Appl. Math. J. Chinese Univ.
2022, 37(2): 213-227

Ideal cotorsion pairs over one point extensions

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Abstract. Let $R[P]$ be the one point extension of a k-algebra R by a projective R-module P. We prove that the extension of a complete ideal cotorsion pair in R-Mod is still a complete ideal cotorsion pair in $R[P]$-Mod. As an application, it is obtainable that the operation $(-)_m[P]$ satisfies the so-called distributive law relating the operations of products and extensions of ideals under appropriate conditions.

§1 Introduction

In the classical approximation theory, the object $A \in \mathcal{A}$ is approximated by a subclass of $\mathcal{A}$. To replace objects and subclasses with morphisms and ideals, Fu, Asensio, Herzog and Torrecillas [6] introduced ideal cotorsion pairs and the ideal approximation theory. Meantime, they established the connection between special precovering ideals and phantom morphisms in exact categories. Furthermore, Fu and Herzog obtained ideal versions of Salce's Lemma, Ghost Lemma and Wakamatsu's Lemma to develop the ideal approximation theory and gave an interesting application in [7].

As a generalization of the classical approximation theory, the ideal approximation theory has attracted more attention. Breaz and Modoi [2] extended the ideal approximation theory to triangulated categories. Asadollahi and Sadeghi investigated so-called " higher ideal approximation theory" in [1]. Tan, Wang and Zhao extended the ideal approximation theory associated to almost n-exact structures in extension closed subcategories of n-angulated categories in [13]. Furthermore, the theory of ideal approximations was developed in [12].

The primary task of this article is to extend the ideal cotorsion pair from a k-algebra R to its one point extension. More precisely, if $R[P]$ is the one point extension of an algebra R by a projective R-module P, we study how to construct an ideal cotorsion pair in $R[P]$-Mod from an ideal cotorsion pair in R-Mod in a natural way. For an ideal $\mathcal{I}$ in R-Mod, $\mathcal{I}[P]$ is the collection of all morphisms $\mathbb{i} = \binom{i_k}{i} : \binom{V_1}{M_1}_{\theta_1} \to \binom{V_2}{M_2}_{\theta_2}$ with $i \in \mathcal{I}$. Unfortunately, for an ideal cotorsion pair $(\mathcal{I}, \mathcal{J})$ in R-Mod, $(\mathcal{I}[P],\mathcal{J}[P])$ is not an ideal cotorsion pair (see Example 3.4). Motivated by the result in [10], we introduce the concept of $\mathcal{I}_m[P]$ (see 2.5 for details).

Now, our main result can be stated as follows.

Received: 2021-02-05.   Revised: 2021-08-19.
MR Subject Classification: 18E10, 18G25, 16S70.
Keywords: one point extension, cotorsion pair, ME-conflation.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-022-4375-z.
Supported by Zhejiang Provincial Natural Science Foundation of China(LY18A010032).
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Theorem 1.1. *Let $R[P]$ be the one point extension of a k-algebra R, and let $(\mathcal{I}, \mathcal{J})$ be a complete ideal cotorsion pair in R-*Mod. *Then $(\mathcal{I}_m[P], \mathcal{J}[P])$ is a complete ideal cotorsion pair in $R[P]$-*Mod.

Let $(\mathcal{A}; \mathcal{E})$ be an exact category, then the category Arr($\mathcal{A}$) has the natural exact structure whose conflations are the morphisms of conflations in $(\mathcal{A}; \mathcal{E})$. We denote this exact category by (Arr($\mathcal{A}$);Arr($\mathcal{E}$)). Fu and Herzog in [7] introduced the concept of ME-conflation (see diagram (2.1)) and proved the ME substructure $(\mathrm{Arr}(\mathcal{A}), ME) \subseteq (\mathrm{Arr}(\mathcal{A}),\mathrm{Arr}(\mathcal{E}))$ is also exact. Moreover, they introduced the concept of an ME-extension $i * j$ of morphisms and then, if $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathcal{A}$, the concept of an extension of ideals $\mathcal{I} \diamond \mathcal{J} = \langle i * j | i \in \mathcal{I}, j \in \mathcal{J} \rangle$. Then, Theorem 1.1 leads to interesting properties of the operation $(-)_m[P]$, which we call the distributive law, that is:

Theorem 1.2. *Let $R[P]$ be the one point extension of a k-algebra R. For special precovering ideals $\mathcal{I}$ and $\mathcal{J}$, we have the following properties (distributive law):*

(1) $(\mathcal{I}\mathcal{J})_m[P] = \mathcal{I}_m[P]\mathcal{J}_m[P]$;

(2) $(\mathcal{I} \diamond \mathcal{J})_m[P] = \mathcal{I}_m[P] \diamond \mathcal{J}_m[P]$.

The contents of this paper is arranged as follows. In section 2, the definitions and properties required in ideal approximation theory are first introduced. In section 3, we construct an extension $\mathcal{I}_m[P]$ of an ideal $\mathcal{I}$, we then give the proof of Theorem 1.1. In section 4, we complete the proof of Theorem 1.2 by applying Theorem 1.1.

§2 Preliminaries

Throughout this paper, $\mathcal{A}$ is an abelian category and k is a field.

2.1. **Ideals**. An *ideal* $\mathcal{I}$ of $\mathcal{A}$ is a collection of morphisms satisfying:

(1) if $f : A \to B$ and $g : A \to B$ are two maps in $\mathcal{I}$, then $f \pm g \in \mathcal{I}$;

(2) if $i : A \to B$ is a map in $\mathcal{I}$, then for any maps $f : X \to A$ and $g : B \to Y$, the composition $gif : X \to Y$ is still in $\mathcal{I}$.

Equivalently, $\mathcal{I}$ is an additive subfunctor of Hom: $\mathcal{A}^{op} \times \mathcal{A} \to$ Ab. The ideal $\mathcal{I}$ associates to every pair A and B of objects in $\mathcal{A}$ a subgroup $\mathcal{I}(A, B) \subseteq \mathrm{Hom}(A, B)$ so that if $f : X \to A$ and $g : B \to Y$ are morphisms in $\mathcal{A}$, then the natural transformation $\mathrm{Hom}(f, g) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(X, Y)$ that assigns to $i \in \mathrm{Hom}(A, B)$.

$$\mathrm{Hom}(f, g)(i) : X \xrightarrow{f} A \xrightarrow{i} B \xrightarrow{g} Y$$

respects $\mathcal{I}$. This is an additive subcategory of $\mathcal{A}$, for every additive subcategory $\mathcal{X} \subseteq \mathcal{A}$ gives rise to the ideal $\mathcal{I}(\mathcal{X})$ generated by morphisms of the form 1_X, where $X \in \mathcal{X}$; this is the ideal of morphisms that factor through an object in $\mathcal{X}$. (see [6]).

2.2. **Homotopy**. Consider an exact category $(\mathcal{A}; \mathcal{E})$ consisting of an additive category $\mathcal{A}$, together with a distinguished collection $\mathcal{E}$ of composable pairs of morphism (i, j) such that i is the kernel of j and j the cokernel of i. Such a pair is depicted by

$$\eta : B \xrightarrow{i} C \xrightarrow{j} A$$

and is called a conflation. The kernel $i : B \to C$ that appears in the conflation η is an inflation, the cokernel $j : C \to A$ is a deflation. The arrow category $\mathrm{Arr}(\mathcal{A})$ of $\mathcal{A}$ is an abelian category whose objects $a : A_0 \to A_1$ are the morphisms (arrows) of $\mathcal{A}$, and $f : a \to b$ in $\mathrm{Arr}(\mathcal{A})$ is given by a pair of morphisms $f = (f_0, f_1)$ of $\mathcal{A}$ for which the diagram

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "a"] & B_0 \arrow[d, "b"] \\ A_1 \arrow[r, "f_1"] & B_1 \end{tikzcd}$$

commutes. And then, if $(\mathcal{A};\mathcal{E})$ is an exact category, then so $(\mathrm{Arr}(\mathcal{A});\mathrm{Arr}(\mathcal{E}))$ [3, Corollary 2.10].

A conflation (i.e. exact sequence) $\xi : a \to b \to c$ in $\mathrm{Arr}(\mathcal{E})$ is *null-homotopic* if there are maps $\alpha : C_0 \to B_1$ and $\beta : B_0 \to A_1$ such that $b = k\beta + \alpha j$.

$$\begin{tikzcd} A_0 \arrow[r, "i"] \arrow[d, "a"'] & B_0 \arrow[r, "j"] \arrow[d, "b"] \arrow[dl, "\beta"] & C_0 \arrow[d, "c"] \arrow[dl, "\alpha"] \\ A_1 \arrow[r, "k"'] & B_1 \arrow[r, "l"'] & C_1 \end{tikzcd}$$

In fact, we have the following well-known result:

Lemma 2.1. *Given the following conflation $\xi : a \to b \to c$*

$$\begin{tikzcd} A_0 \arrow[r, "i"] \arrow[d, "a"'] & B_0 \arrow[r, "j"] \arrow[d, "b"] & C_0 \arrow[d, "c"] \\ A_1 \arrow[r, "k"'] & B_1 \arrow[r, "l"'] & C_1 \end{tikzcd}$$

It is then clear that the following are equivalent:

(1) the conflation ξ is null-homotopic;

(2) there exists $\alpha : C_0 \to B_1$, such that $c = l\alpha$;

(3) there exists $\beta : B_0 \to A_1$, such that $\beta i = a$.

Proof. For completeness of the paper, we give the proof of (2) $\Rightarrow$ (1). Assume that $c = l\alpha$, and so $l\alpha j = jc = lb$. Then we have $l(b - \alpha j) = 0$. By the universal property of the kernel, there exist $\beta : B_0 \to A_1$, such that $k\beta = b - \alpha j$.

$$\begin{tikzcd} & B_0 \arrow[dl, "\beta"'] \arrow[d, "b-\alpha j"] & \\ A_1 \arrow[r, "k"'] & B_1 \arrow[r, "l"'] & C_1 \end{tikzcd}$$

It follows that $b = \alpha j + k\beta$.

□

A conflation $\xi : j \to a \to i$ in $\mathrm{Arr}(\mathcal{E})$ is called ME if there is a factorization

$$\begin{tikzcd}
\Xi_0 : & J_0 \arrow[r] \arrow[d, equal] & A_0 \arrow[r] \arrow[d, "a^1"] & I_0 \arrow[d, "i"] \\
\Xi : & J_0 \arrow[r] \arrow[d, "j"] & A \arrow[r] \arrow[d, "a^2"] & I_1 \arrow[d, equal] \\
\Xi_1 : & J_1 \arrow[r] & A_1 \arrow[r] & I_1
\end{tikzcd} \tag{2.1}$$

of ξ, where the middle row is a conflation in $(\mathcal{A};\mathcal{E})$. Denote by ME $\subseteq \mathrm{Arr}(\mathcal{E})$ the collection of ME conflations in $\mathrm{Arr}(\mathcal{E})$.

2.3. **Ext-orthogonality**. Given objects A and B of $\mathcal{A}$, the isomorphism classes of conflations form an abelian group $\mathrm{Ext}(A, B)$ with respect to the Baer sum operation. Furthermore, we have the additive bifunctor Ext:$\mathcal{A}^{op} \times \mathcal{A} \to Ab$.

A pair (i, j) of morphisms in$\mathcal{A}$ is Ext-*orthogonal*, defined $\mathrm{Ext}(i, j) = 0$ if every ME-extension $\xi : j \to a \to i$ in $\mathrm{Arr}(\mathcal{A})$ is null homotopic. This means that there are morphism $h : A_0 \to J_1$ and $g : I_0 \to A_1$ as in the diagram

$$\begin{tikzcd}
\Xi_0 : & J_0 \arrow[r] \arrow[d, "j"] & A_0 \arrow[r, "e_0"] \arrow[d, "a"] \arrow[dl, "h"] & I_0 \arrow[d, "i"] \arrow[dl, "g"] \\
\Xi_1 : & J_1 \arrow[r, "m_1"'] & A_1 \arrow[r] & I_1
\end{tikzcd}$$

satisfying $a = m_1h + ge_0$.

Proposition 2.2. [7, Proposiotion 5.2] *Let $i : X \to A$ and $j : B \to Y$ be two morphisms in $\mathcal{A}$. The pair (i, j) is said to be* Ext*-orthogonal if* $\mathrm{Ext}(i, j)$:$\mathrm{Ext}(A, B) \to \mathrm{Ext}(X, Y)$ *is a zero morphism.*

2.4. **Special precover**. Let $\mathcal{I} \subseteq \mathcal{A}$ be an ideal and A be an object of $\mathcal{A}$. An $\mathcal{I}$*-precover* of A is a morphism $i \in \mathcal{I}$, $i : X \to A$, such that any other morphism $i' : X' \to A$ in $\mathcal{I}$ factors though i,

$$\begin{tikzcd}
 & X' \arrow[dl, dashed] \arrow[d, "i'"] \\
X \arrow[r, "i"'] & A
\end{tikzcd}$$

In further, an $\mathcal{I}$*-precover* $i : X \to A$ is *special* if it is obtained as the pushout of a conflation η along a morphism $j : Y \to B$ in $\mathcal{I}^{\perp} := \{\ j \mid \mathrm{Ext}(i, j)\text{=}0,\ \forall i \in \mathcal{I}\}$:

$$\begin{tikzcd}
\eta : & Y \arrow[r] \arrow[d, "j"] & Z \arrow[r] \arrow[d] & A \arrow[d, equal] \\
 & B \arrow[r] & X \arrow[r, "i"] & A
\end{tikzcd}$$

The ideal $\mathcal{I}$ is called (*special*) *precovering* if every object $A \in \mathcal{A}$ has an (a special) $\mathcal{I}$-precover $i : X \to A$. Dually, we can define (*special*) $\mathcal{J}$*-preenvelopes* and (*special*) *preenveloping* classes.

A pair of ideals $(\mathcal{I}, \mathcal{J})$ in $\mathcal{A}$ is called an *ideal cotorsion pair* if $\mathcal{I}^{\perp} = \mathcal{J}$ and $\mathcal{I} = {}^{\perp}\mathcal{J}$ with $\mathcal{I}^{\perp} = \{\ j \mid \mathrm{Ext}(i, j)\text{=}0,\ \forall i \in \mathcal{I}\}$ and ${}^{\perp}\mathcal{J} = \{\ i \mid \mathrm{Ext}(i, j)\text{=}0,\ \forall j \in \mathcal{J}\}$. Moreover, an ideal cotorsion pair $(\mathcal{I}, \mathcal{J})$ is *complete* if every object in $\mathcal{A}$ has a special $\mathcal{J}$-preenvelope and a special $\mathcal{I}$-precover(see [6]).

2.5. **One point extension**[11]. Let R be a k-algebra and P be a fixed projective right R-module. We denote by $R[P]$ the *one point extension* of R by P, which the matrix algebra $\begin{pmatrix} k & P \\ 0 & R \end{pmatrix}$ with the ordinary matrix addition and the multiplication induced by the module structure of P. More precisely, the category $R[P]$-Mod can be described as follows: the objects are of form $\binom{V}{M}_\theta$ where $M \in R$-Mod, $V \in k$-Mod and $\theta : P \otimes_R M \to V$; the morphisms are of form $\binom{g}{f} : \binom{V_1}{M_1}_{\theta_1} \to \binom{V_2}{M_2}_{\theta_2}$, such that the following diagram is commutative

$$\begin{tikzcd} P \otimes M_1 \arrow[r, "1\otimes f"] \arrow[d, "\theta_1"] & P \otimes M_2 \arrow[d, "\theta_2"] \\ V_1 \arrow[r, "g"] & V_2 \end{tikzcd}$$

For a class $\mathcal{C}$ of R-Mod, $\mathcal{C}[P]$ denotes the class of all objects $\binom{V}{C}_\theta$ of $R[P]$-Mod with $C \in \mathcal{C}$ and $\mathcal{C}_m[P]$ is the subclass of $\mathcal{C}[P]$ consisting of all objects $\binom{V}{C}_\theta$ with $C \in \mathcal{C}$ and θ a monomorphism. In particular, R-Mod[P]=$R[P]$-Mod.

For an ideal $\mathcal{I}$ of R-Mod, $\mathcal{I}[P]$ denotes the class in $R[P]$-Mod of all morphisms $\mathbb{i} = \binom{i_k}{i}$: $\binom{V_1}{M_1}_{\theta_1} \to \binom{V_2}{M_2}_{\theta_2}$ with $i \in \mathcal{I}$. It is easy to check that $\mathcal{I}[P]$ is an ideal of $R[P]$-Mod. Forthermore, $\mathcal{I}_m[P]$ denotes the ideal of $R[P]$-Mod generated by all morphisms $\mathbb{i} = \binom{i_k}{i}$: $\binom{V_1}{M_1}_{\theta_1} \to \binom{V_2}{M_2}_{\theta_2}$ with $i \in \mathcal{I}$ and θ_1 a monomorphism.

§3 The proof of Theorem 1.1

Lemma 3.1. *If* Ext(i, j)=0 *with* $i : I_1 \to I_2$ *and* $j : J_0 \to J_1$ *in* $\mathcal{A}$*, then* Ext(fig, j)=0 *for every morphism* f *and* g.

Proof. Let $f : I_2 \to F$ and $g : I_0 \to I_1$ be morphisms and $\xi' : fig \to a \to j$ is a conflation. By [6, Proposition 3], we have

$$\begin{aligned} \mathrm{Ext}(fig, j) &= \mathrm{Ext}(fig, J_1)\mathrm{Ext}(F, j) \\ &= \mathrm{Ext}(g, J_1)\mathrm{Ext}(fi, J_1)\mathrm{Ext}(F, j) \\ &= \mathrm{Ext}(g, J_1)\mathrm{Ext}(I_1, j)\mathrm{Ext}(fi, J_0) \\ &= \mathrm{Ext}(g, J_1)\mathrm{Ext}(I_1, j)\mathrm{Ext}(i, J_0)\mathrm{Ext}(f, J_0) \\ &= \mathrm{Ext}(g, J_1)\mathrm{Ext}(i, j)\mathrm{Ext}(f, J_0) \\ &= 0 \end{aligned}$$

Thus, we get Ext(fig, j)=0, as desired. □

Suppose that a morphism of conflations is given, as in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "g"'] & B \arrow[r] \arrow[d, "k"] & C \arrow[d, "f"] \\ Y \arrow[r, "h"'] & Z \arrow[r, "l"'] & X \end{tikzcd}$$

Then the morphism $k : B \to Z$ is called a *coextension* of f by A. Next, we proceed as in the dual proof of [6,Proposition 9], we get the following result:

Lemma 3.2. *If* $\mathcal{M}$ *is a collection of morphism in* $\mathcal{A}$*, then* ${}^\perp\mathcal{M}$ *is an ideal closed under coextension by projective objects.*

Lemma 3.3. *Let $\mathcal{A}$ be an abelian category with enough projective objects and suppose that $\mathcal{J}$ is a special preenveloping class of $\mathcal{A}$. Given an object $A \in \mathcal{A}$, consider a conflation η':$K \to P \to A$ where P is a projective object and take a pushout*

$$\begin{tikzcd}
\eta' : & K \arrow[r] \arrow[d, "j"'] & P \arrow[r] \arrow[d] & A \arrow[d, equal] \\
\eta : & C \arrow[r] & X \arrow[r, "i"'] & A
\end{tikzcd}$$

along a special $\mathcal{J}$-preenvelope $j : K \to C$. The morphism $i : X \to A$ is a special ${}^{\perp}\mathcal{J}$-precover of A.

Proof.Since $j \in \mathcal{J} \subseteq [{}^{\perp}\mathcal{J}]^{\perp}$, it is enough to prove $i \in {}^{\perp}\mathcal{J}$ by [6, Proposition 11]. Then $i : X \to A$ is a special ${}^{\perp}\mathcal{J}$-precover of A. The special $\mathcal{J}$-preenvelope $j : K \to C$ arises from a pullback

$$\begin{tikzcd}
 & & K \arrow[dl, equal] \arrow[dd, dashed] & & \\
\eta' : & K \arrow[rr] \arrow[dd, "j"'] & & P \arrow[r] \arrow[dd] & A \arrow[dd, equal] \\
 & & Z \arrow[dd, dashed] & & \\
\eta : & C \arrow[ur, dashed, "h"] \arrow[rr] \arrow[dd] & & X \arrow[r, "i"] \arrow[dd] & A \\
 & & Y & & \\
 & W \arrow[ur, dashed, "g"] \arrow[rr] & & W &
\end{tikzcd}$$

Let $h : C \to Z$, then the pushout of η' along jh is also the pushout along h of the conflation $\eta : C \to X \to A$. Thus we obtain the commutative diagram

$$\begin{tikzcd}
 & & K \arrow[rr] \arrow[dl, equal] \arrow[dd, dashed, "jh"] & & P \arrow[rr] \arrow[dl, equal] \arrow[dd] & & A \arrow[dl, equal] \arrow[dd, equal] \\
\eta' : & K \arrow[rr] \arrow[dd, "j"'] & & P \arrow[rr] \arrow[dd] & & A \arrow[dd, equal] & \\
 & & Z \arrow[rr, dashed] \arrow[dd, dashed] & & L \arrow[rr, dashed] \arrow[dd] & & A \arrow[dl, equal] \\
\eta : & C \arrow[ur, dashed, "h"] \arrow[rr] \arrow[dd] & & X \arrow[ur, dashed] \arrow[rr, "i"'] \arrow[dd] & & A & \\
 & & Y \arrow[rr, equal] & & Y & & \\
 & W \arrow[ur, dashed, "g"] \arrow[rr] & & W \arrow[ur] & & &
\end{tikzcd}$$

where all rows and columns are conflations. Since $g \in {}^{\perp}\mathcal{J}$ and $k : X \to L$ is a coextension of $g : W \to Y$ by the projective object P, we have $k \in {}^{\perp}\mathcal{J}$ and $i = fk \in {}^{\perp}\mathcal{J}$.

$\square$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Firstly, we show that $\mathrm{Ext}(\mathcal{I}_m[P],\mathcal{J}[P])$=0. By Lemma 3.1, it is sufficient to prove that $\mathrm{Ext}(\mathbb{i},\mathbb{j})=0$, where $\mathbb{i}=\binom{i_k}{i}\colon \binom{V}{M}_\theta \to \binom{V_1}{M_1}_{\theta_1}$ with θ a monomorphism, $i\in\mathcal{I}$ and $\mathbb{j}=\binom{j_k}{j}\in\mathcal{J}[P]$. Consider the ME-conflation in $R[P]$-Mod

$$\begin{tikzcd}
0 \arrow[r] & \binom{U_1}{L_1} \arrow[r] \arrow[d, equal] & \binom{D_1}{C_1} \arrow[r] \arrow[d] & \binom{V}{M}_\theta \arrow[r] \arrow[d, "\mathbb{i}"] & 0 \\
0 \arrow[r] & \binom{U_1}{L_1} \arrow[r] \arrow[d, "\mathbb{j}"'] & \binom{D_2}{C_2} \arrow[r] \arrow[d] & \binom{V_1}{M_1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \binom{U_2}{L_2} \arrow[r, "\binom{u_2}{l_2}"] & \binom{D_3}{C_3}_d \arrow[r, "\binom{d_3}{c_3}"] & \binom{V_1}{M_1}_{\theta_1} \arrow[r] & 0
\end{tikzcd}$$

By Lemma 2.1, there is a morphism $f:M\to C_3$ in R-Mod, such that $c_3f=i$ since $i\in\mathcal{I}$, $j\in\mathcal{J}$. Then we consider the ME-conflation in R-Mod

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r] \arrow[d, equal] & C_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] \arrow[lldd, dashed, "f"'] & 0 \\
0 \arrow[r] & L_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_2 \arrow[r] & C_3 \arrow[r, "c_3"] & M_1 \arrow[r] & 0
\end{tikzcd}$$

Note that d_3 is an epimorphism in k-Mod, then there is a morphism $h:V\to D_3$ such that $d_3h=i_k$. It follows that

$$d_3d(1\otimes f)=\theta_1(1\otimes c_3)(1\otimes f)=\theta_1(1\otimes i)=i_k\theta=d_3h\theta$$

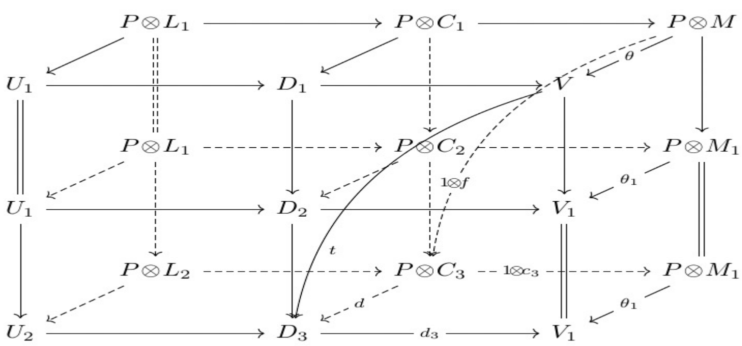
i.e. $d_3[d(1\otimes f)-h\theta]=0$. By the property of kernel, there is a morphism $g:P\otimes M\to U_2$, such that $d(1\otimes f)-h\theta=u_2g$. In addition, since θ is a monomorphism in k-Mod, there is a morphism $e:V\to U_2$, such that $g=e\theta$. Let $t=u_2e+h$. It is easy to find that

$$d_3t=d_3u_2e+d_3h=i_k$$

and

$$t\theta=u_2e\theta+h\theta=u_2g+h\theta=d(1\otimes f)$$

which means that we have the following commutative diagram



Therefore, $\mathrm{Ext}(\mathbb{i},\mathbb{j})=0$ holds, as desired.

In sequel, we will prove that ${}^{\perp}(\mathcal{J}[P]) \subset \mathcal{I}_m[P]$. Let $\mathbb{i} = \binom{i_k}{i} : \binom{V_1}{M_1}_{\theta_1} \to \binom{V_2}{M_2}_{\theta_2} \in {}^{\perp}(\mathcal{J}[P])$. Since $(\mathcal{I}, \mathcal{J})$ is a complete ideal cotorsion pair, there is a special $\mathcal{I}$-precover as follows, with $c_3 \in \mathcal{I}, j \in \mathcal{J}$

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r, "l_1"] \arrow[d, "j"] & C_2 \arrow[r, "c_2"] \arrow[d, "c"] & M_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_2 \arrow[r, "l_2"] & C_3 \arrow[r, "c_3"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

Let $\binom{1\otimes c}{0} : P\otimes C_2 \to (P\otimes C_3) \oplus V_2$ and $(\theta_2(1\otimes c_3), 1) : (P\otimes C_3) \oplus V_2 \to V_2$. It is easy to find $(\theta_2(1\otimes c_3), 1)$ is an epimorphism. Let (U_1, u_1) be the kernel of $(\theta_2(1\otimes c_3), 1)$. Thus, the following exact diagram commutes

$$\begin{tikzcd}
& P\otimes L_1 \arrow[rr, "1\otimes l_1"] \arrow[dl, "l_j"'] \arrow[dd, dashed, "j"] & & P\otimes C_2 \arrow[rr, "1\otimes c_2"] \arrow[dl, dashed, "\binom{1\otimes c}{0}"'] \arrow[dd, "1\otimes c"] & & P\otimes M_2 \arrow[dl, "\theta_2"'] \arrow[dd, equal] \\
U_1 \arrow[rr, "u_1"] \arrow[dd, equal] & & (P\otimes C_3)\oplus V_2 \arrow[rr] \arrow[dd, equal] & & V_2 \arrow[dd, equal] & \\
& P\otimes L_2 \arrow[rr, dashed, "1\otimes l_2"] \arrow[dl, dashed, "l"'] & & P\otimes C_3 \arrow[rr, dashed, "1\otimes c_3"] \arrow[dl, dashed, "\binom{1}{0}"'] & & P\otimes M_2 \arrow[dl, "\theta_2"'] \\
U_1 \arrow[rr, "u_1"] & & (P\otimes C_3)\oplus V_2 \arrow[rr, "(\theta_2(1\otimes c_3),1)"] & & V_2 &
\end{tikzcd}$$

where l, l_j are induced naturally. Consider the pullback of the conflation

$$0 \to \begin{pmatrix} U_1 \\ L_1 \end{pmatrix} \to \begin{pmatrix} (P\otimes C_3)\oplus V_2 \\ C_2 \end{pmatrix} \to \begin{pmatrix} V_2 \\ M_2 \end{pmatrix} \to 0$$

along the morphism $\mathbb{i}$, there is the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \binom{U_1}{L_1} \arrow[r] \arrow[d, equal] & \binom{D_1}{C_1} \arrow[r] \arrow[d] & \binom{V_1}{M_1}_{\theta} \arrow[r] \arrow[d, "\mathbb{i}"] & 0 \\
0 \arrow[r] & \binom{U_1}{L_1} \arrow[r] \arrow[d, "\mathbb{j}"'] & \binom{(P\otimes C_3)\oplus V_2}{C_2} \arrow[r] \arrow[d] & \binom{V_2}{M_2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \binom{U_1}{L_2} \arrow[r] & \binom{(P\otimes C_3)\oplus V_2}{C_3} \arrow[r, "\binom{(\theta_2(1\otimes c_3),1)}{c_3}"] & \binom{V_2}{M_2}_{\theta_1} \arrow[r] & 0
\end{tikzcd}$$

Note that $\mathbb{j} = \binom{1}{j} \in \mathcal{J}[P]$ since $j \in \mathcal{J}$ and $\mathbb{i} \in {}^{\perp}(\mathcal{J}[P])$, we have $\mathrm{Ext}(\mathbb{i}, \mathbb{j}) = 0$. By Lemma 2.1, we have following commutative diagram

$$\begin{tikzcd}
\binom{V_1}{M_1} \arrow[rr] \arrow[dr] & & \binom{V_2}{M_2} \\
& \binom{(P\otimes C_3)\oplus V_2}{C_3} \arrow[ur, "\binom{1}{0}"'] &
\end{tikzcd}$$

Since $\binom{1}{0}$ is a monomorphism and $c_3 \in \mathcal{I}$, $\mathbb{i} \in \mathcal{I}_m[P]$, i.e. ${}^{\perp}(\mathcal{J}[P]) \subset \mathcal{I}_m[P]$.

Next, we claim that $(\mathcal{I}_m[P])^{\perp} \subset \mathcal{J}[P]$. Let $\mathbb{j} = \binom{j_k}{j} : \binom{U_1}{L_1}_{\phi_1} \to \binom{U_2}{L_2}_{\phi_2} \in (\mathcal{I}_m[P])^{\perp}$, it suffices to prove that $j \in \mathcal{J}$. For every ME-conflation with $i : M_1 \to M_2$ in $\mathcal{I}$

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r, "l_1"] \arrow[d, equal] & C_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "l_2"] \arrow[d, "j"] & C_2 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_2 \arrow[r] & C_3 \arrow[r] & M_2 \arrow[r] & 0
\end{tikzcd}$$

In addition, consider the pushout of $\phi_1, 1\otimes l_1$ and $\phi_1, 1\otimes l_2$, we have following commutative diagram

$$\begin{tikzcd}
& P\otimes L_1 \arrow[rr, "1\otimes l_1"] \arrow[dl, "\phi_1"'] \arrow[dd, equal, dashed] & & P\otimes C_1 \arrow[rr] \arrow[dl] \arrow[dd, dashed, "1\otimes c"] & & P\otimes M_1 \arrow[dl, equal] \arrow[dd, "1\otimes i"] \\
U_1 \arrow[rr] \arrow[dd, equal] & & D_1 \arrow[rr] & & P\otimes M_1 \arrow[dd] & \\
& P\otimes L_1 \arrow[rr, dashed, "1\otimes l_2"] \arrow[dl, dashed, "\phi_1"'] & & P\otimes C_2 \arrow[rr, dashed] \arrow[dl, dashed, "d"'] & & P\otimes M_2 \arrow[dl, equal] \\
U_1 \arrow[rr, "u_2"] & & D_2 \arrow[rr] & & P\otimes M_2 &
\end{tikzcd}$$

Since $u_2 \circ 1_{U_1} \circ \phi_1 = d \circ (1\otimes c) \circ (1\otimes l_1)$, there is a morphism from D_1 to D_2 making the following diagram commutative by the universal property of the pushout

$$\begin{tikzcd}
& P\otimes L_1 \arrow[rr, "1\otimes l_1"] \arrow[dl, "\phi_1"'] \arrow[dd, equal, dashed] & & P\otimes C_1 \arrow[rr] \arrow[dl] \arrow[dd, dashed, "1\otimes c"] & & P\otimes M_1 \arrow[dl, equal] \arrow[dd, "1\otimes i"] \\
U_1 \arrow[rr] \arrow[dd, equal] & & D_1 \arrow[rr] \arrow[dd] & & P\otimes M_1 \arrow[dd] & \\
& P\otimes L_1 \arrow[rr, dashed, "1\otimes l_2"] \arrow[dl, dashed, "\phi_1"'] & & P\otimes C_2 \arrow[rr, dashed] \arrow[dl, dashed, "d"'] & & P\otimes M_2 \arrow[dl, equal] \\
U_1 \arrow[rr, "u_2"] & & D_2 \arrow[rr] & & P\otimes M_2 &
\end{tikzcd}$$

Since $i \in \mathcal{I}$ and $1_{P\otimes M_1}$ is a monomorphism, $\mathbb{i} = \binom{1\otimes i}{i} \in \mathcal{I}_m[P]$. Consider the pushout of the conflation $0 \to \binom{U_1}{L_1} \to \binom{D_2}{C_2} \to \binom{P\otimes M_2}{M_2} \to 0$ along the morphism $\mathbb{j}$, then we have the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \binom{U_1}{L_1} \arrow[r] \arrow[d, equal] & \binom{D_1}{C_1} \arrow[r] \arrow[d] & \binom{P\otimes M_1}{M_1} \arrow[r] \arrow[d, "\mathbb{i}"] & 0 \\
0 \arrow[r] & \binom{U_1}{L_1} \arrow[r] \arrow[d, "\mathbb{j}"] & \binom{D_2}{C_2} \arrow[r] \arrow[d] & \binom{P\otimes M_2}{M_2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \binom{U_1}{L_2} \arrow[r] & \binom{D_3}{C_3} \arrow[r] & \binom{P\otimes M_2}{M_2} \arrow[r] & 0
\end{tikzcd}$$

Due to $\mathbb{i} \in \mathcal{I}_m[P]$, we obtain $\mathrm{Ext}(\mathbb{i}, \mathbb{j}) = 0$ which implies $\mathrm{Ext}(i, j) = 0$. That is $j \in {}^{\perp}\mathcal{I} = \mathcal{J}$, as required. Thus, we have $(\mathcal{I}_m[P], \mathcal{J}[P])$ is an ideal cotorsion pair in $R[P]$-Mod.

Finally, we need to show that $(\mathcal{I}_m[P], \mathcal{J}[P])$ is complete. Let $\binom{V_2}{M_2}$ be an object in $R[P]$-Mod. As $(\mathcal{I}, \mathcal{J})$ is complete, M_2 has a special $\mathcal{I}$-precover, i.e. we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r, "l_1"] \arrow[d, "j"] & C_2 \arrow[r, "c_2"] \arrow[d, "c"] & M_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_2 \arrow[r, "l_2"] & C_3 \arrow[r, "c_3"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

with $c_3 \in \mathcal{I}$ and $j \in \mathcal{J}$. As the about proof, we have the following exact commutative diagram with u_1 the kernel of $(\theta_2(1 \otimes c_3), 1)$, $\mathbb{c} = \binom{(\theta_2(1\otimes c_3),1)}{c_3}$ and $\mathbb{j} = \binom{1}{j}$.

$$\begin{tikzcd}
0 \arrow[r] & \binom{U_1}{L_1} \arrow[r] \arrow[d, "\mathbb{j}"] & \binom{(P\otimes C_3)\oplus V_2}{C_2} \arrow[r] \arrow[d] & \binom{V_2}{M_2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \binom{U_1}{L_2} \arrow[r] & \binom{(P\otimes C_3)\oplus V_2}{C_3} \arrow[r, "\mathbb{c}"] & \binom{V_2}{M_2} \arrow[r] & 0
\end{tikzcd}$$

Clearly, $\mathbb{c} \in \mathcal{I}_m[P]$, $\mathbb{j} \in \mathcal{J}[P]$. Therefore, $\binom{(P\otimes C_3)\oplus V_2}{C_2} \xrightarrow{\mathbb{c}} \binom{(P\otimes C_3)\oplus V_2}{C_3}$ is a special $\mathcal{I}$-precover.

In conclusion, $(\mathcal{I}_m[P], \mathcal{J}[P])$ is a complete ideal cotorsion pair.

□

The following example shows that $(\mathcal{I}[P], \mathcal{J}[P])$ is not an ideal cotorsion pair in general.

Example 3.4. *Let $k[k]$ be the one point extension of k by k with k is a field, $\mathcal{H}$ denotes the class of all morphism in k-*Mod*. Clearly, $(\mathcal{H}, \mathcal{H})$ is a unique ideal cotorsion pair in k-*Mod*, but the pair $(\mathcal{H}[k], \mathcal{H}[k])$ is not an ideal cotorsion pair in $k[k]$-*Mod*. In fact, $1_{\binom{0}{k}}$ in $\mathcal{H}[k]$, but there doesn't exist $f : \binom{0}{k} \to \binom{k}{k}$, such that the following diagram commutes*

$$\begin{tikzcd}
 & \binom{0}{k} \arrow[dl, dashed, "f"'] \arrow[d, equal, "1"] \\
\binom{k}{k} \arrow[r, "\binom{0}{1}"'] & \binom{0}{k}
\end{tikzcd}$$

Thus $1_{\binom{0}{k}}$ is not a projective morphism, and hence $\mathcal{H}[k] \neq {}^{\perp}(\mathcal{H}[k])$.

Corollary 3.5. *Let $(\mathcal{F}, \mathcal{C})$ be a complete cotorsion pair in R-Mod. Then we have the following equation in $R[P]$-*Mod

$$(\mathcal{I}(\mathcal{F}_m[P]), \mathcal{I}(\mathcal{C}[P])) = (\mathcal{I}(\mathcal{F})_m[P], \mathcal{I}(\mathcal{C})[P]).$$

*Proof.*By [10, Theorem 1.1], $(\mathcal{F}_m[P], \mathcal{C}[P])$ is a complete cotorsion pair. Then $(\mathcal{I}(\mathcal{F}_m[P]), \mathcal{I}(\mathcal{C}[P]))$ is an ideal cotorsion pair by [6, Theorem 28].

Let $\mathbb{i} = \binom{i_k}{i} : \binom{V_1}{M_1} \to \binom{V_2}{M_2} \in \mathcal{I}(\mathcal{F}_m[P])$. Then $\mathbb{i}$ factors through by $\binom{U}{F}_\theta$ which belongs to $\mathcal{F}_m[P]$. Then we have the following commutative diagram

$$\begin{tikzcd}
P\otimes M_1 \arrow[rr, "1\otimes i"] \arrow[dr] \arrow[dd] & & P\otimes M_2 \arrow[dd] \\
 & P\otimes F \arrow[ur, dashed, "1\otimes f"] \arrow[dd, "\theta"] & \\
V_1 \arrow[rr, dashed, "i_k"] \arrow[dr] & & V_2 \\
 & U \arrow[ur] &
\end{tikzcd}$$

Due to $\binom{U}{F}_\theta \in \mathcal{F}_m[P]$, θ is a monomorphism and $F \in \mathcal{F}$. Then $f \in \mathcal{I}(\mathcal{F})$. In further, $\binom{i_k}{i} \in \mathcal{I}(\mathcal{F})_m[P]$, i.e. $\mathcal{I}(\mathcal{F}_m[P]) \subset \mathcal{I}(\mathcal{F})_m[P]$.

Conversely, let $\mathbb{i} = \binom{i_k}{i} : \binom{V_1}{M_1} \to \binom{V_2}{M_2}_{\theta_2} \in \mathcal{I}(\mathcal{F})_m[P]$, which means that we have following commutative diagram

$$\begin{tikzcd}
P\otimes M_1 \arrow[rr, "1\otimes i"] \arrow[dr] \arrow[dd] & & P\otimes M_2 \arrow[dd, "\theta_2"] \\
& P\otimes M \arrow[ur, "1\otimes m_2"'] \arrow[dd, "\theta"] & \\
V_1 \arrow[rr, dashed, "i_k"] \arrow[dr] & & V_2 \\
& V \arrow[ur, "v_2"'] &
\end{tikzcd}$$

where θ is a monomorphism and m_2 is in $\mathcal{I}(\mathcal{F})$. Then, m_2 can factor through by F with $F \in \mathcal{F}$. Denote the morphism from M to F and the morphism from F to M_2 by m and f, i.e. $m_2 = fm$. Denote by θ', v the pushout of $\theta, 1\otimes m$. Since θ is a monomorphism, θ' is monic i.e. $\binom{U}{F} \in \mathcal{F}_m[P]$. Since $\theta_2(1\otimes m_2) = v_2\theta$, there is a morphism $u : U \to V_2$ such that the following diagram commutative

$$\begin{tikzcd}
P\otimes M_1 \arrow[rr, "1\otimes i"] \arrow[dr] \arrow[dd] & & P\otimes M_2 \arrow[dd, dashed, "\theta_2"] & \\
& P\otimes M \arrow[ur, "1\otimes m_2"] \arrow[rr, "1\otimes m"] \arrow[dd, "\theta"] & & P\otimes F \arrow[ul, "1\otimes f"'] \arrow[dd, "\theta'"] \\
V_1 \arrow[rr, dashed, "i_k"] \arrow[dr] & & V_2 & \\
& V \arrow[ur, dashed, "v_2"] \arrow[rr, "v"] & & U \arrow[ul, dashed, "u"']
\end{tikzcd}$$

Hence $\mathbb{i}$ factors through by $\binom{U}{F}$. It follows $\mathbb{i} \in \mathcal{I}(\mathcal{F}_m[P])$, i.e. $\mathcal{I}(\mathcal{F})_m[P] \subset \mathcal{I}(\mathcal{F}_m[P])$. Thus, $\mathcal{I}(\mathcal{F})_m[P] = \mathcal{I}(\mathcal{F}_m[P])$.

□

§4 The proof of Theorem 1.2

Let i and j be morphisms in $\mathcal{A}$. The concept of $i * j$ is introduced by ME-extension $j \to i * j \to i$. To prove the Theorem 1.2, we need the following crucial result.

Proposition 4.1. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. Then $\mathcal{I}[P]\mathcal{J}[P] = (\mathcal{I}\mathcal{J})[P]$ and $\mathcal{I}[P] \diamond \mathcal{J}[P] = (\mathcal{I} \diamond \mathcal{J})[P]$.*

Proof. At first, we claim that $\mathcal{I}[P]\mathcal{J}[P] = (\mathcal{I}\mathcal{J})[P]$. For any $\mathbb{i}\mathbb{j} \in \mathcal{I}[P]\mathcal{J}[P]$, i.e.

$$\binom{V_1}{M_1} \xrightarrow{\mathbb{j}} \binom{V_2}{M_2} \xrightarrow{\mathbb{i}} \binom{V_3}{M_3}$$

with $\mathbb{i} \in \mathcal{I}[P]$ and $\mathbb{j} \in \mathcal{J}[P]$, which means that there is the following diagram

$$\begin{tikzcd}
P\otimes M_1 \arrow[r, "1\otimes j"] \arrow[d, "\theta_1"] & P\otimes M_2 \arrow[r, "1\otimes i"] \arrow[d, "\theta_2"] & P\otimes M_3 \arrow[d, "\theta_3"] \\
V_1 \arrow[r, "j_k"'] & V_2 \arrow[r, "i_k"'] & V_3
\end{tikzcd}$$

Since $\mathbb{i} \in \mathcal{I}[P]$ and $\mathbb{j} \in \mathcal{J}[P]$, i is in $\mathcal{I}$ and j is in $\mathcal{J}$. Therefore, ij is in $\mathcal{I}\mathcal{J}$. Hence, we have $\mathbb{i}\mathbb{j} \in (\mathcal{I}\mathcal{J})[P]$, which means $\mathcal{I}[P]\mathcal{J}[P] \subset (\mathcal{I}\mathcal{J})[P]$.

Next, for any $\binom{m_k}{m} \in (\mathcal{I}\mathcal{J})[P]$, we get the following commutative diagram

$$\begin{tikzcd}
P\otimes M_1 \arrow[r, "1\otimes m"] \arrow[d, "\theta_1"] & P\otimes M_3 \arrow[d, "\theta_3"] \\
V_1 \arrow[r, "m_k"] & V_3
\end{tikzcd}$$

with $m \in \mathcal{I}\mathcal{J}$. Then we have $m = ij$ where $i \in \mathcal{I}$ and $j \in \mathcal{J}$, i.e. we obtain the following commutative diagram

$$\begin{tikzcd}
P\otimes M_1 \arrow[r, "1\otimes j"] \arrow[d, "\theta_1"] & P\otimes M_2 \arrow[r, "1\otimes i"] \arrow[d, "\theta_3(1\otimes i)"] & P\otimes M_3 \arrow[d, "\theta_3"] \\
V_1 \arrow[r, "m_k"'] & V_3 \arrow[r, equal] & V_3
\end{tikzcd}$$

Due to $\binom{m_k}{j} \in \mathcal{J}[P]$ and $\binom{1_{V_3}}{i} \in \mathcal{I}[P]$, we have $\binom{m_k}{m} = \binom{1_{V_3}}{i}\binom{m_k}{j} \in \mathcal{I}[P]\mathcal{J}[P]$ which means $(\mathcal{I}\mathcal{J})[P] \subset \mathcal{I}[P]\mathcal{J}[P]$. Therefore, we come to the conclusion that $(\mathcal{I}\mathcal{J})[P] = \mathcal{I}[P]\mathcal{J}[P]$.

In sequel, we will show that $\mathcal{I}[P]\diamond\mathcal{J}[P] = (\mathcal{I}\diamond\mathcal{J})[P]$. For any $\mathbb{a}(\mathbb{i}*\mathbb{j})\mathbb{b} = \binom{a_k}{a}\left(\binom{i_k}{i}*\binom{j_k}{j}\right)\binom{b_k}{b}$ in $\mathcal{I}[P]\diamond\mathcal{J}[P]$ where $\mathbb{j} \in \mathcal{J}[P]$ and $\mathbb{i} \in \mathcal{I}[P]$, there is the following commutative diagram.

$$\begin{tikzcd}
 & & \binom{V_2}{M_2} \arrow[d, "\mathbb{b}"] & & \\
0 \arrow[r] & \binom{V_1}{M_1} \arrow[r] \arrow[d, "\mathbb{j}"] & \binom{V_0}{M_0} \arrow[r] \arrow[d, "\mathbb{i}*\mathbb{j}"] & \binom{V_3}{M_3} \arrow[r] \arrow[d, "\mathbb{i}"] & 0 \\
0 \arrow[r] & \binom{V_1'}{M_1'} \arrow[r] & \binom{V_0'}{M_0'} \arrow[r] \arrow[d, "\mathbb{a}"] & \binom{V_3'}{M_3'} \arrow[r] & 0 \\
 & & \binom{V_2'}{M_2'} & &
\end{tikzcd}$$

Since $0 \to j \to i*j \to i \to 0$ is an exact sequence, we have $a(i*j)b \in \mathcal{I}\diamond\mathcal{J}$. It follows that $\mathbb{a}(\mathbb{i}*\mathbb{j})\mathbb{b}$ in $(\mathcal{I}\diamond\mathcal{J})[P]$ which implies $\mathcal{I}[P]\diamond\mathcal{J}[P] \subset (\mathcal{I}\diamond\mathcal{J})[P]$.

Next, assume that for any $\binom{c}{a(i*j)b} \in (\mathcal{I}\diamond\mathcal{J})[P]$, where $i*j \in \mathcal{I}\diamond\mathcal{J}$, i.e. there is an exact sequence

$$0 \longrightarrow j \xrightarrow{\overline{\alpha}_1=\binom{\alpha_1'}{\alpha_1}} i*j \xrightarrow{\overline{\alpha}_3=\binom{\alpha_3'}{\alpha_3}} i \longrightarrow 0$$

Then we have the following commutative diagram

$$\begin{tikzcd}
 & & P\otimes M_2 \arrow[dl, "\theta_2"'] \arrow[dd, "1\otimes b"] & \\
 & V_2 \arrow[dddd, "c"] & & \\
P\otimes M_1 \arrow[rr, "1\otimes\alpha_1"] \arrow[d, "1\otimes j"] & & P\otimes M_0 \arrow[r, "1\otimes\alpha_3"] \arrow[d, "1\otimes(i*j)"] & P\otimes M_3 \arrow[d, "1\otimes i"] \\
P\otimes M_1' \arrow[rr, "1\otimes\alpha_1'"] & & P\otimes M_0' \arrow[r, "1\otimes\alpha_3'"] \arrow[d, "1\otimes a"] & P\otimes M_3' \\
 & & P\otimes M_2' \arrow[dl, "\theta_2'"'] & \\
 & V_2' & &
\end{tikzcd}$$

There are morphisms

$$\begin{array}{c}\varphi=\theta_2'(1\otimes a)(1\otimes(i*j)):P\otimes M_0\to V_2'\\ \varphi'=\theta_2'(1\otimes a):P\otimes M_0'\to V_2'\end{array}$$

From this construction, we can get the following commutative diagram

$$\begin{tikzcd}
 & & P\otimes M_2 \arrow[dl, "\theta_2"'] \arrow[dd, "1\otimes b"] & \\
 & V_2 \arrow[dd, "c"] & & \\
P\otimes M_1 \arrow[rr, "1\otimes\alpha_1"] \arrow[dd, "1\otimes j"] & & P\otimes M_0 \arrow[dl, "\varphi"'] \arrow[r, "1\otimes\alpha_3"] \arrow[dd, "1\otimes(i*j)"] & P\otimes M_3 \arrow[dd, "1\otimes i"] \\
 & V_2' \arrow[dd, equal] & & \\
P\otimes M_1' \arrow[rr, "1\otimes\alpha_1'"] & & P\otimes M_0' \arrow[dl, "\varphi'"'] \arrow[r, "1\otimes\alpha_3'"] \arrow[dd, "1\otimes a"] & P\otimes M_3' \\
 & V_2' \arrow[dd, equal] & & \\
 & & P\otimes M_2' \arrow[dl, "\theta_2'"'] & \\
 & V_2' & &
\end{tikzcd}$$

Denote$\binom{\varphi'}{\varphi}$ by $\overline{\varphi}$. Therefore, there is a diagram as follows

$$\begin{tikzcd} 0 \arrow[r] & 1\otimes j \arrow[r, "1\otimes\overline{\alpha}_1"] & 1\otimes(i*j) \arrow[r, "1\otimes\overline{\alpha}_3"] \arrow[d, "\overline{\varphi}"] & 1\otimes i \arrow[r] & 0 \\ & & 1_{V_2'} & & \end{tikzcd}$$

Denoted by $\overline{\varphi}', \overline{\alpha}_3'$ and the pushout of $\overline{\varphi}, 1\otimes\overline{\alpha}_3$. Let j_k be the kernel of $\overline{\varphi}'$, and then we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & 1\otimes j \arrow[r, "1\otimes\overline{\alpha}_1"] \arrow[d, "\overline{\theta}"] & 1\otimes(i*j) \arrow[r, "1\otimes\overline{\alpha}_3"] \arrow[d, "\overline{\varphi}"] & 1\otimes i \arrow[r] \arrow[d, "\overline{\alpha}_3'"] & 0 \arrow[d, equal] \\ 0 \arrow[r] & j_k \arrow[r] & 1_{V_2'} \arrow[r, "\overline{\varphi}'"] & i_k \arrow[r] & 0 \end{tikzcd}$$

with $\overline{\theta} : 1\otimes j \to j_k$ induced. Due to $j \in \mathcal{J}$ and $i \in \mathcal{I}$, $\binom{j_k}{j}$ is in $\mathcal{J}[P]$ and $\binom{i_k}{i}$ is in $\mathcal{I}[P]$. Hence

$$\binom{c}{a(i*j)b} = \binom{1_{V_2'}}{a}\left(\binom{i_k}{i} * \binom{j_k}{j}\right)\binom{c}{b} \in \mathcal{I}[P] \diamond \mathcal{J}[P]$$

So, $\mathcal{I}[P]\diamond\mathcal{J}[P] \subset (\mathcal{I}\diamond\mathcal{J})[P]$ and the conclusion $\mathcal{I}[P]\diamond\mathcal{J}[P] = (\mathcal{I}\diamond\mathcal{J})[P]$ can be reached finally. □

Lemma 4.2. $(\mathcal{I}, \mathcal{I}^{\perp})$ *is a complete cotorsion pair if and only if* $\mathcal{I}$ *is a special precovering ideal.*

Lemma 4.3. *Let* $(\mathcal{I}, \mathcal{I}^{\perp})$ *and* $(\mathcal{J}, \mathcal{J}^{\perp})$ *be complete cotorsion pairs. Then,* $(\mathcal{I}\mathcal{J}, \mathcal{J}^{\perp}\diamond\mathcal{I}^{\perp})$ *and* $(\mathcal{I}\diamond\mathcal{J}, \mathcal{J}^{\perp}\mathcal{I}^{\perp})$ *are complete cotorsion pairs.*

Proof of Theorem 1.2. Since $\mathcal{I}, \mathcal{J}$ are special precovering ideals, $(\mathcal{I}, \mathcal{I}^{\perp})$ and $(\mathcal{J}, \mathcal{J}^{\perp})$ are complete ideal cotorsion pairs by Lemma 4.2.

Then, according to Theorem 1.1, $(\mathcal{I}_m[P], \mathcal{I}^{\perp}[P])$ and $(\mathcal{J}_m[P], \mathcal{J}^{\perp}[P])$ are complete ideal cotorsion pairs. Hence, $(\mathcal{I}_m[P]\mathcal{J}_m[P], \mathcal{J}^{\perp}[P]\diamond\mathcal{I}^{\perp}[P])$ and $(\mathcal{I}_m[P]\diamond\mathcal{J}_m[P], \mathcal{J}^{\perp}[P]\mathcal{I}^{\perp}[P])$ are complete ideal cotorsion pairs by Lemma 4.3.

On the other hand, $(\mathcal{I}\mathcal{J}, \mathcal{J}^{\perp}\diamond\mathcal{I}^{\perp})$ and $(\mathcal{I}\diamond\mathcal{J}, \mathcal{J}^{\perp}\mathcal{I}^{\perp})$ are complete cotorsion pairs. Therefore, $((\mathcal{I}\mathcal{J})_m[P], (\mathcal{J}^{\perp}\diamond\mathcal{I}^{\perp})[P])$ and $((\mathcal{I}\diamond\mathcal{J})_m[P], (\mathcal{J}^{\perp}\mathcal{I}^{\perp})[P])$ are complete cotorsion pairs. By Proposition 4.1 , we have

$$\mathcal{I}_m[P]\mathcal{J}_m[P] = {}^{\perp}(\mathcal{J}^{\perp}[P]\diamond\mathcal{I}^{\perp}[P]) = {}^{\perp}((\mathcal{J}^{\perp}\diamond\mathcal{I}^{\perp})[P])(\mathcal{I}\mathcal{J})_m[P]$$

and

$$\mathcal{I}_m[P]\diamond\mathcal{J}_m[P] = {}^{\perp}(\mathcal{J}^{\perp}[P]\mathcal{I}^{\perp}[P]) = {}^{\perp}((\mathcal{J}^{\perp}\mathcal{I}^{\perp})[P]) = (\mathcal{I}\diamond\mathcal{J})_m[P]$$

□

Acknowledgement

The authors are grateful to the referees for reading the paper carefully and for many suggestions on mathematics and English expressions.

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