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Modified approximation and error estimation for King's type (p,q)-BBH operators

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Abstract. In this paper, the King's type modification of (p,q)-Bleimann-Butzer and Hahn operators is defined. Some results based on Korovkin's approximation theorem for these new operators are studied. With the help of modulus of continuity and the Lipschitz type maximal functions, the rate of convergence for these new operators are obtained. It is shown that the King's type modification have better rate of convergence, flexibility than classical (p,q)-BBH operators on some subintervals. Further, for comparisons of the operators, we presented some graphical examples and the error estimation in the form of tables through MATLAB (R2015a)

§1 Introduction

In the year 1980, Bleimann-Butzer and Hahn [6] generalized the following Bernstein type operators which approximate continuous functions on unbounded interval $[0,\infty)$.

$$L_n(f;x)=\frac{1}{(1+x)^n}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]x^k f\left(\frac{k}{n-k+1}\right),\quad n\in\mathbb{N}\ \ and\ \ x\in[0,\infty). \tag{1}$$

For detailed study, one can refer ([12], [15]).

In the field of approximation theory, first of all Lupaş used q-integers of the Bernstein polynomials and after that Phillips gave another q-based generalization of these polynomials. In 2015, Mursaleen et al. firstly, used (p,q)-integers in this field and constructed (p,q)-analogue of the classical Bernstein polynomials and studied the approximation properties of the operators. After that many researchers used (p,q)-integers for other operators and introduced their approximation properties (see [1]–[3], [13], [14], [16], [17], [20]–[32]).

In the sequel, we recall basic definitions and some notations from post quantum calculus [(p,q)-calculus].

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For $0 < q < p \leq 1$ and non-negative integer j, $j \in \mathbb{N}$, we have

$$[j]_{p,q} = \begin{cases} \frac{p^j - q^j}{p-q}, & \text{if } q \neq p \neq 1 \\ jp^{j-1}, & \text{if } q = p \neq 1 \\ [j]_q, & \text{if } p = 1 \\ j, & \text{if } q = p = 1, \end{cases}$$

$$[j]_{p,q}! = \begin{cases} [j]_{p,q}[j-1]_{p,q} \cdots [1]_{p,q}, & j \in \mathbb{N} \\ 1, & j = 0 \end{cases}$$

and

$$(ax+by)^n_{p,q} = \sum_{j=0}^{n} p^{\frac{(n-j)(n-j-1)}{2}} q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} a^{n-j} b^j x^{n-j} y^j,$$

$$(x+y)^n_{p,q} = \sum_{j=0}^{n} (p^{j-1}x + q^{j-1}y). \tag{2}$$

If we take $y = 0$ in (2), then

$$(x)^j_{p,q} = p^{\frac{j(j-1)}{2}} x^j.$$

$$\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[j]_{p,q}![n-j]_{p,q}!}, \quad 0 \leq j \leq n.$$

For more details, one can refer [11].

Let $C_B[0,\infty)$ denote the set of all bounded and continuous functions defined on $[0,\infty)$ endowed with the norm

$$\| f \|_{C_B} = \sup_{x \geq 0} |f(x)|,$$

and let ω be a modulus of continuity that satisfies:

(i) ω is a non-negative increasing function on the interval $[0,\infty)$.

(ii) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ and ω satisfy the following inequality

$$\omega(r\delta) \leq r\omega(\delta), \quad for \quad r \in \mathbb{N}.$$

(iii) $\omega(\delta) = 0$, as $\delta \to 0$.

Let H_w be the set of all real valued functions on $[0,\infty)$ and $H_w \subseteq C_B[0,\infty)$. For any $u, v \in [0,\infty)$, we have

$$|f(u) - f(v)| \leq \omega\left(\left|\frac{u}{1+u} + \frac{v}{1+v}\right|\right).$$

Gadjiev and Çakar [10] studied the properties of the operators $A_n : H_w \to C_B[0,\infty)$ and obtained the following result.

Theorem 1.1. *Let $\{A_n\}$ be a sequence of positive linear operators $A_n : H_w \to C_B[0,\infty)$ and satisfy*

$$\lim_{n\to\infty} \left\| A_n\left(\left(\frac{t}{1+t}\right)^j; x\right) - \left(\frac{x}{1+x}\right)^j \right\|_{C_B} = 0, \quad j = 0, 1, 2.$$

Then

$$\lim_{n\to\infty} \|A_n(f;x) - f(x)\|_{C_B} = 0, \ for\ any\ f \in H_w.$$

Mursaleen et al. [25] introduced the following (p,q)-BBH operators. For $0<q<p\leq 1$ and $x\in[0,\infty)$, we have

$$L_n^{p,q}(f;x)=\frac{1}{l_n^{p,q}(x)}\sum_{k=0}^{n} f\left(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right) q^{\frac{k(k-1)}{2}} p^{\frac{(n-k)(n-k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k, \tag{3}$$

where f is defined on $[0,\infty)$ and $l_n^{p,q}(x)=\prod_{k=0}^{n-1}(p^k+q^k x)$.
For the above operators (3), we recall the following lemma as proved in [25].

Lemma 1.2. *For $x\geq 0$ and $0<q<p\leq 1$. Then the following properties are true.*

1. $L_n^{p,q}(e_0;x)=e_0$,
2. $L_n^{p,q}(\frac{t}{1+t};x)=\frac{p[n]_{p,q}}{[n+1]_{p,q}}\left(\frac{x}{1+x}\right)$,
3. $L_n^{p,q}((\frac{t}{1+t})^2;x)=\frac{pq^2[n]_{p,q}[n-1]_{p,q}}{[n+1]^2_{p,q}}\frac{x^2}{(1+x)(p+qx)}+\frac{p^{n+1}[n]_{p,q}}{[n+1]^2_{p,q}}\frac{(x)}{(1+x)}$

$$=\frac{q^2[n]_{p,q}[n-1]_{p,q}}{[n+1]^2_{p,q}}\frac{(x)^2}{(1+x)^2}+\frac{p^{n+1}[n]_{p,q}}{[n+1]^2_{p,q}}\frac{(x)}{(1+x)}.$$

In the above Lemma 1.2, we see that the operators (3) preserve only e_0 but do not preserve the test functions e_1 and e_2. So after scaling $f\left(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)$ by $f\left(\frac{[n+1]_{p,q}}{p[n]_{p,q}}\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)$ in (3), we have

$$L_n^{p,q}(f;x)=\frac{1}{l_n^{p,q}(x)}\sum_{k=0}^{n} f\left(\frac{p^{n-k}[n+1]_{p,q}[k]_{p,q}}{[n]_{p,q}[n-k+1]_{p,q}q^k}\right) p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k. \tag{4}$$

The operators (4) satisfy the following properties.

Lemma 1.3. *For $0<q<p\leq 1$ and for any $x\in[0,\infty)$, we obtain*

1. $L_n^{p,q}(e_0;x)=e_0$,
2. $L_n^{p,q}(\frac{t}{1+t};x)=\frac{x}{1+x}$,
3. $L_n^{p,q}((\frac{t}{1+t})^2;x)=\frac{q^2[n-1]_{p,q}}{p^2[n]_{p,q}}\frac{(x)^2}{(1+x)^2}+\frac{p^{n-1}}{[n]_{p,q}}\frac{(x)}{(1+x)}$.

In Lemma 1.3, the operators (4) preserve the test functions e_0 and e_1 but do not preserve the monomial e_2. In 2003, King [18] established a new technique to get better approximation for the Bernstein operators. In this method, the new operators approximate every continuous function $f\in C[0,1]$, while preserving the function $e_2=x^2$. Several standard linear positive operators preserve e_0 and e_1 (i.e. preserve constant as well as linear function), but this technique helps in reproducing the quadratic function as well. For more details about King type (see [8], [9]).

§2 Construction of the operators $H_n^{p,q}$

Applying King's technique on the operators (4), we give modified (p,q)-Bleimann-Butzer and Hahn operators as follows:

$$H_n^{p,q}(f;x)=\frac{1}{h_n^{p,q}(x)}\sum_{j=0}^{n} f\left(\frac{p^{n-j}[n+1]_{p,q}[j]_{p,q}}{[n]_{p,q}[n-j+1]_{p,q}q^j}\right)p^{\frac{(n-j)(n-j-1)}{2}}q^{\frac{j(j-1)}{2}}\left[\begin{array}{c} n \\ j \end{array}\right]_{p,q}(r_n^{p,q}(x))^j, \quad (5)$$

where $h_n^{p,q}(x)=\prod_{j=0}^{n-1}(p^j+q^j r_n^{p,q}(x))$, for $0<q<p\leq 1$.

Now using King's approach [18] and after some simplifications, $r_n^{p,q}(x)$ is defined as

$$r_n^{p,q}(x)=\frac{\left(\sqrt{p^{2n+2}+p^2q^2[n]_{p,q}[n-1]_{p,q}\frac{4(x)^2}{(1+x)^2}}-p^{n+1}\right)}{2q^2[n-1]_{p,q}}, \qquad n\geq 2.$$

Obviously $r_n^{p,q}(x)\geq 0$, since

$$\sqrt{p^{2n+2}+4p^2q^2[n]_{p,q}[n-1]_{p,q}\frac{(x)^2}{(1+x)^2}}-p^{n+1}\geq 0.$$

If we take $r_n^{p,q}(x)=\frac{x}{1+x}$, then the operators (5) turn out to be the previous operators (4), and the operators (5) satisfy the following relations.

Lemma 2.1. *For $0<q<p\leq 1$ and $x\in C_B[0,\infty)$, the following equality holds.*

1. $H_n^{p,q}(e_0;x)=e_0$,

2. $H_n^{p,q}(\frac{t}{1+t};x)=r_n^{p,q}(x)$,

3. $H_n^{p,q}((\frac{t}{1+t})^2;x)=\frac{q^2[n-1]_{p,q}}{p^2[n]_{p,q}}\left(r_n^{p,q}(x)\right)^2+\frac{p^{n-1}}{[n]_{p,q}}r_n^{p,q}(x)=\frac{(x)^2}{(1+x)^2}$.

§3 Main results

By using the operators $H_n^{p,q}(f;x)$, we get the Korovkin's type result with the help of Theorem 1.1.

We now take $p=p_n$, $q=q_n$, where $q_n\in(0,1)$, $p_n\in(q_n,1)$ such that

$$\lim_{n\to\infty}q_n=1, \quad \lim_{n\to\infty}q_n^n=c_1, \tag{6}$$

$$\lim_{n\to\infty}p_n=1 \text{ and } \lim_{n\to\infty}p_n^n=c_2, \tag{7}$$

where c_1, c_2 are constants.

Theorem 3.1. *Let (6) and (7) hold and $H_n^{p,q}(f;x)$ be defined by (5). Then, we get*
$$\lim_{n\to\infty}\|H_n^{p_n,q_n}(f;x)-f(x)\|_{C_B}=0, \text{ for any } f\in H_w.$$

Proof. By using Theorem 1.1 and Lemma 2.1, we have

$$\lim_{n\to\infty}\left\|H_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^j;x\right)-\left(\frac{x}{1+x}\right)^j\right\|_{C_B}=0, \quad j=0,1,2. \tag{8}$$

For $j=0$, (8) is fulfilled by using Lemma 2.1. Now for $j=1$

$$\left\|H_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^j;x\right)-\left(\frac{x}{1+x}\right)^j\right\|_{C_B} = \left\|r_n^{p_n,q_n}(x)-\frac{x}{1+x}\right\|,$$

where, $r_n^{p_n,q_n}(x)$ is defined as above and we know that $\lim\limits_{n\to\infty} r_n^{p_n,q_n}(x)=\frac{x}{1+x}$. Hence the condition holds for $j=1$.
Finally, for $j=2$, we have

$$\begin{aligned}\left\|H_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right)-\left(\frac{x}{1+x}\right)^2\right\|_{C_B} &= \left\|\frac{(x)^2}{(1+x)^2}-\left(\frac{x}{1+x}\right)^2\right\| \\ &= 0,\end{aligned}$$

since $H_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right)=\frac{(x)^2}{(1+x)^2}$. So, equation (8) holds for $j=2$. Thus, the proof is completed. □

§4 Rates of convergence

Let $f\in H_w$. The modulus of continuity of f is given by

$$w(f,\delta)=w_f(\delta)=\sum_{\left|\frac{\tau}{1+\tau}-\frac{x}{1+x}\right|\leq\delta}|f(\tau)-f(x)|,\ \text{for each}\ \ \tau,x\geq 0,$$

and for all f in $H_w[0,\infty)$, we have
(i) $\lim\limits_{n\to\infty} w_f(\delta)=0$,
(ii) $|f(\tau)-f(x)|\leq w_f(\delta)\left|\left(1+\frac{\left|\frac{\tau}{1+\tau}-\frac{x}{1+x}\right|}{\delta}\right)\right|$.

Recall that the rate of convergence for the operators (3) were obtained by Mursaleen et al. [25]. After some modification in (3), we get the new operators (4) and calculate the rate of convergence. For $\delta>0$ and for every $f\in C[0,\infty)$, we get

$$|L_n^{p,q}(f;x)-f(x)|\leq 2\tilde{w}\left(f;\sqrt{\delta_n(x)}\right), \tag{9}$$

where

$$\delta_n(x)=\frac{x^2}{(1+x)^2}\left(\frac{q_n^2[n-1]_{p_n,q_n}}{p_n^2[n]_{p_n,q_n}}\frac{1+x}{p_n+q_nx}-1\right)+\frac{p_n^{n-1}}{[n]_{p_n,q_n}}\frac{x}{1+x}.$$

Further, for the operators (5), we estimate the rate of convergence and also show that it is better than the rate of convergence of the operators (4) on some subinterval.

Theorem 4.1. *Let p_n, q_n, for $0<q_n<p_n\leq 1$ satisfying (6) and (7). Then for any $f\in H_w$ and for each $x\geq 0$, we obtain*

$$|H_n^{p_n,q_n}(f;x)-f(x)|\leq 2w\left(f;\sqrt{\gamma_n(x)}\right),\quad n\geq 2$$

where

$$\begin{aligned}\gamma_n(x) &= \frac{x}{1+x}\left\{\frac{p_n^{n+1}-\sqrt{p_n^{2n+2}+p_n^2q_n^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}\frac{4(x)^2}{(1+x)^2}}}{q_n^2[n-1]_{p_n,q_n}}\right\} \\ &+ \frac{2(x)^2}{(1+x)^2}.\end{aligned}$$

Proof. For all $n \geq 2$, $x \in [0, \infty)$ and $n \in \mathbb{N} - \{0, 1\}$, we have

$$\begin{aligned} |H_n^{p_n,q_n}(f;x) - f(x)| &\leq H_n^{p_n,q_n}(|f(t) - f(x)|;x) \\ &\leq w(f,\delta)\left\{1 + \frac{1}{\delta} H_n^{p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|;x\right)\right\}. \end{aligned}$$

By applying the Cauchy Schwarz inequality in above expression, we have

$$|H_n^{p_n,q_n}(f;x) - f(x)| \leq w(f,\delta)\left\{1 + \frac{1}{\delta}\sqrt{H_n^{p_n,q_n}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2;x\right)}\right\}.$$

Since

$$\begin{aligned} H_n^{p_n,q_n}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2;x\right) &= H_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right) + \left(\frac{x}{1+x}\right)^2 H_n^{p_n,q_n}(1;x) \\ &- 2\left(\frac{x}{1+x}\right) H_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right);x\right) \\ &= \frac{(x)^2}{(1+x)^2} + \left(\frac{x}{1+x}\right)^2 - 2\left(\frac{x}{1+x}\right)(r_n^{p_n,q_n}(x)) \\ &= 2\frac{(x)^2}{(1+x)^2} - 2\left(\frac{x}{1+x}\right)(r_n^{p_n,q_n}(x)), \end{aligned} \tag{10}$$

where, $r_n^{p_n,q_n}(x)$ is defined in (5).
By using the expression (10), we omit the details. □

If we choose the sequences p_n and q_n, where $0 < q_n < p_n \leq 1$ satisfying (6) and (7), then

$$\lim_{n\to\infty} \frac{1}{[n]_{p_n,q_n}} = 0.$$

Hence

$$\begin{aligned} &\lim_{n\to\infty} \gamma_n(x) \\ &= \lim_{n\to\infty}\left\{2\left(\tfrac{x}{1+x}\right)^2 + \tfrac{x}{1+x}\left\{\frac{p_n^{n+1} - \sqrt{p_n^{2n+2} + 4p_n^2 q_n^2 [n]_{p_n,q_n}[n-1]_{p_n,q_n}\frac{(x)^2}{(1+x)^2}}}{q_n^2[n-1]_{p_n,q_n}}\right\}\right\} \\ &= \lim_{n\to\infty}\left\{2\left(\tfrac{x}{1+x}\right)^2 + \tfrac{x}{1+x}\left\{\frac{p_n^{n+1}}{q_n^2[n-1]_{p_n,q_n}} - \sqrt{\frac{p_n^{2n+2}}{q_n^4[n-1]_{p_n,q_n}^2} + \frac{4p_n^2[n]_{p_n,q_n}\frac{(x)^2}{(1+x)^2}}{q_n^2[n-1]_{p_n,q_n}}}\right\}\right\} \\ &= \lim_{n\to\infty}\left\{\tfrac{2(x)^2}{(1+x)^2} + \tfrac{x}{1+x}\left\{0 - \sqrt{0 + \frac{4p_n^2[n]_{p_n,q_n}\frac{(x)^2}{(1+x)^2}}{q_n^2[n-1]_{p_n,q_n}}}\right\}\right\} \\ &= \tfrac{2(x)^2}{(1+x)^2} + \tfrac{x}{1+x}\left\{-2\left(\tfrac{x}{1+x}\right)\right\} \\ &= 0. \end{aligned}$$

Since $[n]_{p,q} = p^{n-1} + q[n-1]_{p,q}$, it is obvious that $\lim_{n\to\infty} \gamma_n(x) = 0$. Theorem 4.1 gives the rate of pointwise convergence of the operators $H_n^{p,q}(f;x)$ to $f(x)$.

Now if we take

$$\frac{x}{(1+x)} \leq \frac{p^{n-1}}{3[n]_{p,q}},$$

(or)

$$x \leq \frac{p^{n-1}}{3[n]_{p,q} - p^{n-1}},$$

then it is shown that for all $n \geq 2$, Theorem 4.1 have better rate of convergence for the new operators than the rate of convergence given by (9) for the operators (4). Since

$$\gamma_n(x) \leq \delta_n(x) \text{ for all n} \geq 2.$$

$$2\left(\frac{x}{1+x}\right)^2 + \frac{x}{1+x}\left\{\frac{p^{n+1} - \sqrt{p^{2n+2} + p^2q^2[n]_{p,q}[n-1]_{p,q}\frac{4(x)^2}{(1+x)^2}}}{q^2[n-1]_{p,q}}\right\}$$

$$\leq \frac{x^2}{(1+x)^2}\left(\frac{q^2[n-1]_{p,q}}{p^2[n]_{p,q}}\frac{1+x}{(p+qx)} - 1\right) + \frac{p^{n-1}}{[n]_{p,q}}\frac{x}{1+x}$$

$$\Rightarrow 2\left(\frac{x}{1+x}\right)^2 + \frac{x}{1+x}\left\{\frac{p^{n+1} - \sqrt{p^{2n+2}+0}}{q^2[n-1]_{p,q}}\right\} \leq \frac{x^2}{(1+x)^2}(0-1) + \frac{p^{n-1}}{[n]_{p,q}}\frac{x}{1+x}$$

$$\Rightarrow 2\left(\frac{x}{1+x}\right)^2 + \frac{x}{1+x}\left\{\frac{0}{q^2[n-1]_{p,q}}\right\} \leq -\frac{x^2}{(1+x)^2} + \frac{p^{n-1}}{[n]_{p,q}}\frac{x}{1+x}$$

$$\Rightarrow 3\left(\frac{x}{1+x}\right)^2 \leq \frac{p^{n-1}}{[n]_{p,q}}\frac{x}{1+x}$$

$$\Rightarrow 3\left(\frac{x}{1+x}\right) \leq \frac{p^{n-1}}{[n]_{p,q}}$$

$$\Rightarrow \frac{x}{1+x} \leq \frac{p^{n-1}}{3[n]_{p,q}}$$

$$\Rightarrow x \leq \frac{p^{n-1}}{3[n]_{p,q} - p^{n-1}}.$$

On this subinterval

$$\left[0, \frac{p^{n-1}}{3[n]_{p,q} - p^{n-1}}\right],$$

King's type approach for approximation is better than (p,q)-Bleimann-Butzer and Hahn operators. Further we will obtain an estimate concerning the rate of convergence of the operators $H_n^{p,q}(f;x)$ by means of Lipschitz type maximal function.
We consider the following space [4]

$$T_{\beta,E} = \{f : \sup(1+x)^\beta f_\beta(x) \leq K\frac{1}{(1+z)^\beta} : x \in [0,\infty) \text{ and } z \in E \subset [0,\infty)\},$$

where $K > 0$ and f is a continuous and bounded function on $[0,\infty)$, $0 < \beta \leq 1$.
A Lipschitz type maximal function was defined by Lenze [19] and is defined as

$$f_\beta(x,t) = \sum_{t>0}\frac{|f(t) - f(x)|}{|x-t|^\beta}, \quad x \neq t$$

and

$$d(x,E) = \inf\{|x-z| : z \in E\}.$$

Theorem 4.2. *For every $f \in T_{\beta,E}$, we have*

$$|H_n^{p_n,q_n}(f;x) - f(x)| \leq K\left(\gamma_n^{\frac{\beta}{2}}(x) + 2(d(x,E))^\beta\right),$$

where $\gamma_n(x)$ is shown in Theorem 4.1.

Proof. Let $\overline{E}$ be the closure of the space E. Then for $x \geq 0$, there exists a point $x_1 \in E$ such that

$$|x - x_1| = d(x, E).$$

Thus, we have

$$|f - f(x)| \leq |f - f(x_1)| + |f(x_1) - f(x)|.$$

Since $H_n^{p,q}$ are linear positive operators, $f \in T_{\beta,E}$. Now applying above inequality, we get

$$\begin{aligned} |H_n^{p,q}(f;x) - f(x)| &\leq H_n^{p,q}(|f - f(x_1)|;x) + |f(x_1) - f(x)| H_n^{p,q}(1;x) \\ &\leq K\left(\frac{|x-x_1|^\beta}{(1+x)^\beta(1+x_1)^\beta} + H_n^{p,q}\left(\left|\frac{t}{1+t} - \frac{x_1}{1+x_1}\right|^\beta ; x\right)\right). \end{aligned} \tag{11}$$

Since $H_n^{p,q}(1;x) = 1$, by using the inequality $(c+d)^\beta \leq c^\beta + d^\beta$ for $c \geq 0$, $d \geq 0$, we can write

$$\left|\frac{t}{1+t} - \frac{x_1}{1+x_1}\right|^\beta \leq \left|\frac{t}{1+t} - \frac{x}{1+x}\right|^\beta + \left|\frac{x}{1+x} - \frac{x_1}{1+x_1}\right|^\beta.$$

Consequently, we get

$$\begin{aligned} H_n^{p,q}\left(\left|\frac{t}{1+t} - \frac{x_1}{1+x_1}\right|^\beta ; x\right) &\leq H_n^{p,q}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^\beta ; x\right) \\ &+ H_n^{p,q}\left(\left|\frac{x}{1+x} - \frac{x_1}{1+x_1}\right|^\beta ; x\right) \\ &\leq H_n^{p,q}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^\beta ; x\right) + \frac{|x-x_1|^\beta}{(1+x)^\beta(1+x_1)^\beta} H_n^{p,q}(1;x). \end{aligned} \tag{12}$$

By using the Hölder's inequality, we obtain

$$H_n^{p,q}\left(\left|\frac{t}{1+t} - \frac{x_1}{1+x_1}\right|^\beta ; x\right) \leq H_n^{p,q}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2 ; x\right)^{\frac{\beta}{2}} + \frac{|x-x_1|^\beta}{(1+x)^\beta(1+x_1)^\beta} H_n^{p,q}(1;x).$$

In view of (11), we omit the details. □

In particular, if we take $E^* = [0, \infty)$ in Theorem 4.2, then the following holds.

Corollary 4.3. *For every $f \in T_{\beta,E^*}$, we get*

$$|H_n^{p_n,q_n}(f;x) - f(x)| \leq K\gamma_n^{\frac{\beta}{2}}(x),$$

where $\gamma_n(x)$ is defined as above.

§5 Generalization of the operators $H_n^{p,q}$

Here, we obtain some generalization for the new operators $H_n^{p,q}$ similar to [4], [7] and [25]. We consider

$$H_n^{(p,q),\mu} = \frac{1}{h_n^{p,q}(x)} \sum_{j=0}^{n} f\left(\frac{p^{n-j}[n+1]_{p,q}[j]_{p,q} + \mu}{b_{n,j}}\right) p^{\frac{(n-j)(n-j-1)}{2}} q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} (r_n^{p,q}(x))^j, \tag{13}$$

where $\mu \in \mathbb{R}$ and $b_{n,j}$ satisfies

$$b_{n,j} + p^{n-j}[k]_{p,q} = a_n,$$

and

$$\frac{[n]_{p,q}^2}{a_n} \longrightarrow 1, \quad as \quad n \longrightarrow \infty.$$

If $b_{n,j} = [n]_{p,q}[n-j+1]_{p,q}q^j + \alpha$ for any n, j and $0 < q < p \leq 1$ then $a_n = d_n + \alpha$, where

$$\begin{aligned} d_n &= [n]_{p_n,q_n}[n-j+1]_{p_n,q_n}q_n^j + p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n} \\ &= [n]_{p_n,q_n}[n+1]_{p_n,q_n} + p_n^{n-j}q_n^n[j]_{p_n,q_n} \end{aligned}$$

as $[n-j+1]_{p,q}q^j = [n+1]_{p,q} - p^{n-j+1}[j]_{p,q}$. For the operators (13), we have the following result.

Theorem 5.1. *Let $p = p_n$ and $q = q_n$ which satisfy (6) and (7) for $0 < q_n < p_n \leq 1$. Then for any $f \in T_{\beta,[0,\infty)}$, we obtain*

$$\lim_{n\to\infty} \left\| H_n^{(p_n,q_n),\mu}(f;x) - f(x) \right\|_{C_B} \leq 3K$$

$$\times \max\left\{ \left(\frac{[n]_{p_n,q_n}^2}{a_n+\mu}\right)^\beta \left(\frac{\mu}{[n]_{p_n,q_n}^2}\right)^\beta, \left|1 - \frac{d_n}{a_n+\mu}\right|^\beta (M_1)^\beta, 2 - \sqrt{\frac{4p_n^2[n]_{p_n,q_n}}{q_n^2[n-1]_{p_n,q_n}}} \right\},$$

where

$$M_1 = \frac{\sqrt{p_n^{2n+2} + 4p_n^2q_n^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}} - p_n^{n+1}}{2q_n^2[n-1]_{p_n,q_n}}.$$

Proof. By using (5) and (13), we get

$$\begin{aligned} &|H_n^{(p_n,q_n),\mu}(f;x) - f(x)| \\ &\leq \frac{1}{h_n^{p_n,q_n}(x)} \sum_{j=0}^n \left| f\left(\frac{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}+\mu}{b_{n,j}}\right) - f\left(\frac{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}}{\mu+b_{n,j}}\right) \right| \left[\begin{array}{c} n \\ j \end{array}\right]_{p_n,q_n} \\ &\times p_n^{\frac{(n-j)(n-j-1)}{2}} q_n^{\frac{j(j-1)}{2}} (r_n^{p,q}(x))^j \\ &+ \frac{1}{h_n^{p_n,q_n}(x)} \sum_{j=0}^n \left| f\left(\frac{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}}{\mu+b_{n,j}}\right) - f\left(\frac{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}+\mu}{[n]_{p_n,q_n}[n-j+1]_{p_n,q_n}q_n^j}\right) \right| \left[\begin{array}{c} n \\ j \end{array}\right]_{p_n,q_n} \\ &\times p_n^{\frac{(n-j)(n-j-1)}{2}} q_n^{\frac{j(j-1)}{2}} (r_n^{p,q}(x))^j + |H_n^{p_n,q_n}(f;x) - f(x)|. \end{aligned}$$

Since $f \in T_{\beta,E^*}$ and using Corollary 4.3, we obtain

$$\begin{aligned} &|H_n^{(p_n,q_n),\mu}(f;x) - f(x)| \\ &\leq \frac{K}{h_n^{p_n,q_n}(x)} \sum_{j=0}^n \left| \frac{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}+\mu}{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}+\mu+b_{n,j}} - \frac{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}}{\mu+p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}+b_{n,j}} \right|^\beta \left[\begin{array}{c} n \\ j \end{array}\right]_{p_n,q_n} \\ &\times p_n^{\frac{(n-j)(n-j-1)}{2}} q_n^{\frac{j(j-1)}{2}} (r_n^{p,q}(x))^j \\ &+ \frac{K}{h_n^{p_n,q_n}(x)} \sum_{j=0}^n \left| \frac{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}}{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}+\mu+b_{n,j}} - \frac{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}}{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}+[n]_{p_n,q_n}[n-j+1]_{p_n,q_n}q_n^j} \right|^\beta \\ &\times \left[\begin{array}{c} n \\ j \end{array}\right]_{p_n,q_n} p_n^{\frac{(n-j)(n-j-1)}{2}} q_n^{\frac{j(j-1)}{2}} (r_n^{p,q}(x))^j + K\gamma_n^{\frac{\beta}{2}}(x). \end{aligned}$$

This implies that

$$\begin{aligned} &|H_n^{(p_n,q_n),\mu}(f;x) - f(x)| \leq K \left(\frac{[n]_{p_n,q_n}^2}{a_n+\mu}\right)^\beta \left(\frac{\mu}{[n]_{p_n,q_n}^2}\right)^\beta \\ &+ \frac{K}{h_n^{p_n,q_n}(x)} \left|1 - \frac{d_n}{a_n+\mu}\right|^\beta \sum_{j=0}^n \left[\frac{p_n^{n-j}[n+1]_{p_n,q_n}[j]_{p_n,q_n}}{d_n}\right]^\beta p_n^{\frac{(n-j)(n-j-1)}{2}} q_n^{\frac{j(j-1)}{2}} \left[\begin{array}{c} n \\ j \end{array}\right]_{p_n,q_n} (r_n^{p,q}(x))^j \end{aligned}$$

$+|H_n^{p_n,q_n}(f;x)-f(x)|$
$= K\left(\frac{[n]^2_{p_n,q_n}}{a_n+\mu}\right)^\beta\left(\frac{\mu}{[n]^2_{p_n,q_n}}\right)^\beta + K\left|1-\frac{d_n}{a_n+\mu}\right|^\beta H_n^{p_n,q_n}\left((\frac{t}{1+t})^\beta;x\right)+K\gamma_n^{\frac{\beta}{2}}(x).$

Now using the Hölder inequality and condition (2) of Lemma 2.1, we obtain
$|H_n^{(p_n,q_n),\mu}(f;x)-f(x)|\le K\left(\frac{[n]^2_{p_n,q_n}}{a_n+\mu}\right)^\beta\left(\frac{\mu}{[n]^2_{p_n,q_n}}\right)^\beta + K\left|1-\frac{d_n}{a_n+\mu}\right|^\beta (r_n^{p,q}(x))^\beta+K\gamma_n^{\frac{\beta}{2}}(x).$
Thus the proof is completed. □

§6 Some graphical analysis with error estimation

By using the MATLAB(R2015a), we illustrate here the comparison between the operators defined in (4) and the modified operators (5) to the function x^2 for different parameters. For our convenience, we give some graphical examples and tables (rate of convergence, error estimation) on the interval $[0,1]$ and its subintervals. Error estimation is denoted by $\parallel L_n^{p,q}(f;x)-f(x)\parallel$ and $\parallel H_n^{p,q}(f;x)-f(x)\parallel$ for the operators (4) and (5), respectively.

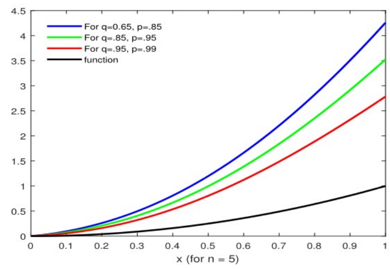

Figure 1. Approximation of the function $f(x)=x^2$ by (p,q)-BBH operators for $n=5$.

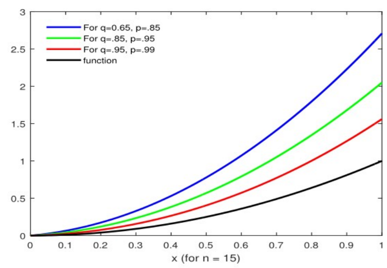

Figure 2. Approximation of the function $f(x)=x^2$ by (p,q)-BBH operators for $n=15$.

Example 6.1 Figures 1 and 2 show the convergence of (p,q)-BBH operators (4) for $n=5,15$. We see that for different values of the integers p,q, as n increases, the operators (4) converge towards the function.

Example 6.2 Figures 3 and 4 show the convergence of the operators (5) to the function $f(x)=x^2$. Here, we can see that for different values of p and q, as n increases, King's type modification gives better approximation on the subinterval

$$\left[0,\frac{p^{n-1}}{3[n]_{p,q}-p^{n-1}}\right]. \tag{14}$$

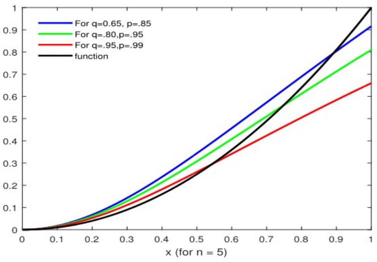

Figure 3. Approximation of the function $f(x) = x^2$ by King's type modification of (p, q)-BBH operators for $n = 5$.

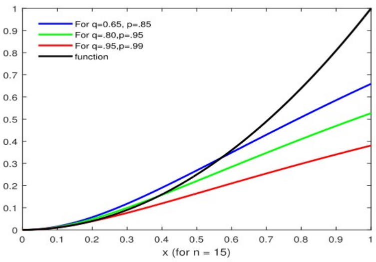

Figure 4. Approximation of the function $f(x) = x^2$ by King's type modification of (p, q)-BBH operators for $n = 15$.

6.1 Rate of convergence

Let us take $f(x) = x^2$. We now calculate the rate of convergence for the operators (4) and (5) which are denoted by $\delta_n(x)$ and $\gamma_n(x)$, respectively. We compare rate of convergence for different values of p, q and n on the above subinterval (14) which is shown in the following:

Tables $1(a), 1(b)$ show the comparison between rate of convergence of the operators (4) and (5), respectively.

Table $1(a)$.(For fixed $p = 0.9$, $q = 0.7$)

n	rate of conv. at x=0.03	rate of conv. at x=0.05	rate of conv. at x=0.08
15	0.0064	0.0103	0.0155
16	0.0064	0.0102	0.0154
17	0.0064	0.0102	0.0154

Table $1(b)$.(For fixed $p = 0.9$, $q = 0.7$)

n	rate of conv. at x=0.03	rate of conv. at x=0.05	rate of conv. at x=0.08
15	0.0054	0.0078	0.0105
16	0.0036	0.0048	0.0054
17	0.0022	0.0024	0.0014

Tables $2(a), 2(b)$ show the comparison between rate of convergence of the operators (4) and (5), respectively.

Table 2(a). (For fixed $p = 0.95$, $q = 0.85$)

n	rate of conv. at x=0.02	rate of conv. at x=0.03	rate of conv. at x=0.04
40	0.0020	0.0030	0.0039
41	0.0020	0.0030	0.0039
42	0.0020	0.0030	0.0039

Table 2(b).(For fixed $p = 0.95$, $q = 0.85$)

n	rate of conv. at x=0.02	rate of conv. at x=0.03	rate of conv. at x=0.04
40	0.0015	0.0021	0.0028
41	0.0012	0.0016	0.0020
42	0.0009	0.0011	0.0013

6.2 Error estimation

Here, we calculate the error estimation for (p, q)-Bleimann-Butzer and Hahn operators and King's type (p, q)-Bleimann-Butzer and Hahn operators to the function $f(x) = x^2$.
In Tables 3 and 4, we can easily see that the error estimation of King's type operators is better than (p, q)-Bleimann-Butzer and Hahn operators.
For any $x, t \geq 0$ such that $|t - x| < \delta$, the absolute error bound (a.e.b) is denoted by

$$2\omega(f; \delta) = \sup |f(t) - f(x)|.$$

Table 3. For the operators (4).

n(fixed $p = 0.9$, $q = 0.7$)	$\sqrt{\delta_n(x)}$	(a. e. b.) $= 2\omega(f; \sqrt{\delta_n})$
15	$\sqrt{\delta_{15}(0.08)}$=0.1244	0.0310
	$\sqrt{\delta_{15}(0.05)}$=0.1014	0.0206
	$\sqrt{\delta_{15}(0.03)}$=0.080	0.0128
16	$\sqrt{\delta_{16}(0.08)}$=0.1240	0.0308
	$\sqrt{\delta_{16}(0.05)}$=0.1009	0.0204
	$\sqrt{\delta_{16}(0.03)}$=0.080	0.0128
17	$\sqrt{\delta_{17}(0.08)}$=0.1240	0.0308
	$\sqrt{\delta_{17}(0.05)}$=0.1009	0.0204
	$\sqrt{\delta_{17}(0.03)}$=0.0800	0.0128

Table 4. For New Operators (5).

n(fixed $p = 0.9$, $q = 0.7$)	$\sqrt{\gamma_n(x)}$	(a. e. b.) $= 2\omega(f; \sqrt{\gamma_n})$
15	$\sqrt{\delta_{15}(0.08)}$=0.1024	0.0201
	$\sqrt{\delta_{15}(0.05)}$=0.0883	0.0156
	$\sqrt{\delta_{15}(0.03)}$=0.0734	0.0108
16	$\sqrt{\delta_{16}(0.08)}$=0.0734	0.0108
	$\sqrt{\delta_{16}(0.05)}$=0.0692	0.0096
	$\sqrt{\delta_{16}(0.03)}$=0.0600	0.0072
17	$\sqrt{\delta_{17}(0.08)}$=0.0374	0.0028
	$\sqrt{\delta_{17}(0.05)}$=0.0489	0.0048
	$\sqrt{\delta_{17}(0.03)}$=0.0469	0.0044

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