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Existence and uniqueness of S-asymptotically periodic α-mild solutions for neutral fractional delayed evolution equation

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Abstract. In this paper, we investigate a class of abstract neutral fractional delayed evolution equation in the fractional power space. With the aid of the analytic semigroup theories and some fixed point theorems, we establish the existence and uniqueness of the S-asymptotically periodic α-mild solutions. The linear part generates a compact and exponentially stable analytic semigroup and the nonlinear parts satisfy some conditions with respect to the fractional power norm of the linear part, which greatly improve and generalize the relevant results of existing literatures.

§1 Introduction

Let X be a Banach space provided with norm $\|\cdot\|$ and let $r > 0$ be a constant. We denote $\mathcal{B} := C([-r, 0], X)$ as the Banach space of continuous functions from $[-r, 0]$ to X with the norm $\|\phi\|_B = \max_{s\in[-r,0]} \|\phi(s)\|$. The aim of this paper is to study the following abstract neutral fractional evolution equation with delay

$$\begin{cases} {}^cD_t^q(u(t) - G(t, u_t)) + Au(t) = F(t, u_t), \quad t \geq 0, \\ u(t) = \varphi(t), \quad t \in [-r, 0], \end{cases} \tag{1.1}$$

where ${}^cD_t^q$ is the Caputo fractional derivation of order $q \in (0, 1)$, $A : D(A) \subset X \to X$ is a closed linear operator, and $-A$ generates a compact and exponentially stable analytic semigroup $T(t)(t \geq 0)$ in Banach space X; $F, G : \mathbb{R}^+ \times \mathcal{B} \to X$ are given functions which will be specified later, $\varphi \in \mathcal{B}$. For $t \geq 0$, u_t denotes the history function defined by $u_t(s) = u(t+s)$ for $s \in [-r, 0]$, where u is a continuous function from $[-r, \infty)$ into X.

The theory of delayed partial differential equations has a wide range of physical background and practical mathematical models (see [11, 33]). During the last few decades, the neutral differential equations, in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, have attracted many scholars' attention. Neutral

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differential equations have many applications, such as population dynamics, transmission lines, immune response or distribution of albumin in the blood etc. The existence and uniqueness of mild solutions to the integer order neutral evolution equations with delay have been considered by many authors. Here we only mention [1, 8, 10, 12, 17, 18]. In the past decades, in view of the wide practical backgrounds and application prospects of fractional calculus, many concepts and methods are developed to solve the fractional differential equations (see [2, 21, 35]). Because the fractional evolution equations are abstract formulations in many practical applications, the research of them has been paid more and more attention by many scholars, and has made great progress (see [3, 4, 7, 9, 14, 30–32, 36] and the references therein).

The problem concerning periodic solutions of partial differential equations is an important area of investigation since it can take into account seasonal fluctuations occurring in the phenomena appearing in the models. However, because fractional derivative has genetic or memory properties, the solutions of periodic boundary value problems for fractional differential equations cannot be extended periodically to time t in $\mathbb{R}^+$. Specially, the nonexistence of nontrivial periodic solutions of fractional evolution equations had been shown in [22]. Since 2008, Henríquez et al.[13] introduced the concept of S-asymptotically periodic function, the existence and uniqueness of the S-asymptotically periodic solutions for fractional evolution equations have been widely studied (see [16, 20, 22, 23, 26, 27]). Specially, Shu et al [26, 27] studied the semilinear neutral Caputo fractional evolution equation (1.1) and obtained the existence of the S-asymptotically periodic solutions under the assumption that the nonlinear functions F and G satisfy the Lipschitz-type conditions. Li and Wang [16] studied the neutral Caputo fractional evolution equation with finite delay

$$\begin{cases} {}^cD_t^q(u(t) - G(t, u_t)) = A(u(t) - G(t, u_t)) + F(t, u_t), \quad t \geq 0, \\ u(t) = \varphi(t), \quad t \in [-r, 0], \end{cases} \tag{1.2}$$

and established the existence of S-asymptotically periodic solutions without the assumptions that the nonlinearity F satisfies a Lipschitz type condition.

Fractional power operator can simulate fractional power dispersion (see [25]), and many researchers study the existence and regularity of solutions for differential equations in fractional power spaces (see [18, 31, 32]). For instance, using singular version of Gronwall inequality and Leray-Schauder fixed point theorem, Wang and Zhou [32] obtained the existence and uniqueness of α-mild solutions for the fractional evolution equation

$$\begin{cases} {}^cD_t^q u(t) + Au(t) = F(t, u_t), \quad t \in [0, T], \\ u(0) = u_0 \in X_\alpha, \end{cases} \tag{1.3}$$

where $-A$ generates a compact analytic semigroup $T(t)(t \geq 0)$, $0 \in \rho(A)$ and $X_\alpha(\alpha \in (0,1))$ is a fractional power space. Clearly, the condition $F : [0,T] \times X_\alpha \to X$ is weaker than $F : [0,T] \times X \to X$.

Inspired by the above literature, by means of analytic semigroups and fixed point theorem, we consider the abstract neutral fractional delayed evolution equation (1.1) in the fractional power space. In Section 3, we establish the existence of S-asymptotically periodic α-mild solutions without the assumptions that the nonlinearity F satisfies a Lipschitz condition by Sadovskii fixed point theorem and obtain the uniqueness result via contraction mapping principle when F satisfies Lipschitz condition. Our results will improve the main results in [16, 26, 27]. In Section 2, some notions, definitions, and preliminary facts are introduced and at last, an example of time fractional partial differential equation is given to illustrate the application of our results.

§2 Preliminaries

Throughout this paper, we assume that X is a Banach space with norm $\|\cdot\|$, $A : D(A) \subset X \to X$ is a closed linear operator with $0 \in \rho(A)(\rho(A)$ is the resolvent set of $A)$, and $-A$ generates an exponentially stable analytic semigroup $T(t)(t \geq 0)$ in Banach space X.

For a general C_0-semigroup $T(t)(t \geq 0)$, there exist $M \geq 1$ and $\nu \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\nu t}, \quad t \geq 0. \tag{2.1}$$

Let

$$\nu_0 = \inf\{\gamma \in \mathbb{R} | \text{ There exists } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\nu t},\ \forall t \geq 0\}, \tag{2.2}$$

then ν_0 is called the growth exponent of the semigroup $T(t)(t \geq 0)$ and $T(t)$ is said to be an exponentially stable C_0-semigroup if $\nu_0 < 0$. Clearly, the exponentially stable C_0-semigroup $T(t)(t \geq 0)$ is uniformly bounded. If C_0-semigroup $T(t)$ is continuous in the uniform operator topology for every $t > 0$ in X, then ν_0 can also be determined by $\sigma(A)$ (the spectrum of A),

$$\nu_0 = -\inf\{\mathrm{Re}\lambda |\ \lambda \in \sigma(A)\}. \tag{2.3}$$

We know that $T(t)(t \geq 0)$ is continuous in the uniform operator topology for $t > 0$ if $T(t)(t \geq 0)$ is compact semigroup (see [19, 29]). In particular, if $T(t)(t \geq 0)$ is an analytic semigroup generated by A with $0 \in \rho(A)$, then for any $\alpha > 0$, we can define $A^{-\alpha}$ by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) dt.$$

It follows that each $A^{-\alpha}$ is an injective continuous endomorphism of X. Hence we can define A^α by $A^\alpha := (A^{-\alpha})^{-1}$, which is a closed bijective linear operator in X. Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression $\|x\|_\alpha := \|A^\alpha x\|$ defines a norm on $D(A^\alpha)$ for $x \in D(A^\alpha)$.

Hereafter we represent X_α to the space $D(A^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$ and we denote $\mathcal{B}_\alpha := C([-r, 0], X_\alpha)$ as the Banach space of continuous functions from $[-r, 0]$ to X_α with the norm $\|\phi\|_{B_\alpha} = \max_{s\in[-r,0]} \|\phi(s)\|_\alpha$.

Lemma 2.1.([19]) *If $T(t)(t \geq 0)$ is analytic semigroup with infinitesimal generator $-A$ satisfying $0 \in \rho(A)$, then*
(i) X_α is a Banach space for $0 \leq \alpha \leq 1$;
(ii) $A^{-\alpha}$ is a bounded linear operator for $0 \leq \alpha \leq 1$ in X;
(iii) $T(t) : X \to X_\alpha$ for $t > 0$, and $A^\alpha T(t)x = T(t)A^\alpha x$ for $x \in D(A^\alpha)$ and $t \geq 0$;
(vi) for every $t > 0$, $A^\alpha T(t)$ is bounded in X and there exists $M_\alpha > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha t^{-\alpha} e^{\nu_0 t},$$

(v) for $0 \leq \alpha \leq \beta \leq 1$, $X_\beta \hookrightarrow X_\alpha$ (with $X_0 = X$ and $X_1 = D(A)$), and the embedding $X_\beta \hookrightarrow X_\alpha$ is compact whenever the resolvent operator of A is compact.

Observe by Lemma 2.1 (iii) that the restriction $T_\alpha(t)$ of $T(t)$ to X_α is exactly the part of $T(t)$ in X_α. Moreover, for any $x \in X_\alpha$, we have

$$\|T_\alpha(t)x\|_\alpha = \|A^\alpha T(t)x\| = \|T(t)A^\alpha x\| \leq \|T(t)\| \cdot \|A^\alpha x\| = \|T(t)\| \cdot \|x\|_\alpha, \tag{2.4}$$

and

$$\|T_\alpha(t)x - x\|_\alpha = \|A^\alpha T(t)x - A^\alpha x\| = \|T(t)A^\alpha x - A^\alpha x\| \to 0, \quad t \to 0, \tag{2.5}$$

it follows that $T_\alpha(t)(t \geq 0)$ is a strongly continuous semigroup on X_α and $\|T_\alpha(t)\|_\alpha \leq \|T(t)\|$ for all $t \geq 0$.

For the definition of Caputo fractional derivation, we can refer to many references (see [4, 32] and so on), so we will not repeat it here. Next, we define operator families $U(t)(t \geq 0)$ and

$V(t)(t\geq 0)$ in X as following

$$U(t)=\int_0^\infty \xi_q(\tau)T(t^q\tau)d\tau,\ \ V(t)=q\int_0^\infty \tau\xi_q(\tau)T(t^q\tau)d\tau, \tag{2.6}$$

where

$$\xi_q(\tau)=\frac{1}{\pi q}\sum_{n=1}^{\infty}(-\tau)^{n-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \qquad \tau\in(0,\infty) \tag{2.7}$$

is a probability density function satisfying $\xi_q(\tau)\geq 0$ for $\tau\in(0,\infty)$ and

$$\int_0^\infty \tau^\alpha \xi_q(\tau)d\tau=\frac{\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}, \quad \alpha\in(-\infty,1]. \tag{2.8}$$

Lemma 2.2([32]) *Let $T(t)(t\geq 0)$ be a C_0-semigroup. Then the operator families $U(t)(t\geq 0)$ and $V(t)(t\geq 0)$ defined by (2.6) have the following properties:*

(i) $U(t)(t\geq 0)$ and $V(t)(t\geq 0)$ are strongly continuous operators.

(ii) If $T(t)(t\geq 0)$ is uniformly bounded , then $U(t)$ and $V(t)$ are linear bounded operators for any fixed $t\in\mathbb{R}^+$, i.e,

$$\|U(t)x\|\leq M\|x\|, \qquad \|V(t)x\|\leq \frac{M}{\Gamma(q)}\|x\|, \qquad x\in E.$$

(iii) If $T(t)(t\geq 0)$ is compact, then $U(t)$ and $V(t)$ are compact operators for every $t>0$.

(iv) For any $x\in X$, $\alpha,\beta\in(0,1)$, $AV(t)x=A^{1-\beta}V(t)A^\beta x$ and

$$\|A^\alpha V(t)\|\leq \frac{qM_\alpha\Gamma(1-\alpha)}{t^{q\alpha}\Gamma(q(1-\alpha))}, \quad t>0.$$

(v) For fixed $t\geq 0$ and any $x\in X_\alpha$,

$$\|U(t)x\|_\alpha\leq M\|x\|_\alpha,\ \ \|V(t)x\|_\alpha\leq \frac{qM}{\Gamma(1+q)}\|x\|_\alpha.$$

(vi) If $T(t)(t\geq 0)$ is equicontinuous, then $U_\alpha(t)$ and $V_\alpha(t)$ are uniformly continuous for $t>0$, i.e. for every fixed $t>0$ and $\varepsilon>0$, there exists $h>0$ such that for $t+\epsilon\geq 0$ and $|\epsilon|<h$

$$\|U_\alpha(t+\epsilon)-U_\alpha(t)\|_\alpha<\varepsilon,\ \|V_\alpha(t+\epsilon)-V_\alpha(t)\|_\alpha<\varepsilon,$$

where $U_\alpha(t)=\int_0^\infty \xi_q(\tau)T_\alpha(t^q\tau)d\tau$ and $V_\alpha(t)=q\int_0^\infty \tau\xi_q(\tau)T_\alpha(t^q\tau)d\tau$.

Let $C_b(\mathbb{R}^+,X)$ (or $C_b(\mathbb{R}^+,X_\alpha)$) denote the Banach space of all bounded and continuous functions from $\mathbb{R}^+$ to X (or X_α) equipped with the norm $\|u\|_C=\sup_{t\in\mathbb{R}^+}\|u(t)\|$ (or $\|u\|_{C_\alpha}=\sup_{t\in\mathbb{R}^+}\|u(t)\|_\alpha$). According to [5, Lemma 4], we have the following result.

Lemma 2.3 *Let $\Omega\subset C_b(\mathbb{R}^+,X)$ be bounded. Then Ω is relatively compact if and only if*

(C1) for all $t\in\mathbb{R}^+$, $\Omega(t)$ is compact in X;

(C2) Ω is equicontinuous, i.e. for any $u\in\Omega$ and $\varepsilon>0$, there exists $\delta>0$ such that $\|u(t_2)-u(t_1)\|<\varepsilon$ whenever $|t_2-t_1|<\delta$ for $t_1,t_2\in\mathbb{R}^+$;

(C3) $\forall\ \varepsilon>0$, there exists $N=N(\varepsilon)>0$ such that $\chi(\Omega|_{[N,\infty)})<\varepsilon$, where $\chi(\cdot)$ is the Hausdorff measure of non-compactness.

Next, we introduce the notions of S-asymptotically ω-periodic functions, which are important in this paper.

Definition 2.4([13]) *A function $u\in C_b(\mathbb{R}^+,X)$ is called S-asymptotically ω-periodic if there exists $\omega>0$ such that $\lim\limits_{t\to\infty}\|u(t+\omega)-u(t)\|=0$. In this case, we say that ω is an asymptotic period of u. It is clear that if ω is an asymptotic period for u, then every $k\omega$, $k=1,2,\cdots,$ is also an asymptotic period of u.*

Definition 2.5 ([13]) *A continuous function $f:[0,\infty)\times X\to X$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded set D of X, the set $\{f(t,x)|t\geq$*

$0, x \in D\}$ is bounded and $\lim_{t\to\infty} \|f(t+\omega, x) - f(t,x)\| = 0$ uniformly in $x \in D$.

Definition 2.6 ([13]) *A continuous function $f : [0,\infty) \times X \to X$ is said to be asymptotically uniformly continuous on bounded sets if for every $\varepsilon > 0$ and every bounded set D of X, there exist $L_{\varepsilon,D} \geq 0$ and $\delta_{\varepsilon,D} > 0$ such that $\|f(t,x) - f(t,y)\| \leq \varepsilon$ for all $t \geq L_{\varepsilon,D}$ and all $x, y \in D$ with $\|x - y\| \leq \delta_{\varepsilon,D}$.*

Lemma 2.7 ([13]) *Let X, Y be two Banach spaces and $f : [0,\infty) \times Y \to X$ be uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. If $u : [0,\infty) \to Y$ is an S-asymptotically ω-periodic function, then the function $f(t, u(t)) \in SAP_\omega(X)$.*

Let $SAP_\omega(X_\alpha)$ represent the subspace of $C_b(\mathbb{R}^+, X_\alpha)$ consisting of all the X_α-value S-asymptotically ω-periodic functions endowed with the uniform convergence norm denoted by $\|\cdot\|_{C_\alpha}$. Then $SAP_\omega(X_\alpha)$ is a Banach space(see [13, Proposition 3.5]). If $u \in SAP_\omega(X_\alpha)$, then it is not difficult to test and verify that the function $t \to u_t$ belongs to $SAP_\omega(\mathcal{B}_\alpha)$ (see [15, 16]).

For a given $\varphi \in \mathcal{B}_\alpha$, define

$$C_\varphi(X_\alpha) := \{u \in C([-r,\infty), X_\alpha) |\ u|_{[-r,0]} = \varphi \text{ and } u|_{\mathbb{R}^+} \in C_b(\mathbb{R}^+, X_\alpha)\}.$$

It is easy to see that $C_\varphi(X_\alpha)$ is a Banach space equipped with the norm $\|u\|_{C_{\varphi,\alpha}} = \|\varphi\|_{\mathcal{B}_\alpha} + \|u\|_{C_\alpha}$. Now, according to [7, 34], we define the α-mild solution for the equation (1.1) as follows.

Definition 2.8 *A function $u \in C([-r,\infty), X_\alpha)$ is said to be an α-mild solution for Eq.(1.1) if the function $s \mapsto (t-s)^{q-1} AV(t-s)G(s,u_s)$ is integrable for $t > s$ and u satisfies $u(t) = \varphi(t)$ for $\varphi \in \mathcal{B}_\alpha, t \in [-r, 0]$ and*

$$\begin{aligned} u(t) &= U(t)(u(0) - G(0,u_0)) + G(t,u_t) - \int_0^t (t-s)^{q-1} AV(t-s)G(s,u_s)ds \\ &\quad + \int_0^t (t-s)^{q-1} V(t-s)F(s,u_s)ds \end{aligned}$$

for $t \geq 0$. Moreover, if $u|_{\mathbb{R}^+} \in SAP_\omega(X_\alpha)$, then u is called S-asymptotically ω-periodic α-mild solution for Eq.(1.1).

§3 Main results

Theorem 3.1. *Let $-A$ generate a compact and exponentially stable analytic semigroup $T(t)(t \geq 0)$ in Banach space X, whose growth exponent denotes $\nu_0 < 0$. For $\alpha \in [0, 1)$, assume that $\varphi \in \mathcal{B}_\alpha$, $F : \mathbb{R}^+ \times \mathcal{B}_\alpha \to X$ and $G : \mathbb{R}^+ \times \mathcal{B}_\alpha \to X_1$ are continuous functions and $G(t, \theta) = \theta$ for $t \geq 0$. If the following conditions hold:*

(H1) for every $t \in \mathbb{R}^+, x \in X_\alpha$ and $\phi \in \mathcal{B}_\alpha$, there exists $\omega > 0$ such that

$$\lim_{t\to\infty} \|F(t+\omega,\phi) - F(t,\phi)\| = 0, \quad \lim_{t\to\infty} \|AG(t+\omega,\phi) - AG(t,\phi)\| = 0,$$

(H2) for any $R > 0$, there exists a positive function $h_R : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\sup_{\|\phi\|_{\mathcal{B}_\alpha} \leq R} \|F(t,\phi)\| \leq h_R(t), \quad t \in \mathbb{R}^+,$$

the function $s \mapsto (t-s)^{q-(q\alpha+1)} h_R(s)$ belongs to $L(\mathbb{R}^+, \mathbb{R}^+)$ for all $t > s$, and there exists a constant $0 < \gamma < \infty$ such that

$$\liminf_{R\to\infty} \frac{1}{R} \int_0^t (t-s)^{q-(q\alpha+1)} h_R(s)ds = \gamma, \quad t \in \mathbb{R}^+,$$

(H3) there exists a constant $L_G > 0$ *such that*

$$\|AG(t,\phi_2) - AG(t,\phi_1)\| \le L_G\|\phi_2 - \phi_1\|_{\mathcal{B}_\alpha}, \quad t \in \mathbb{R}^+, \phi_1, \phi_2 \in \mathcal{B}_\alpha,$$

(H4) $C_{\alpha-1}L_G + \frac{L_G M_\alpha \Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}} + \frac{\gamma M_\alpha \Gamma(1-\alpha)}{\Gamma(q-q\alpha))} < 1$, *where* $C_{\alpha-1} = \|A^{\alpha-1}\|$,

then the problem (1.1) has at least one S*-asymptotic* ω*-periodic* α*-mild solution.*

Proof Define the operator $Q : C_\varphi(X_\alpha) \to C_\varphi(X_\alpha)$ by

$$Qu(t) = \begin{cases} U(t)(\varphi(0) - G(0,\varphi)) + G(t,u_t) - \displaystyle\int_0^t (t-s)^{q-1}V(t-s)AG(s,u_s)ds \\ + \displaystyle\int_0^t (t-s)^{q-1}V(t-s)F(s,u_s)ds, & t \ge 0, \\ \varphi(t), & t \in [-r,0]. \end{cases} \tag{3.1}$$

For any $u \in C_\varphi(X_\alpha)$, from the condition (H3) and Lemma 2.2, we find that

$$\begin{aligned} & \|(t-s)^{q-1}V(t-s)AG(s,u_s)\|_\alpha \\ \le\ & q(t-s)^{q-1}\Big\|\int_0^\infty \tau\xi_q(\tau)A^\alpha T((t-s)^q\tau)d\tau\Big\| \cdot L\|u_s - \theta\|_{\mathcal{B}_\alpha} \\ \le\ & L_G M_\alpha (t-s)^{q-(q\alpha+1)} \cdot \int_0^\infty e^{\nu_0(t-s)^q\tau}\tau^{1-\alpha}\xi_q(\tau)d\tau \cdot \|u\|_{C_{\varphi,\alpha}}, \end{aligned}$$

thus $s \mapsto (t-s)^{q-1}V(t-s)AG(s,u_s)$ is integrable on $[0,t)$ for every $t > 0$. Similarly, we can deduce that $s \mapsto (t-s)^{q-1}V(t-s)F(s,u_s)$ is integrable on $[0,t)$ for every $t > 0$. Therefore, it is easy to test that $Q : C_\varphi(X_\alpha) \to C_\varphi(X_\alpha)$ is well defined. By Definition 2.10, we can assert that the fixed points of Q are mild solutions to the problem (1.1).

For any $R > 0$, we define a set by

$$\overline{\Omega}_R = \{u \in C_\varphi(X_\alpha) |\ \ \|u\|_{C_{\varphi,\alpha}} \le R\}. \tag{3.2}$$

Clearly, $\overline{\Omega}_R \subset C_\varphi(X_\alpha)$ is a closed ball whose center is θ and radius is R. We show that there is a positive constant R such that $Q(\overline{\Omega}_R) \subset \overline{\Omega}_R$. In fact, from the condition (H4), we can choose a constant $R > 0$ big enough such that

$$R \ge \frac{(M + C_{\alpha-1}L_G M)\|\varphi\|_{\mathcal{B}_\alpha}}{1 - (C_{\alpha-1}L_G + \frac{L_G M_\alpha\Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}} + \frac{\gamma M_\alpha \Gamma(1-\alpha)}{\Gamma(q-q\alpha))})}. \tag{3.3}$$

For every $u \in \overline{\Omega}_R$, if $t \in [-r,0]$, then from (3.1), it follows that $Qu(t) = \varphi(t)$,

$$\|Qu(t)\|_\alpha = \|\varphi(t)\|_\alpha \le \|\varphi\|_{\varphi_\alpha} \le \|u\|_{C_{\varphi,\alpha}} \le R. \tag{3.4}$$

For $t \ge 0$, by (2.8) and (3.1), one can find

$$\begin{aligned} \|Qu(t)\|_\alpha \le\ & \|U(t)(\varphi(0) - G(0,\varphi))\|_\alpha + \|G(t,u_t)\|_\alpha \\ & + \Big\|\int_0^t (t-s)^{q-1}V(t-s)AG(s,u_s)ds\Big\|_\alpha + \Big\|\int_0^t (t-s)^{q-1}V(t-s)F(s,u_s)ds\Big\|_\alpha \\ \le\ & \|U(t)\|(\|A^\alpha\varphi(0)\| + \|A^{\alpha-1}\| \cdot \|AG(0,\varphi)\|) + \|A^{\alpha-1}\| \cdot \|AG(t,u_t)\| \\ & + \int_0^t (t-s)^{q-1}\|A^\alpha V(t-s)\| \cdot \|AG(s,u_s)\|ds \\ & + \int_0^t (t-s)^{q-1}\|A^\alpha V(t-s)\| \cdot \|F(s,u_s)\|ds \\ \le\ & (M + C_{\alpha-1}L_G M)\|\varphi\|_{\mathcal{B}_\alpha} + C_{\alpha-1}L_G\|u\|_{C_{\varphi,\alpha}} \end{aligned}$$

$$
\begin{aligned}
& +LM_\alpha \int_0^\infty \tau^{-\alpha}\xi_q(\tau)d\tau \int_0^\infty s^{-q\alpha}e^{\nu_0 s}ds \cdot \|u\|_{C_{\varphi,\alpha}} \\
& +\frac{M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\int_0^t (t-s)^{q-(q\alpha+1)}h_R(s)ds \\
\leq\ & (M+C_{\alpha-1}L_G M)\|\varphi\|_{\mathcal{B}_\alpha}+C_{\alpha-1}L_G R+\frac{L_G M_\alpha\Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}}R+\frac{\gamma M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}R \\
\leq\ & R.
\end{aligned}
$$

Hence, for every $t \in [-r,\infty)$, there is a positive constant R such that $Q(\overline{\Omega}_R)\subset\overline{\Omega}_R$.

Next, we show that the operator Q has a fixed point on $\overline{\Omega}_R$. From the definition of Q, it follows that $Qu\equiv\varphi$ for every $u\in\overline{\Omega}_R$ on $[-r,0]$. Hence, it only remains to consider the case under $t\geq 0$. For this purpose, we introduce the decomposition $Q=Q_1+Q_2$, where the operators Q_1 and Q_2 are defined on $\overline{\Omega}_R$, respectively, by

$$Q_1u(t)=U(t)\varphi(0)+\int_0^t (t-s)^{q-1}V(t-s)F(s,u_s)ds, \tag{3.5}$$

$$Q_2u(t)=G(t,u_t)-U(t)G(0,\varphi)-\int_0^t (t-s)^{q-1}V(t-s)AG(s,u_s)ds \tag{3.6}$$

for $t\geq 0$. In what follows, we will prove that Q_1 is a compact operator while Q_2 is a contraction.

To prove that Q_1 is a compact operator, firstly we show that Q_1 is continuous on $\overline{\Omega}_R$. Let $\{u^{(n)}\}\subset\overline{\Omega}_R$ and $u^{(n)}\to u$ in $\overline{\Omega}_R$ as $n\to\infty$. Hence, for every $t\geq 0$, $\|u_t^{(n)}-u_t\|_{\mathcal{B}_\alpha}\to 0$, as $n\to\infty$, and from the continuity of F, $\|F(t,u_t^{(n)})-F(t,u_t)\|\to 0$, as $n\to\infty$. Hence, the dominated convergence theorem ensure that

$$
\begin{aligned}
\|Q_1u_n(t)-Q_1u(t)\|_\alpha &\leq \int_0^t (t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot\|F(t,u_t^{(n)})-F(t,u_t)\|ds \\
&\to 0,\quad \text{as } n\to\infty,
\end{aligned}
$$

which implies that Q_1 is continuous on $\overline{\Omega}_R$.

Secondly, we verify that $Q_1(\overline{\Omega}_R)$ is equicontinuous on $[0,\infty)$. Without loss of generality, let $0\leq t_1<t_2$. For any $u\in\overline{\Omega}_R$, by (3.5), one can see

$$
\begin{aligned}
& \|Q_1u(t_2)-Q_1u(t_1)\|_\alpha \\
\leq\ & \|U(t_2)-U(t_1)\|\cdot\|A^\alpha u(0)\| \\
& +\int_0^{t_1}((t_1-s)^{q-1}-(t_2-s)^{q-1})\cdot\|A^\alpha V(t_2-s)\|\cdot\|F(s,u_s)\|ds \\
& +\int_0^{t_1}(t_1-s)^{q-1}\cdot\|A^\alpha(V(t_2-s)-V(t_1-s))\|\cdot\|F(s,u_s)\|ds \\
& +\int_{t_1}^{t_2}(t_2-s)^{q-1}\cdot\|A^\alpha V(t_2-s)\|\cdot\|F(s,u_s)\|ds \\
:=\ & J_1+J_2+J_3+J_4.
\end{aligned}
$$

Next, we check $\|J_i\|$ tend to 0 independently of $u\in\overline{\Omega}_{R_0}$ as $t_2-t_1\to 0, i=1,2,3,4$. By Lemma 2.2(i),

$$J_1 \leq \|U(t_2)-U(t_1)\|\cdot\|\varphi\|_{\mathcal{B}_\alpha}\to 0,\ \text{as } t_2-t_1\to 0.$$

By Lemma 2.2, we can obtain

$$J_2 \leq \frac{M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\int_0^{t_1}\Big((\frac{t_1-s}{t_2-s})^{q-1}-1\Big)\cdot\frac{h_R(s)}{(t_2-s)^{1+q\alpha-q}}ds$$

$$
\begin{aligned}
&\to \ 0 \ \text{ as } \ t_2-t_1\to 0,\\
J_4 \ &\le \ \frac{M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\int_{t_1}^{t_2}\frac{h_R(s)}{(t_2-s)^{1+q\alpha-q}}ds\\
&\to \ 0 \ \text{ as } \ t_2-t_1\to 0.
\end{aligned}
$$

If $t_1=0$ and $t_2>0$, then $J_3=0$. If $t_1>0$ and $\epsilon\in(0,t_1)$, by Lemma 2.5, then we get that

$$
\begin{aligned}
J_3 \ &\le \ \int_0^{t_1-\epsilon}(t_1-s)^{q-1}\cdot\|A^\alpha(V(t_2-s)-V(t_1-s))\|\cdot h_R(s)ds\\
&\quad+\int_{t_1-\epsilon}^{t_1}(t_1-s)^{q-1}\cdot\|A^\alpha(V(t_2-s)-V(t_1-s))\|\cdot h_R(s)ds\\
&\le \ \frac{M_{\alpha+1}\Gamma(1-\alpha)}{\alpha\Gamma(q-q\alpha)}\int_0^{t_1-\epsilon}\Big(1-(\frac{t_2-s}{t_1-s})^{-q\alpha}\Big)\frac{h_R(s)}{(t_1-s)^{1+q\alpha-q}}ds\\
&\quad+\frac{2M_\alpha\Gamma(1-\alpha)}{\Gamma(q-q\alpha)}\int_{t_1-\epsilon}^{t_1}\frac{h_R(s)}{(t_1-s)^{1+q\alpha-q}}ds\\
&\to \ 0 \ \text{ as } \ t_2-t_1\to 0,\epsilon\to 0.
\end{aligned}
$$

As a result, $\|Q_1u(t_2)-Q_1u(t_1)\|$ tends to 0 independently of $u\in\overline{\Omega}_{R_0}$ as $t_2-t_1\to 0$, which means that $Q_1(\overline{\Omega}_R)$ is equicontinuous on is equicontinuous on $[0,\infty)$.

Thirdly, we show that $Q_1(\overline{\Omega}_R)(t)$ is relatively compact in X_α for all $t\in[0,\infty)$. Clear, $Q_1(\overline{\Omega}_R)(0)=\{Q_1u(0)|u\in\overline{\Omega}_R\}=\{u(0)\}$ is compact. For $t>0$, we define a set $Q_{\varepsilon,\delta}(\overline{\Omega}_R)(t)$ by

$$
Q_{\varepsilon,\delta}(\overline{\Omega}_R)(t):=\{(Q^{(0)}_{\varepsilon,\delta}u)(t)|\ u\in\overline{\Omega}_R,\ 0<\varepsilon<t,\delta>0\}, \tag{3.7}
$$

where

$$
\begin{aligned}
Q_{\varepsilon,\delta}u(t) \ &= \ U(t)u(0)+q\int_0^{t-\varepsilon}\int_\delta^\infty\tau(t-s)^{q-1}\xi_q(\tau)T((t-s)^q\tau)F(s,u_s)d\tau ds\\
&= \ U(t)u(0)\\
&\quad+qT(\varepsilon^q\delta)\int_0^{t-\varepsilon}\int_\delta^\infty\tau(t-s)^{q-1}\xi_q(\tau)T((t-s)^q\tau-\varepsilon^q\delta)F(s,u_s)d\tau ds.
\end{aligned}
$$

From the compactness of $U(t)$ and $T(\varepsilon^q\delta)$, we know that $(Q_{\varepsilon,\delta}\overline{\Omega}_R)(t)$ is relatively compact in X_α. Moreover, by Lemma 2.2 we can find

$$
\begin{aligned}
&\|Qu(t)-Q_{\varepsilon,\delta}u(t)\|_\alpha\\
\le \ &\Big\|q\int_0^t\int_0^\delta\tau(t-s)^{q-1}\xi_q(\tau)T((t-s)^q\tau)F(s,u_s)d\tau ds\Big\|_\alpha\\
&+\Big\|q\int_{t-\varepsilon}^t\int_\delta^\infty\tau(t-s)^{q-1}\xi_q(\tau)T((t-s)^q\tau)F(s,u_s)d\tau ds\Big\|_\alpha\\
\le \ &\int_0^t\int_0^\delta\tau(t-s)^{q-1}\xi_q(\tau)\cdot\|A^\alpha T((t-s)^q\tau)\|\cdot h_R(s)d\tau ds\\
&+\int_{t-\varepsilon}^t\int_\delta^\infty\tau(t-s)^{q-1}\xi_q(\tau)\cdot\|A^\alpha T((t-s)^q\tau)\|\cdot h_R(s)d\tau ds\\
\le \ &M_\alpha\int_0^t\int_0^\delta\frac{\tau^{1-\alpha}\xi_q(\tau)h_R(s)}{(t-s)^{q\alpha+1-q}}d\tau ds+M_\alpha\int_{t-\varepsilon}^t\int_\delta^\infty\frac{\tau^{1-\alpha}\xi_q(\tau)h_R(s)}{(t-s)^{q\alpha+1-q}}d\tau ds
\end{aligned}
$$

$$
\begin{aligned}
&\le M_\alpha \int_0^t \frac{h_R(s)}{(t-s)^{q\alpha+1-q}}ds \int_0^\delta \tau^{1-\alpha}\xi_q(\tau)d\tau + M_\alpha \int_{t-\varepsilon}^t \frac{h_R(s)}{(t-s)^{q\alpha+1-q}}ds \int_0^\infty \tau^{1-\alpha}\xi_q(\tau)d\tau \\
&\to 0 \text{ as } \varepsilon \to 0, \delta \to 0,
\end{aligned}
$$

which implies that the set $Q_1(\overline{\Omega}_R)(t)$ is relatively compact in X_α for $t>0$. Therefore, $Q_1(\overline{\Omega}_R)(t)$ is relatively compact in X_α for all $t \in [0,\infty)$.

Finally, we prove that for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that $\chi(Q_1(\overline{\Omega}_R)|_{[N,\infty)}) < \varepsilon$, where $\chi(\cdot)$ is the Hausdorff measure of non-compactness. By the condition (H2), there exists $N > 0$ such that

$$
\int_N^\infty (t-s)^{q-(1+q\alpha)}h(s)ds < \frac{\Gamma(q(1-\alpha))\varepsilon}{M_\alpha\Gamma(1-\alpha)}. \tag{3.8}
$$

Set $\Pi := \{\Phi_u | u \in \overline{\Omega}_R\} \subset C_b([N,\infty), X_\alpha)$, where

$$
\Phi_u(t) = U(t)u(0) + \int_0^N (t-s)^{q-1}V(t-s)F(s,u_s)ds.
$$

By (H2) and the compactness of $U(t)$ and $V(t)$, it is readily derived that Π is relatively compact. Moreover, by (3.8), one can find that

$$
\begin{aligned}
\|Q_1u(t) - \Phi_u(t)\|_\alpha &\le \int_N^t (t-s)^{q-1}\|A^\alpha V(t-s)\| \cdot \|F(s,u_s)\|ds \\
&\le \frac{M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\int_N^\infty (t-s)^{q-(1+q\alpha)}h(s)ds \\
&< \varepsilon
\end{aligned}
$$

for any $u \in \overline{\Omega}_R$ and $t > N$. Hence, $\chi(Q_1(\overline{\Omega}_R)|_{[N,\infty)}) < \varepsilon$.

In summary, Lemma 2.3 guarantees that Q_1 is a compact operator. Now, it remains to prove that Q_2 is a contraction. Let $u, v \in \overline{\Omega}_R$, by (2.8),(3.6) and (H3), one can see

$$
\begin{aligned}
&\|Q_2u(t) - Q_2v(t)\|_\alpha \\
\le\ & \|G(t,u_t) - G(t,v_t)\|_\alpha \\
&+ \Big\| \int_0^t (t-s)^{q-1}V(t-s)A(G(s,u_s) - G(s,v_s))ds \Big\|_\alpha \\
\le\ & \|A^{\alpha-1}\| \cdot \|A(G(t,u_t) - G(t,v_t))\| \\
&+ \int_0^t (t-s)^{q-1}\|A^\alpha V(t-s)\| \cdot \|A(G(s,u_s) - G(s,v_s))\|ds \\
\le\ & C_{\alpha-1}L_G\|u_t - v_t\|_{\mathcal{B}_\alpha} \\
&+ L_GM_\alpha \int_0^t (t-s)^{q-(q\alpha+1)} \int_0^\infty \tau^{1-\alpha}\xi_q(\tau)e^{\nu_0(t-s)^q\tau} \cdot \|u_s - v_s\|_{\mathcal{B}_\alpha} d\tau ds \\
\le\ & C_{\alpha-1}L_G\|u_t - v_t\|_{\mathcal{B}_\alpha} + L_GM_\alpha \int_0^\infty \tau^{-\alpha}\xi_q(\tau)d\tau \int_0^\infty s^{-q\alpha}e^{\nu_0 s}ds \cdot \|u-v\|_{C_{\varphi,\alpha}} \\
\le\ & \Big(C_{\alpha-1}L_G + \frac{L_GM_\alpha\Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}}\Big) \cdot \|u-v\|_{C_{\varphi,\alpha}},
\end{aligned}
$$

which implies that

$$
\|Q_2u - Q_2v\|_{C_{\varphi,\alpha}} \le \Big(C_{\alpha-1}L_G + \frac{L_GM_\alpha\Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}}\Big) \cdot \|u-v\|_{C_{\varphi,\alpha}}. \tag{3.9}
$$

Thus, from (H4), we can deduce Q_2 is a contraction.

Next, we show that Q is $SAP_{\omega,\varphi}(X_\alpha)$-valued, where we identify the elements $u\in SAP_{\omega,\varphi}(X_\alpha)$ with its extension to $[-r,\infty)$ given by $u(t)=\varphi(t)$ for $t\in[-r,0]$. Obviously, $SAP_{\omega,\varphi}(X_\alpha)$ is a closed subspace of $C_\varphi(X_\alpha)$ and $u|_{[0,\infty)}\in SAP_\omega(X_\alpha)$ implies that the function $t\mapsto u_t$ belongs to $SAP_\omega(\mathcal{B}_\alpha)$.

Choosing $u\in SAP_{\omega,\varphi}(X_\alpha)$, from the definition of Q, it follows that $\mathcal{Q}u|_{[-r,0]}\in\mathcal{B}_\alpha$. Thus it suffices to show that the function

$$\begin{aligned} f:t &\mapsto U(t)(\varphi(0)-G(0,\varphi))+G(t,u_t)-\int_0^t(t-s)^{q-1}V(t-s)AG(s,u_s)ds\\ &\quad+\int_0^t(t-s)^{q-1}V(t-s)F(s,u_s)ds\\ &\in SAP_\omega(X_\alpha). \end{aligned}$$

To this end, we consider

$$\begin{aligned} &f(t+\omega)-f(t)\\ =\;&(U(t+\omega)-U(t))\cdot(\varphi(0)-G(0,\varphi))+(G(t+\omega,u_{t+\omega})-G(t,u_t))\\ &+\int_0^\omega(t+\omega-s)^{q-1}V(t+\omega-s)AG(s,u_s)ds\\ &+\int_0^t(t-s)^{q-1}V(t-s)\Big(AG(s+\omega,u_{s+\omega})-AG(s,u_s)\Big)ds\\ &+\int_0^\omega(t+\omega-s)^{q-1}V(t+\omega-s)F(s,u_s)ds\\ &+\int_0^t(t-s)^{q-1}V(t-s)\Big(F(s+\omega,u_{s+\omega})-F(s,u_s)\Big)ds\\ :=\;&I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6(t). \end{aligned}$$

Then

$$\|f(t+\omega)-f(t)\|_\alpha\le\sum_{i=1}^6\|I_i(t)\|_\alpha. \tag{3.10}$$

First of all, from the exponential stability of semigroup $T(t)(t\ge0)$, it follows that $\|T(t)\|\le Me^{\nu_0 t}$ with $\nu_0<0$. Thus, combing this with $U(t)=\int_0^\infty\xi_q(\tau)T(t^q\tau)d\tau$ and $\int_0^\infty\xi_q(\tau)d\tau=1$, one can deduce that

$$\begin{aligned} \|I_1(t)\| &\le 2\|U(t)\|\cdot\|A^\alpha\varphi(0)-A^\alpha G(0,\varphi)\|\\ &\to 0,\quad\text{as } t\to\infty. \end{aligned}$$

Since $u|_{[0,\infty)}\in SAP_\omega(X_\alpha)$ and $u_t\in SAP_\omega(\mathcal{B}_\alpha)$ for all $t\ge0$, hence, there exist positive constants R and $t_{\varepsilon,1}$, such that $\|u_t\|_{\mathcal{B}_\alpha}\le R$ and $\|u_{t+\omega}-u_t\|_{\mathcal{B}}\le\varepsilon$ for any $t\ge t_{\varepsilon,1}$ and $\varepsilon>0$. Thus, from the continuity of F and G, one can see for $t\ge t_{\varepsilon,1}$

$$\|F(t,u_{t+\omega})-F(t,u_t)\|\le\varepsilon, \tag{3.11}$$

$$\|AG(t,u_{t+\omega})-AG(t,u_t)\|\le\varepsilon. \tag{3.12}$$

Furthermore, from the condition (H1), it follows that there exits a positive constant $t_{\varepsilon,2}$ sufficiently large such that for $t\ge t_{\varepsilon,2}$,

$$\|F(t+\omega,u_{t+\omega})-F(t,u_{t+\omega})\|\le\varepsilon, \tag{3.13}$$

$$\|AG(t+\omega,u_{t+\omega})-AG(t,u_{t+\omega})\|\le\varepsilon. \tag{3.14}$$

Obviously, for $t>t_\varepsilon:=\max\{t_{\varepsilon,1},t_{\varepsilon,2}\}$,

$$\|I_2(t)\|_\alpha\le\|AG(t+\omega,u_{t+\omega})-AG(t,u_{t+\omega})\|+\|AG(t,u_{t+\omega})-AG(t,u_t)\|$$

$$< \quad \varepsilon.$$

Noting that $t+\omega-s \geq \frac{t+\omega}{\omega}(\omega-s)$, the conditions (H2), (H3), and Lemma 2.2 (iv), we have

$$\begin{aligned}
\|I_3(t)\|_\alpha &\leq \int_0^\omega (t+\omega-s)^{q-1}\|A^\alpha V(t+\omega-s)\|\cdot\|AG(s,u_s)\|ds \\
&\leq \frac{L_G\|u\|_{C_{\varphi,\alpha}}M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\int_0^\omega (t+\omega-s)^{q-1-q\alpha}ds \\
&\leq \frac{L_G\|u\|_{C_{\varphi,\alpha}}M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\times\Big(\frac{\omega}{t+\omega}\Big)^{1-(q-q\alpha)}\int_0^\omega (\omega-s)^{q-q\alpha-1}ds \\
&= \frac{L_G\|u\|_{C_{\varphi,\alpha}}M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\times\frac{\omega}{q-q\alpha}\times\frac{1}{(t+\omega)^{1-(q-q\alpha)}} \\
&\to 0, \quad \text{as} \quad t\to\infty,
\end{aligned}$$

$$\begin{aligned}
\|I_5(t)\|_\alpha &\leq \int_0^\omega (t+\omega-s)^{q-1}\|A^\alpha V(t+\omega-s)\|\cdot\|F(s,u_s)\|ds \\
&\leq \frac{M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\int_0^\omega (t+\omega-s)^{q-1-q\alpha}h_R(s)ds \\
&\leq \frac{M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\times\Big(\frac{\omega}{t+\omega}\Big)^{1-(q-q\alpha)}\int_0^\omega (\omega-s)^{q-q\alpha-1}h_R(s)ds \\
&\to 0, \quad \text{as} \quad t\to\infty.
\end{aligned}$$

From the fact $t-s \geq \frac{t}{t_\varepsilon}(t_\varepsilon - s)$, the conditions (H2), (H3) and (2.8), (3.11)-(3.14), we have

$$\begin{aligned}
\|I_4\|_\alpha &\leq \int_0^t (t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot\|AG(s+\omega,u_{s+\omega})-AG(s,u_s)\|ds \\
&\leq \int_0^t (t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot\|AG(s+\omega,u_{s+\omega})-AG(s,u_{s+\omega})\|ds \\
&\quad +\int_0^t (t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot\|AG(s,u_{s+\omega})-AG(s,u_s)\|ds \\
&\leq \int_0^{t_\varepsilon} (t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot(\|AG(s+\omega,u_{s+\omega})\|+\|AG(s,u_{s+\omega})\|)ds \\
&\quad +\int_{t_\varepsilon}^t (t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot\|AG(s+\omega,u_{s+\omega})-AG(s,u_{s+\omega})\|ds \\
&\quad \int_0^{t_\varepsilon} (t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot(\|AG(s,u_{s+\omega})\|+\|AG(s,u_s)\|)ds \\
&\quad +\int_{t_\varepsilon}^t (t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot\|AG(s,u_{s+\omega})-AG(s,u_s)\|ds \\
&\leq \frac{4L_G\|u\|_{C_{\varphi,\alpha}}M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\int_0^{t_\varepsilon}(t-s)^{q-1-q\alpha}ds \\
&\quad +2\varepsilon M_\alpha\int_{t_\varepsilon}^t (t-s)^{q-(q\alpha+1)}\int_0^\infty \tau^{1-\alpha}\xi_q(\tau)e^{\nu_0(t-s)^q\tau}d\tau ds \\
&\leq \frac{4L_G\|u\|_{C_{\varphi,\alpha}}M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}\Big(\frac{t_\varepsilon}{t}\Big)^{1+q\alpha-q}\int_0^{t_\varepsilon}(t_\varepsilon-s)^{q-1-q\alpha}ds
\end{aligned}$$

$$\begin{aligned}
& +2\varepsilon M_\alpha \int_0^t (t-s)^{q-(q\alpha+1)} \int_0^\infty \tau^{1-\alpha}\xi_q(\tau)e^{\nu_0(t-s)^q\tau}d\tau ds \\
\leq\ & \frac{4L_G\|u\|_{C_{\varphi,\alpha}}M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \times \frac{t_\varepsilon}{q-q\alpha} \times \frac{1}{t^{1+q\alpha-q}} + \frac{2\varepsilon M_\alpha\Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}} \\
\to\ & 0, \text{ as } t\to\infty, \varepsilon\to 0,
\end{aligned}$$

$$\begin{aligned}
\|I_6\|_\alpha \leq\ & \int_0^t (t-s)^{q-1}\|A^\alpha V(t-s)\| \cdot \|F(s+\omega,u_{s+\omega}) - F(s,u_s)\|ds \\
\leq\ & \int_0^{t_\varepsilon} (t-s)^{q-1}\|A^\alpha V(t-s)\| \cdot (\|F(s+\omega,u_{s+\omega})\| + \|F(s,u_{s+\omega})\|)ds \\
& + \int_{t_\varepsilon}^t (t-s)^{q-1}\|A^\alpha V(t-s)\| \cdot \|F(s+\omega,u_{s+\omega}) - F(s,u_{s+\omega})\|ds \\
& \int_0^{t_\varepsilon} (t-s)^{q-1}\|A^\alpha V(t-s)\| \cdot (\|F(s,u_{s+\omega})\| + \|F(s,u_s)\|)ds \\
& + \int_{t_\varepsilon}^t (t-s)^{q-1}\|A^\alpha V(t-s)\| \cdot \|F(s,u_{s+\omega}) - F(s,u_s)\|ds \\
\leq\ & \frac{4M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \int_0^{t_\varepsilon} (t-s)^{q-1-q\alpha}h_R(s)ds \\
& +2\varepsilon M_\alpha \int_{t_\varepsilon}^t (t-s)^{q-(q\alpha+1)} \int_0^\infty \tau^{1-\alpha}\xi_q(\tau)e^{\nu_0(t-s)^q\tau}d\tau ds \\
\leq\ & \frac{4M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \Big(\frac{t_\varepsilon}{t}\Big)^{1+q\alpha-q} \int_0^{t_\varepsilon} (t_\varepsilon-s)^{q-1-q\alpha}h_R(s)ds \\
& +2\varepsilon M_\alpha \int_0^t (t-s)^{q-(q\alpha+1)} \int_0^\infty \tau^{1-\alpha}\xi_q(\tau)e^{\nu_0(t-s)^q\tau}d\tau ds \\
\leq\ & \frac{4M_\alpha\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \Big(\frac{t_\varepsilon}{t}\Big)^{1+q\alpha-q} \int_0^{t_\varepsilon} (t_\varepsilon-s)^{q-1-q\alpha}h_R(s)ds + \frac{2\varepsilon M_\alpha\Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}} \\
\to\ & 0, \text{ as } t\to\infty, \varepsilon\to 0.
\end{aligned}$$

Thus, according to the above estimations for $I_i (i = 1, 2, \cdots, 6)$, we can obtain

$$\begin{aligned}
f: t \mapsto\ & U(t)(\varphi(0) - G(0,\varphi)) + G(t,u_t) - \int_0^t (t-s)^{q-1}V(t-s)AG(s,u_s)ds \\
& + \int_0^t (t-s)^{q-1}V(t-s)F(s,u_s)ds \\
\in\ & SAP_\omega(X_\alpha).
\end{aligned}$$

Combining this with the definition of Q, we know that Q is $SAP_{\omega,\varphi}(X_\alpha)$-valued.

From above argument, one can deduce that Q is a condensing operator from $\overline{\Omega}_R \cap SAP_{\omega,\varphi}(X_\alpha)$ into $\overline{\Omega}_R \cap SAP_{\omega,\varphi}(X_\alpha)$. It follows from the famous Sadovskii fixed point theorem [24] that Q has at least one fixed point $u \in \overline{\Omega}_R \cap SAP_{\omega,\varphi}(X_\alpha)$, which implies that u is an S-asymptotic ω-periodic α-mild solution. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $-A$ generate an exponentially stable analytic semigroup $T(t)(t \geq 0)$ in Banach space X. For $\alpha \in [0, 1)$, assume that $\varphi \in \mathcal{B}_\alpha$, $G : \mathbb{R}^+ \times \mathcal{B}_\alpha \to X_1$ and $F : \mathbb{R}^+ \times \mathcal{B}_\alpha \to X$ are bounded and continuous functions. If the conditions (H3) and*

(H5) there exists a constant $L_F > 0$ such that

$$\|F(t,\phi_2) - F(t,\phi_1)\| \le L_F \|\phi_2 - \phi_1\|_{\mathcal{B}_\alpha},$$

for any $t \ge 0$ and $\phi_1, \phi_2 \in \mathcal{B}_\alpha$,

(H6) $C_{\alpha-1}L_G + \frac{(L_G+L_F)M_\alpha\Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}} < 1$

hold, then the problem (1.1) has a unique S-asymptotic ω-periodic α-mild solution.

Proof Let the operator Q and the space $SAP_{\omega,\varphi}(X_\alpha)$ be as in the proof of the Theorem 3.1. According to the proof of Theorem 3.1, we can conclude that the operator $Q : C_\varphi(X_\alpha) \to C_\varphi(X_\alpha)$ is well defined, and the S-asymptotic ω-periodic α-mild solution of the problem (1.1) is the fixed point of Q in $SAP_{\omega,\varphi}(X_\alpha)$. In what follows, we will show that $Q : SAP_{\omega,\varphi}(X_\alpha) \to SAP_{\omega,\varphi}(X_\alpha)$ is a contraction mapping.

Choosing $u \in SAP_{\omega,\varphi}(X_\alpha)$, from the definition of Q, it follows that $\mathcal{Q}u|_{[-r,0]} \in \mathcal{B}_\alpha$. Next, we show that the function

$$\begin{aligned} f : t \quad &\mapsto \quad U(t)(\varphi(0) - G(0,\varphi)) + G(t,u_t) - g(t) + h(t) \\ &\in \quad SAP_\omega(X_\alpha). \end{aligned}$$

where $g(t) = \int_0^t (t-s)^{q-1} V(t-s) AG(s,u_s)ds$ and $h(t) = \int_0^t (t-s)^{q-1} V(t-s) F(s,u_s)ds$. From the proof of Theorem 3.1, we know that $U(t)(\varphi(0) - G(0,\varphi)) \in SAP_\omega(X_\alpha)$. Since $u|_{[0,\infty)} \in SAP_\omega(X_\alpha)$ and $u_t \in SAP_\omega(\mathcal{B}_\alpha)$ for all $t \ge 0$, taking into account the asymptotic uniform continuity of F and G on bounded sets, it follows from Lemma 2.7, that the functions $t \to F(t,u_t)$ and $t \to G(t,u_t)$ belong to $SAP_\omega(X)$ and $SAP_\omega(X_1) \hookrightarrow SAP_\omega(X_\alpha)$, respectively. Therefore, there exist constants M_F and M_G such that

$$\sup_{t\in[0,\infty)} \|F(t,u_t)\| \le M_F, \quad \sup_{t\in[0,\infty)} \|AG(t,u_t)\| \le M_G. \tag{3.15}$$

Thus, by (2.8), we have the following estimate

$$\begin{aligned} \|g(t)\|_\alpha &= \Big\| \int_0^t (t-s)^{q-1} V(t-s) AG(s,u_s)ds \Big\|_\alpha \\ &\le \int_0^t (t-s)^{q-1} \|A^\alpha V(t-s)\| \cdot \|AG(s,u_s)\| ds \\ &\le M_G M_\alpha \int_0^t (t-s)^{q-(q\alpha+1)} \int_0^\infty \tau^{1-\alpha} \xi_q(\tau) e^{\nu_0 (t-s)^q \tau} d\tau ds \\ &\le M_G M_\alpha \int_0^\infty \tau^{-\alpha} \xi_q(\tau) d\tau \int_0^\infty s^{-q\alpha} e^{\nu_0 s} ds \\ &\le \frac{M_G M_\alpha \Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}}, \end{aligned}$$

and, similarly

$$\|h(t)\|_\alpha \le \frac{M_F M_\alpha \Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}}.$$

Thus, we get $g(t), h(t) \in C_b([0,\infty), X_\alpha)$, which implies that

$$\Big\| \int_a^\infty (t-s)^{q-1} V(t-s) AG(s,u_s)ds \Big\|_\alpha \to 0, \ \Big\| \int_a^\infty (t-s)^{q-1} V(t-s) F(s,u_s)ds \Big\|_\alpha \to 0$$

as $a \to \infty$, uniformly for $t \ge a$.

Now, fixing a, we can see that the sets $\{G(s,u_s) | s \in [0,a]\}$ and $\{F(s,u_s) | s \in [0,a]\}$ are compact, and

$$(t+\omega-s)^{q-1} V(t+\omega-s) AG(s,u_s) \to (t-s)^{q-1} V(t-s) AG(s,u_s), \quad t \to \infty,$$

$$(t+\omega-s)^{q-1}V(t+\omega-s)F(s,u_s)\to(t-s)^{q-1}V(t-s)F(s,u_s),\quad t\to\infty,$$
uniformly for $s\in[0,a]$. Combining the properties above with the decomposition below

$$\begin{aligned}&g(t+\omega)-g(t)\\ =\;&\int_0^a((t+\omega-s)^{q-1}V(t+\omega-s)-(t-s)^{q-1}V(t-s))AG(s,u_s)ds\\ &+\int_a^{t+\omega}(t+\omega-s)^{q-1}V(t+\omega-s)AG(s,u_s)ds\\ &-\int_a^t(t-s)^{q-1}V(t-s)AG(s,u_s)ds,\end{aligned}$$

and

$$\begin{aligned}&h(t+\omega)-h(t)\\ =\;&\int_0^a((t+\omega-s)^{q-1}V(t+\omega-s)-(t-s)^{q-1}V(t-s))F(s,u_s)ds\\ &+\int_a^{t+\omega}(t+\omega-s)^{q-1}V(t+\omega-s)F(s,u_s)ds\\ &-\int_a^t(t-s)^{q-1}V(t-s)F(s,u_s)ds,\end{aligned}$$

we can easily obtain that $\|g(t+\omega)-g(t)\|_\alpha\to 0$, $\|h(t+\omega)-h(t)\|_\alpha\to 0$ as $t\to\infty$, which imply that $g(t),h(t)\in SAP_\omega(X_\alpha)$.

Therefore, we can draw $Q:SAP_{\omega,\varphi}(X_\alpha)\to SAP_{\omega,\varphi}(X_\alpha)$ from (3.1) and the above discussion. For any $u,v\in SAP_{\omega,\varphi}(X_\alpha)$, it follows from (3.1) that $Qu(t)=Qv(t)=\varphi(t)$ for $t\in[-r,0]$. Moreover, under the conditions (H3) and (H5), Lemma 2.2 yields

$$\begin{aligned}&\|Qu(t)-Qv(t)\|_\alpha\\ \le\;&\|A^{\alpha-1}\|\cdot\|AG(t,u_t)-AG(t,v_t)\|\\ &+\int_0^t(t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot\|AG(s,u_s)-AG(s,v_s)\|ds\\ &+\int_0^t(t-s)^{q-1}\|A^\alpha V(t-s)\|\cdot\|F(s,u_s)-F(s,v_s)\|ds\\ \le\;&C_{\alpha-1}L_G\|u_t-v_t\|_{\mathcal{B}_\alpha}+L_GM_\alpha\int_0^\infty\tau^{-\alpha}\xi_q(\tau)d\tau\int_0^\infty s^{-q\alpha}e^{\nu_0 s}ds\|u_t-v_t\|_{\mathcal{B}_\alpha}\\ &+L_FM_\alpha\int_0^\infty\tau^{-\alpha}\xi_q(\tau)d\tau\int_0^\infty s^{-q\alpha}e^{\nu_0 s}ds\|u_t-v_t\|_{\mathcal{B}_\alpha}\\ \le\;&\Big(C_{\alpha-1}L_G+\frac{(L_G+L_F)M_\alpha\Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}}\Big)\|u-v\|_{C_{\varphi,\alpha}}.\end{aligned}$$

Thus, by the condition (H6),

$$\|Qu-Qv\|_{C_{\varphi,\alpha}}\le\Big(C_{\alpha-1}L_G+\frac{(L_G+L_F)M_\alpha\Gamma(1-\alpha)}{|\nu_0|^{1-q\alpha}}\Big)\|u-v\|_{C_{\varphi,\alpha}}<\|u-v\|_{C_{\varphi,\alpha}}$$

for any $u,v\in SAP_{\omega,\varphi}(X_\alpha)$. Therefore, $Q:SAP_{\omega,\varphi}(X_\alpha)\to SAP_{\omega,\varphi}(X_\alpha)$ is a contraction mapping. The contraction mapping principle guarantees that Q has a unique fixed point $u\in SAP_{\omega,\varphi}(X_\alpha)$ which is a unique S-asymptotic ω-periodic α-mild solution of the problem (1.1). This completes the proof of Theorem 3.2. □

§4 Application

In this section, we present one example, which indicates how our abstract results can be applied to concrete problems.

Consider the following parabolic boundary value problem with delays

$$\begin{cases} \dfrac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\Big(u(\xi,t)-e^{-t}\displaystyle\int_0^1 g(\xi,\eta,u(\eta,t+\tau))d\eta\Big)-\dfrac{\partial^2}{\partial\xi^2}u(\xi,t) \\ \qquad\qquad = e^{-t}f(\xi,u(\xi,t+\tau)), \quad \xi\in[0,1],\ t\in\mathbb{R}^+,\tau\in[-r,0], \\ u(0,t)=u(1,t)=0, \quad t\in\mathbb{R}^+, \\ u(\xi,\tau)=\varphi(\tau)(\xi), \quad \tau\in[-r,0],\xi\in[0,1], \end{cases} \tag{4.1}$$

where $\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}$ is a Caputo fractional partial derivative of order $\frac{1}{2}$, $r>0$ is a constant, $\varphi\in C([-r,0],L^2([0,1],\mathbb{R}))$, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$ and $g\in C([0,1]\times[0,1]\times\mathbb{R},\mathbb{R})$.

To treat this system in the abstract form (1.1), we choose the space $X=L^2([0,1],\mathbb{R})$ endowed with the L^2-norm $\|\cdot\|_{L^2}$. Define operator $A:D(A)\subset X\to X$ by $Au=-\frac{\partial^2 u}{\partial\xi^2}$ with

$$D(A)=\{u\in X|u,u' \text{ are absolutely continuous}, u''\in X, u(0)=u(1)=0\}.$$

As everyone knows that $-A$ generates a compact and exponentially stable analytic semigroup $T(t)(t\geq 0)$ in X. It follows from $0\in\rho(A)$ that the fractional power of A is well defined. In addition, A has a discrete spectrum with eigenvalues of the form $n^2\pi^2$, $n\in\mathbb{N}$, and the associated normalized eigenfunctions are given by $e_n(\xi)=\sqrt{2}\sin(n\pi\xi)$ for $x\in[0,1]$. Thus, the associated semigroup $T(t)(t\geq 0)$ is explicitly given by

$$T(t)u=\sum_{n=1}^{\infty}e^{-n^2\pi^2 t}(u,e_n)e_n, \quad t\geq 0, u\in X,$$

where $(\cdot,\cdot)$ is an inner product on X, and it is not difficult to verify that $\|T(t)\|\leq e^{-\pi^2 t}$ for all $t\geq 0$. Hence, we take $M=1$, $M_{\frac{1}{2}}=\Gamma(\frac{1}{2})=\sqrt{\pi}$. Moreover, the following results are also well known.

(e1) If $u\in D(A)$, then

$$Au=\sum_{n=1}^{\infty}n^2\pi^2(u,e_n)e_n.$$

(e2) For every $u\in X$,

$$A^{-\frac{1}{2}}u=\sum_{n=1}^{\infty}\frac{1}{n}(u,e_n)e_n.$$

(e3) For every $u\in D(A^{\frac{1}{2}}):=\{u\in X|\ \sum_{n=1}^{\infty}n(u,e_n)e_n\in X\}$,

$$A^{\frac{1}{2}}u=\sum_{n=1}^{\infty}n(u,e_n)e_n,$$

and $\|A^{-\frac{1}{2}}\|=1$.

The proof of the following lemma can be found in [28].

Lemma 4.1. *If $v\in D(A^{\frac{1}{2}})$, then v is absolutely continuous with $v'\in X$ and $\|v'\|_{L^2}=\|A^{\frac{1}{2}}v\|_{L^2}$.*

According to Lemma 4.1, we define the Banach space $X_{\frac{1}{2}}:=(D(A^{\frac{1}{2}}),\|\cdot\|_{\frac{1}{2}})$, where $\|v\|_{\frac{1}{2}}=\|A^{\frac{1}{2}}v\|_{L^2}$ for all $v\in X_{\frac{1}{2}}$. Under the above assumptions we discuss the existence and uniqueness of time S-asymptotic 1-periodic α-mild solutions of the system (4.1).

Theorem 4.2. If the following conditions

(F1) there exists a positive constant a such that for every $(\xi, v) \in [0,1] \times \mathbb{R}$

$$|f(\xi, v)| \leq a|v|,$$

(F2) there is a positive function $b : [0,1] \times [0,1] \to \mathbb{R}^+$ such that

$$|g(\xi, \eta, v_2) - g(\xi, \eta, v_1)| \leq b(\xi, \eta)|v_2 - v_1|$$

for all $\xi, \eta \in [0,1]$ and $v_1, v_2 \in \mathbb{R}$, moreover, $(\xi, \eta) \mapsto \frac{\partial^2}{\partial \xi^2} b(\xi, \eta)$ is well defined and measurable with

$$l^2 := \int_0^1 \int_0^1 \Big(\frac{\partial^2}{\partial \xi^2} b(\xi, \eta)\Big)^2 d\eta d\xi < \infty,$$

(F3) $l + l\pi^{-\frac{1}{2}} + a\pi < 1$,

hold, then the system (4.1) has at least one time S-asymptotic 1-periodic $\frac{1}{2}$-mild solution.

Proof Denote $X_1 = D(A)$, $\mathcal{B}_{\frac{1}{2}} = C([-r,0], X_{\frac{1}{2}})$. For $\xi \in [0,1]$, $\tau \in [-r,0]$, $t \geq 0$, we set $u(t)(\xi) = u(\xi, t)$, $u_t(\tau)(\xi) = u(\xi, t+\tau)$, $\varphi(\tau)(\xi) = \varphi(\xi, \tau)$ and

$$F(t, \phi)(\xi) = e^{-t} f(\xi, \phi(\xi)),$$

$$G(t, \phi)(\xi) = e^{-t} \int_0^1 g(\xi, \eta, \phi(\eta)) d\eta.$$

Therefore, the system (4.1) can be reformulated as the abstract (1.1).

By the definition of F and (F1), we see that $F : \mathbb{R} \times \mathcal{B}_{\frac{1}{2}} \to X$ is continuous and

$$\begin{aligned} \|F(t, \phi)\|_{L^2} &= \Big(\int_0^1 \Big(e^{-t} f(t, \phi(\xi, \tau))\Big)^2 d\xi\Big)^{\frac{1}{2}} \\ &\leq e^{-t}\Big(\int_0^1 a^2 |\phi(\xi, \tau)|^2 d\xi\Big)^{\frac{1}{2}} \\ &\leq a e^{-t} \max_{\tau \in [-r,0]} \|\phi(\tau)\|_{L^2} \leq a e^{-t} \|\phi\|_{\mathcal{B}_{\frac{1}{2}}} \end{aligned} \tag{4.2}$$

for each $\phi \in \mathcal{B}_{\frac{1}{2}}$. Thus, from the following inequality

$$\int_0^t (t-s)^{-\frac{3}{4}} e^{-s} ds \leq \Gamma(\frac{1}{4}), \tag{4.3}$$

we can deduce there exists constant $\gamma = a\Gamma(\frac{1}{4})$ such that the condition (H2) in Theorem 3.1 holds. By the definition of G and assumption (F2), we see that $G : \mathbb{R} \times \mathcal{B}_{\frac{1}{2}} \to X$ is continuous and

$$\begin{aligned} & \|AG(t, \phi) - AG(t, \psi)\|_{L^2}^2 \\ \leq & \int_0^1 \int_0^1 \Big(\frac{\partial^2}{\partial x^2} e^{-t} b(\xi, \eta) \cdot \Big|\phi(\eta, \tau) - \psi(\eta, \tau)\Big|\Big)^2 dy dx \\ \leq & e^{-2t} \int_0^1 \int_0^1 \Big(\frac{\partial^2}{\partial x^2} b(\xi, \eta)\Big)^2 dy dx \cdot \|\phi - \psi\|_{L^2}^2 \\ \leq & l^2 \|\phi - \psi\|_{\mathcal{B}_{\frac{1}{2}}}^2, \end{aligned} \tag{4.4}$$

for each $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{B}_{\frac{1}{2}}$. Thus, the condition (H3) in Theorem 3.1 holds. Moreover, by (4.2) and (4.4), we can find

$$\|F(t+1, \phi) - F(t, \phi)\|_{L^2} \leq \|F(t+1, \phi)\|_{L^2} + \|F(t, \phi)\|_{L^2} \leq \frac{2a}{e^t} \|\phi\|_{\mathcal{B}_{\frac{1}{2}}},$$

$$\|AG(t+1, \phi) - AG(t, \phi)\|_{L^2} \leq \|AG(t+1, \phi)\|_{L^2} + \|AG(t, \phi)\|_{L^2} \leq \frac{2l}{e^t} \|\phi\|_{\mathcal{B}_{\frac{1}{2}}},$$

hence, the condition (H1) in Theorem 3.1 are fulfilled with $\omega = 1$. Finally, by (F3), $\|A^{-\frac{1}{2}}\| = 1$, $|\nu_0| = \pi^2$, $M_{\frac{1}{2}} = \Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\gamma = a\Gamma(\frac{1}{4})$, it is easy to test that the condition (H4) holds.

Therefore, it from Theorem 3.1 follows that the system (4.1) has at least one time S-asymptotic 1-periodic $\frac{1}{2}$-mild solution. The proof is completed. □

From Theorem 4.2, Definition 2.5, Definition 2.6, Lemma 2.7 and Theorem 3.2, it is easy to prove the following uniqueness.

Theorem 4.3. If the condition (F2) and the following conditions

(F4) there exists a positive constant a such that for every $(\xi, v), (\xi, w) \in [0,1] \times \mathbb{R}$

$$|f(\xi, v) - f(\xi, w)| \leq a|v - w|,$$

(F5) $l + (l + a)\pi^{-\frac{1}{2}} < 1$,

hold, then the system (4.1) has a unique time S-asymptotic 1-periodic $\frac{1}{2}$-mild solution.

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