1}^{x}\Big|\sum_{j=0,3,4}J_j(t)+\sum_{j=1}^{2}I_NJ_j(t)\Big|dt+\Big|\sum_{j=0,3,4}J_j(t)+\sum_{j=1}^{2}I_NJ_j(t)\Big|.$$

Using lemma 3.6, we have

$$\begin{aligned}&\|e\|_{L^2_{\omega^c}(-1,1)} \qquad (21)\\ \le\ & C\Big(\|J_0\|_{L^2_{\omega^c}(-1,1)}+\|I_NJ_1\|_{L^2_{\omega^c}(-1,1)}+\|I_NJ_2\|_{L^2_{\omega^c}(-1,1)}+\|J_3\|_{L^2_{\omega^c}(-1,1)}+\|J_4\|_{L^2_{\omega^c}(-1,1)}\Big).\end{aligned}$$

Now we estimate each item from left to right for the above inequality. For $J_0(x)$, the first conclusion in lemma 3.1 shows that

$$\|J_0(x)\|_{L^2_{\omega^c}(-1,1)} = \|u - I_N u\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m}|u|_{H^{m;N}_{\omega^c}(-1,1)}. \tag{22}$$

For the estimation of $\|I_N J_1\|_{L^2_{\omega^c}(-1,1)}$, using lemma 3.7 and the inequality (16), we obtain

$$\|I_N J_1(x)\|_{L^2_{\omega^c}(-1,1)} \leq C \max_{-1\leq x\leq 1} |J_1(x)| \leq CN^{-m}K_1^*\Big(\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}\Big), \tag{23}$$

then let $m = 1$ in the conclusion of theorem 4.1 and we have

$$\|e\|_{L^\infty(-1,1)} \leq C\Big(\|u^{'}\|_{L^2_{\omega^c}(-1,1)} + (K_1^* + K_2^*)\|u\|_{L^\infty(-1,1)}\Big),$$

in the end (23) turns to

$$\|I_N J_1(x)\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m}K_1^*\Big(\|u^{'}\|_{L^2_{\omega^c}(-1,1)} + (K_1^* + K_2^* + 1)\|u\|_{L^\infty(-1,1)}\Big). \tag{24}$$

In the same way,

$$\|I_N J_2(x)\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m}K_2^*\Big(\|u^{'}\|_{L^2_{\omega^c}(-1,1)} + (K_1^* + K_1^* + 1)\|u\|_{L^\infty(-1,1)}\Big). \tag{25}$$

Now we turn to the estimation of $\|J_3(x)\|_{L^2_{\omega^c}(-1,1)}$. With the help of (7) from lemma 3.1, let $m = 1$, like the analysis of inequality (19) and we have

$$\|J_3\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-1}\|e\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-1}\|e\|_{L^\infty(-1,1)}.$$

Due to the conclusion in theorem 4.1, we get

$$\|J_3\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-\frac{1}{2}-m}\Big(|u|_{H^{m;N}_{\omega^c}(-1,1)} + (K_1^* + K_2^*)\|u\|_{L^\infty(-1,1)}\Big). \tag{26}$$

The similar result for $\|J_4(x)\|_{L^2_{\omega^c}(-1,1)}$ is:

$$\|J_4\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-\frac{1}{2}-m}\Big(\|u\|_{H^{m;N}_{\omega^c}(-1,1)} + (K_1^* + K_1^*)\|u\|_{L^\infty(-1,1)}\Big). \tag{27}$$

Jointing $(21), (22), (24) - (27)$, we can get the desired estimation as follows:

$$\|e\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m}(K_1^* + K_2^* + 1)\Big(|u|_{H^{m;N}_{\omega^c}(-1,1)} + \|u^{'}\|_{L^2_{\omega^c}(-1,1)} + \|u\|_{L^\infty(-1,1)}\Big).$$

§5 Numerical experiments

5.1 Example 1

In order to compare our method with Legendre spectral collocation method used in [4], for (3), we let the second kernel be zero. Details are

$$\begin{aligned} \widetilde{K_1}(x,s) &= e^{xs}, \\ \widetilde{K_2}(x,t) &= 0, \\ g(x) &= e^{3x} - \frac{1}{x+3}\Big(-e^{x+3} + e^{x(x+3)}\Big), \\ q &= \frac{1}{2}. \end{aligned}$$

The exact solution of the above problem is

$$u(x) = e^{3x}, x \in [-1,1].$$

The left figure plots the errors for $6 \leqslant N \leqslant 24$ in both L^∞ and $L^2_{\omega^c}$ norms. The approximate solution ($N = 24$) and the exact solution are displayed in the right figure. Moreover, the corresponding errors versus several values of N are displayed in Table 1. As expected, the errors decay exponentially which confirmed our theoretical predictions. Further, if we compare the errors in Table 1 with the errors in Table 5.1 in [4], we can find that the accuracy obtained by our method is higher than the Legendre spectral collocation method.

Table 1. The errors $u - u^N$ versus the number of collocation points in L^∞ and $L^2_{\omega^c}$ norms.

N	6	8	10	12	14
$L^\infty - error$	0.0049198	9.1148e-005	1.299e-006	1.4454e-008	1.2208e-010
$L^2_{\omega^c} - error$	0.0050825	0.00010295	1.4986e-006	1.6964e-008	1.5054e-010
N	16	18	20	22	24
$L^\infty - error$	8.6686e-013	1.0658e-014	1.0658e-014	1.4211e-014	2.4869e-014
$L^2_{\omega^c} - error$	1.0669e-012	2.4242e-014	3.3694e-014	2.9712-014	3.7506e-014

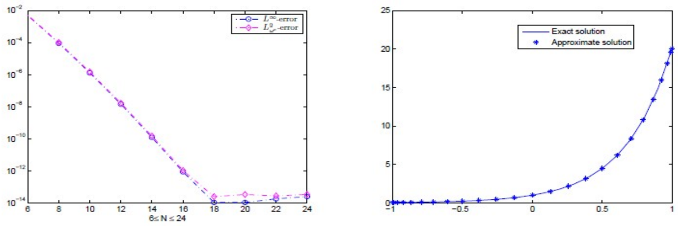

Figure 1. The left figure shows the errors $u - u^N$ versus the number of collocation points in L^∞ and $L^2_{\omega^c}$ norms. The right figure shows the comparison between approximate solution u^N and the exact solution u.

5.2 Example 2

Without losing of generality, we give a general example to verify our theoretical prediction. For (1), let

$$\begin{aligned} R_1(\tau,\xi) &= -(\tau-\xi), \\ R_2(\tau,\eta) &= -(\tau+\eta), \\ T &= 2, \\ q &= \frac{1}{2}, \\ \phi(\tau) &= \tau^2, \\ f(\tau) &= 2\tau(\tau+1)\sin\frac{1}{2}\tau^2 + 4\cos\frac{1}{2}\tau^2 - 3. \end{aligned}$$

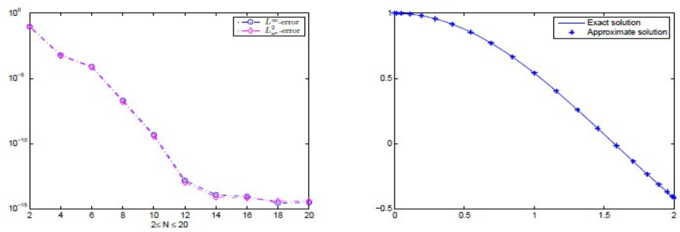

Figure 2. The left figure shows the errors $u - u^N$ versus the number of collocation points in L^∞ and $L^2_{\omega^c}$ norms. The right figure shows the comparison between approximate solution u^N and the exact solution u.

The corresponding exact solution is $y(\tau) = \cos\tau$, $\tau \in [0, 2]$.

Table 2. The errors $u - u^N$ versus the number of collocation points in L^∞ and $L^2_{\omega^c}$ norms.

N	2	4	6	8	10
$L^\infty - error$	0.094571	0.0006152	7.9180e-005	2.0942e-007	4.7323e-010
$L^2_{\omega^c} - error$	0.090112	0.0005386	7.2256e-005	1.6176e-007	3.6854e-010
N	12	14	16	18	20
$L^\infty - error$	1.5138e-013	1.1768e-014	8.8818e-015	2.7200e-015	2.9976e-015
$L^2_{\omega^c} - error$	1.0371e-013	7.8362e-015	6.6881e-015	3.8413e-015	3.7273e-015

In Figure 2, the left figure plots the errors for $2 \leqslant N \leqslant 20$ in both L^∞ and $L^2_{\omega^c}$ norms. The approximate solution ($N = 20$) and the exact solution are displayed in the right figure. Moreover, the corresponding errors versus several values of N are displayed in Table 2. As expected, the errors decay exponentially which confirmed our theoretical predictions.

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