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On ideal convergence of double sequences in 2–fuzzy n–normed linear space

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Abstract. The purpose of this paper is to define the notions of convergence, Cauchy st–convergence, st–Cauchy, I–convergence and I–Cauchy for double sequences in 2–fuzzy n–normed spaces with respect to α–n–norms and study certain classical and standard properties related to these notions.

§1 Introduction

Let A be a subset of natural numbers $\mathbb{N}$. Then, the asymptotic density of A, denoted by $\delta(A)$, is defined as $\delta(A) = \lim_m \frac{1}{m}|\{n \leq m : n \in A\}|$, where the vertical bars denote the cardinality of the enclosed set. Based on the notion of asymptotic density, Fast [4] and Steinhaus [22] were the first ones who defined the concept of statistical convergence of sequence as, a sequence $x = (x_n)$ is said to be statistically convergent to the number ℓ if, for each $\epsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \epsilon\}$ has asymptotic density zero. Kostyrko et. al [13] introduced the notion of ideal convergence as a generalization of statistical convergence, where ideal is a family $I \subset P(X)$ of subsets of a non–empty set X satisfying the following conditions (i) $\emptyset \in I$, (ii) for each $A, B \in I$ we have $A \cup B \in I$, (iii) for each $A \in I$ and $B \subset A$ we have $B \in I$. The ideal I is called an admissible in X if and only if it contains all singletons, that is, $I \supset \{\{x\} : x \in X\}$. A filter on X is a non–empty family of sets $\mathcal{F} \subset P(X)$ satisfying (i) $\emptyset \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) for each $A \in \mathcal{F}$ and $B \supset A$ we have $B \in \mathcal{F}$. For each ideal I there is a filter associated with the ideal I, that is, $\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}$, where $K^c = X \backslash K$. Some properties of ideal convergence were studied by Šalát et. al [20]. Later, Das et. al [3] defined the notion of ideal convergence of double sequences in the vector space $\mathbb{R}$ as well as in general metric spaces. The notion of ideal convergence for both single and double sequences was defined in various settings (see [1, 5-12, 16, 18, 19]).

Park and Alaca [17] introduced the concept of 2–fuzzy n–normed linear space. Afterwards, Wenwen Yang [23] defined the notions of Cauchy and convergent sequences in such 2–fuzzy n–normed spaces and studied some basic properties of these notions. In this paper, we put an effort to define the notions of I–convergence and I–Cauchy for double sequences in 2–fuzzy

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n–normed linear spaces and some new concepts that basically are related to these two notions. Furthermore, we study some properties of these new results.

Now we recall some of the definitions and lemmas that will be used throughout the paper.

Definition 1.1. [15] Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $\|\cdot, \ldots, \cdot\|$ on $\underbrace{X \times X \times \cdots \times X}_{n} = X^n$ satisfying the following properties:

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation,

(iii) $\|x_1, x_2, \ldots, cx_n\| = |c|\|x_1, x_2, \ldots, x_n\|$ for any $c \in \mathbb{R}$,

(iv) $\|x_1, x_2, \ldots, x_{n-1}, y + z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$,

is called n–norm on X and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called a n–normed linear space.

Definition 1.2. [17] Let X be a non–empty set and $F(X)$ be the set of all fuzzy sets in X. If $f \in F(X)$, then, $f = \{(x, \mu) : x \in X \text{ and } \mu \in (0, 1]\}$. Clearly, f is a bounded function for $|f(x)| \leq 1$. Then, $F(X)$ is a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), where the addition and scalar multiplication are defined by

$$f + g = \{(x, \mu) + (y, \nu)\} = \{(x + y, \mu \wedge \nu) : (x, \mu) \in f \text{ and } (y, \nu) \in g\},$$
$$where\ (\mu \wedge \nu)(x) = min(\mu(x), \nu(x))$$

and

$$\lambda f = \{(\lambda x, \mu)\} \text{ such that } (x, \mu) \in f \text{ where } \lambda \in \mathbb{K}\ .$$

The linear space $F(X)$ is said to be a normed linear space if for every $f \in F(X)$ there is associated a non–negative real number $\|f\|$ (called the norm of f) in such a way that

(i) $\|f\| = 0$ if and only if $f = 0$,

(ii) $\|\lambda f\| = |\lambda|\|f\|$, $\lambda \in \mathbb{K}$,

(iii) $\|f + g\| \leq \|f\| + \|g\|$ for every $f, g \in F(X)$.

Then, $(F(X), \|\cdot\|)$ is a normed linear space.

Definition 1.3. [21] A 2–fuzzy set on X is a fuzzy set on $F(X)$.

Definition 1.4. [17] Let X be a real vector space of dimension $d \geq n$, (Here we allow d to be infinite.) and $F(X)$ be the set of all fuzzy sets in X. If a $[0, 1]$–valued function $\|\cdot, \ldots, \cdot\|$ on $\underbrace{F(X) \times F(X) \times \cdots \times F(X)}_{n}$ satisfies the following properties:

(i) $\|f_1, f_2, \ldots, f_n\| = 0$ if and only if $f_1, f_2, \ldots, f_n$ are linearly dependent,

(ii) $\|f_1, f_2, \ldots, f_n\|$ is invariant under any permutation,

(iii) $\|f_1, f_2, \ldots, \lambda f_n\| = |\lambda|\|f_1, f_2, \ldots, f_n\|$ for any $\lambda \in \mathbb{K}$,

(iv) $\|f_1, f_2, \ldots, f_{n-1}, y + z\| \leq \|f_1, x_2, \ldots, f_{n-1}, y\| + \|f_1, f_2, \ldots, f_{n-1}, z\|$.

Then, $(F(X), \|\cdot,\dots,\cdot\|)$ is a fuzzy n–normed linear space or $(X, \|\cdot,\dots,\cdot\|)$ is a 2–fuzzy n–normed linear space.

Definition 1.5. [17] Let $F(X)$ be a linear space over the real field $\mathbb{K}$. A fuzzy subset N of $\underbrace{F(X)\times F(X)\times\cdots\times F(X)}_{n}\times\mathbb{R}$ is called a 2–fuzzy n–norm on X (or fuzzy n–norm on $F(X)$) if and only if,

(2–N1) for all $t\in\mathbb{R}$, with $t\le 0$, $N(f_1,f_2,\dots,f_n,t)=0$,

(2–N2) for all $t\in\mathbb{R}$, with $t>0$, $N(f_1,f_2,\dots,f_n,t)=1$ if and only if, $f_1,f_2,\dots,f_n$ are linearly dependent,

(2–N3) $N(f_1,f_2,\dots,f_n,t)$ is invariant under any permutation,

(2–N4) for all $t\in\mathbb{R}$, with $t>0$, $N(f_1,f_2,\dots,\lambda f_n,t)=N(f_1,f_2,\dots,f_n,t/|\lambda|)$ if $\lambda\ne 0$, $\lambda\in\mathbb{K}$,

(2–N5) for all $s,t\in\mathbb{R}$, $N(f_1,f_2,\dots,f_n+f_n^{'},s+t)\ge\min\{N(f_1,f_2,\dots,f_n,s),N(f_1,f_2,\dots,f_n^{'},t)\}$,

(2–N6) $N(f_1,f_2,\dots,f_n,\cdot):(0,\infty)\to[0,1]$ is continuous,

(2–N7) $\lim_{t\to\infty}N(f_1,f_2,\dots,f_n,t)=1$.

Then, $(F(X),N)$ is a fuzzy n–normed linear space or (X,N) is a 2–fuzzy an n–normed linear space.

Lemma 1.1. [[17], Theorem 3.1] Let $(F(X),N)$ is a fuzzy n–normed linear space. Assume that, (2–N8) $N(f_1,f_2,\dots,f_n,t)>0$. For all $t>0$ implies $f_1,f_2,\dots,f_n$ are linearly dependent. Define

$$\|f_1,f_2,\dots,f_n\|_\alpha=\inf\{t:N(f_1,f_2,\dots,f_n,t)\ge\alpha,\alpha\in(0,1)\}.$$

Then, $\{\|\cdot,\dots,\cdot\|_\alpha,\alpha\in(0,1)\}$ is an ascending family of n–norms on $F(X)$. These n–norms are called α–n–norms on $F(X)$ corresponding to the 2–fuzzy n–norm on X.

Example 1.1. [17] Let $(F(X),\|\cdot,\dots,\cdot\|)$ be an n–normed linear space. Define

$$N(f_1,f_2,\dots,f_n,t)=\begin{cases}\frac{t}{t+\|f_1,f_2,\dots,f_n\|_\alpha} & \text{if } t>0,\ t\in\mathbb{R},\\ 0, & \text{if } t\le 0.\end{cases}$$

for all $(f_1,f_2,\dots,f_n)\in F(x)$. Then, $(F(X),N)$ is a fuzzy n–normed linear space (or (X,N) is a 2–fuzzy n–normed linear space).

Remark 1.1. [17] In a 2–fuzzy n–normed linear space (X,N), $N(f_1,f_2,\dots,f_n,\cdot)$ is a nondecreasing function of $\mathbb{R}$ for all $f_1,f_2,\dots,f_n\in F(X)$.

Remark 1.2. [17] From (2-N4) and (2-N5), it follows that in a 2-fuzzy n-normed linear space, (2-N4) for all $t\in\mathbb{R}$ with $t>0$,

$$N\bigg(f_1,f_2,\dots,\lambda f_i,\dots,f_n,t)=N(f_1,f_2,\dots,f_i,\dots,f_n,\frac{t}{|\lambda|}\bigg) \text{ if } \lambda\ne 0,\ \lambda\in\mathbb{K},$$

(2-N5) for all $s,t\in\mathbb{R}$,

$N(f_1,f_2,\dots,f_i+f_i^{'},\dots,f_n,s+t)\ge\min\{N(f_1,f_2,\dots,f_i,\dots,f_n,s),N(f_1,f_2,\dots,f_i^{'},\dots,f_n,t)\}$.

Definition 1.6. [23] A sequence $\{f_k\}$ in a fuzzy n–normed linear space $(F(X),N)$ is said to be convergent to f with respect to α–n–norm if $\lim_{k\to\infty}\|f_k-f,\omega_2,\dots,\omega_n\|_\alpha=0$ for all $\omega_2,\dots,\omega_n\in F(X)$.

Definition 1.7. [23] A sequence $\{f_k\}$ in a fuzzy 2–normed linear space $(F(X),N)$ is said to be a Cauchy sequence with respect to α–n–norm if $\lim_{k\to\infty} \|f_k - f_m, \omega_2, \dots, \omega_n\|_\alpha = 0$.

Definition 1.8. [2, 14] An admissible ideal $I \subset 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, \dots\}$ of mutually disjoint sets of I, there is a sequence $\{B_1, B_2, \dots\}$ of subsets of $\mathbb{N}$ such that, each symmetric difference $A_i \triangle B_i$ $(i = 1, 2, \dots)$ is finite and $\bigcup_{i=1}^{\infty} B_i \in I$.

Lemma 1.2. [2] Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of subsets of N such that, $A_i \in F(I)$ for each i, where I is an admissible ideal with the property (AP). Then, there exists a set $A \subset \mathbb{N}$ such that, $A \in F(I)$ and the set $A \backslash A_i$ is finite for all i.

Lemma 1.3. [2] Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal with the property (AP) and (X, ρ) be a metric space. Then, $I\text{–}\lim_{k\to\infty} x_k = x_o$ if and only if there exists a set $P \in F(I)$, $P = \{p_1 < p_2 < \dots < p_k < \dots\}$ such that, $\lim_{k\to\infty} x_{p_k} = x_o$.

§2 Main Results

In this section, we define and study the concepts of convergence, Cauchy, statistical convergence, statistical Cauchy, I–convergence and I–Cauchy for double sequences in 2–fuzzy n–normed spaces $(F(X), N)$ with respect to α–n–norms and present some basic results to these definitions. Throughout this paper, we assume that I is an admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.1. A double sequence $\{f_{ij}\}$ in a fuzzy n–normed linear space $(F(X), N)$ is said to be convergent to $f \in F(X)$ with respect to α–n–norm if for every $\epsilon > 0$, $\alpha \in (0, 1)$ and for all $\omega_2, \dots, \omega_n \in F(X)$, there exists a positive integer $k = k(\epsilon)$ such that $\|f_{ij} - f, \omega_2, \dots, \omega_n\|_\alpha < \epsilon$ for all $i, j \geq k$. In this case we write $\lim_{i,j\to\infty} \|f_{ij} - f, \omega_2, \dots, \omega_n\|_\alpha = 0$. The element f is called the limit of $\{f_{ij}\}$ in $F(X)$.

Definition 2.2. A double sequence $\{f_{ij}\}$ in a fuzzy n–normed linear space $(F(X), N)$ is said to be statistically convergent to $f \in F(X)$ with respect to α–n–norm if for every $\epsilon > 0$, $\alpha \in (0, 1)$ and for all $\omega_2, \dots, \omega_n \in F(X)$, the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_2, \dots, \omega_n\|_\alpha \geq \epsilon\}$ has double asymptotic density zero. In this case we write $st\text{–}\lim_{i,j\to\infty} \|f_{ij} - f, \omega_2, \dots, \omega_n\|_\alpha = 0$. The element f is called the st–limit of $\{f_{ij}\}$ in $F(X)$.

Definition 2.3. A double sequence $\{f_{ij}\}$ in a fuzzy n–normed linear space $(F(X), N)$ is said to be I–convergent to $f \in F(X)$ with respect to α–n–norm if for every $\epsilon > 0$, $\alpha \in (0, 1)$ and for all $\omega_2, \dots, \omega_n \in F(X)$, the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_2, \dots, \omega_n\|_\alpha \geq \epsilon\} \in I$. In this case we write $I\text{–}\lim_{i,j\to\infty} \|f_{ij} - f, \omega_2, \dots, \omega_n\|_\alpha = 0$. The element f is called the I–limit of $\{f_{ij}\}$ in $F(X)$.

Definition 2.4. A double sequence $\{f_{ij}\}$ in a fuzzy n–normed space $(F(X), N)$ is said to be Cauchy with respect to α–n–norm on $F(X)$, $\alpha \in (0, 1)$ if for every $\epsilon > 0$ and $\omega_2, \dots, \omega_n \in F(X)$ which are linearly independent there exists a positive integer $n_0 = n_0(\epsilon)$ such that $\|f_{ij} - f_{st}, \omega_2, \dots, \omega_n\|_\alpha < \epsilon$ and for all $i, j, s, t \geq n_0$.

Definition 2.5. A double sequence $\{f_{ij}\}$ in a fuzzy n–normed space $(F(X), N)$ is said to be statistically Cauchy with respect to α–n–norm on $F(X)$, $\alpha \in (0, 1)$ if for every $\epsilon > 0$ and $\omega_2, \dots, \omega_n \in F(X)$ which are linearly independent, there exist two positive integers $s = s(\epsilon)$, $t = t(\epsilon)$ such that, the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f_{st}, \omega_2, \dots, \omega_n\|_\alpha \geq \epsilon\}$ has double asymptotic density zero.

Definition 2.6. A double sequence $\{f_{ij}\}$ in a fuzzy n–normed linear space $(F(X), N)$ is said to be I–Cauchy with respect to α–n–norm on $F(X)$, $\alpha \in (0, 1)$ if for every $\epsilon > 0$ and $\omega_2, \dots, \omega_n \in F(X)$, there exist two positive integers $s = s(\epsilon)$, $t = t(\epsilon)$ such that, the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f_{st}, \omega_2, \dots, \omega_n\|_\alpha \geq \epsilon\}$ belongs to I.

Example 2.1. Take from I the class I_f of all finite subsets of $\mathbb{N} \times \mathbb{N}$. Then, I_f is an admissible ideal and I_f–convergence coincides with the usual convergence for double sequences with respect to α–n–norm on $F(X)$.

Example 2.2. Let $I_\delta = \{A \subset \mathbb{N} \times \mathbb{N} : \delta(A) = 0\}$. Then, I_δ is an admissible ideal, I_δ–convergence coincides with the statistical convergence for double sequences with respect to α–n–norm on $F(X)$.

Theorem 2.1. Let $F((X), N)$ be a fuzzy n–normed space. Then, the following statements are equivalent:

(i) $I\text{–}\lim_{i,j\to\infty} \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha = 0$ for all $\omega_2, \ldots, \omega_n \in F(X)$,

(ii) $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha \geq \epsilon\} \in I$ for all $\omega_2, \ldots, \omega_n \in F(X)$ and each $\epsilon > 0$, $\alpha \in (0,1)$,

(iii) $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha < \epsilon\} \in \mathcal{F}(I)$ for all $\omega_2, \ldots, \omega_n \in F(X)$ and every $\epsilon > 0$, $\alpha \in (0,1)$.

Proof. The proof is standard.

Theorem 2.2. Let $(F(X), N)$ be a fuzzy n–normed linear space and I be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. If a double sequence $\{f_{ij}\}$ in $F(X)$ is I–convergent with respect to α–n–norm on $F(X)$, then I–limit of $\{f_{ij}\}$ is unique.

Proof. Suppose that $I\text{–}\lim\limits_{i,j\to\infty} \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha = 0$ and $I\text{–}\lim\limits_{i,j\to\infty} \|f_{ij} - h, \omega_2, \ldots, \omega_n\|_\alpha = 0$ be such that, $f \neq h$ for all $\omega_2, \ldots, \omega_n \in F(X)$. Choose

$$\epsilon \in \left(0, \frac{\|f - h, \omega_2, \ldots, \omega_n\|_\alpha}{2}\right), \tag{1}$$

we have

$$A(\epsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha \geq \epsilon\} \in I,$$

$$B(\epsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - h, \omega_2, \ldots, \omega_n\|_\alpha \geq \epsilon\} \in I.$$

Then, by assumption and definition of filter associated with ideal I, we have $A^c(\epsilon), B^c(\epsilon) \in \mathcal{F}(I)$. But then the set $A^c(\epsilon) \cap B^c(\epsilon) \in \mathcal{F}(I)$, too. Hence, there is $(s,t) \in \mathbb{N} \times \mathbb{N}$ such that,

$$\|f_{st} - f, \omega_2, \ldots, \omega_n\|_\alpha < \epsilon \text{ and } \|f_{st} - h, \omega_2, \ldots, \omega_n\|_\alpha < \epsilon.$$

From this, for all $\omega_2, \ldots, \omega_n \in F(X)$ we have

$$\|f - h, \omega_2, \ldots, \omega_n\|_\alpha < \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha + \|f_{ij} - h, \omega_2, \ldots, \omega_n\|_\alpha < 2\epsilon,$$

which is a contradiction to (1).

Theorem 2.3. Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$. If a double sequence $\{f_{ij}\}$ is convergent to f in a fuzzy n–normed space $(F(X), N)$ with respect to α–n–norm, then, it is I–convergent to the same limit. But the converse need not be true.

Proof. Let $\{f_{ij}\}$ is convergent to f in a fuzzy n–normed space $(F(X), N)$ with respect to α–n–norm, then for every $\epsilon > 0$ and for all $\omega_2, \ldots, \omega_n \in F(X)$, there is a positive integer $k = k(\epsilon)$ such that,

$$\|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha < \epsilon \quad \text{for all } i, j \geq k.$$

Since,

$$A = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha \geq \epsilon\} \subseteq \{1,2,3,\ldots,k-1\} \times \{1,2,3,\ldots,k-1\}$$

and the ideal I is admissible, we have $A \in I$. This shows that $I\text{–}\lim\limits_{i,j\to\infty} \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha = 0$.

The following example shows that the converse need not true.

Example 2.3. Let $I = I_\delta = \{A \subset \mathbb{N} \times \mathbb{N} : \delta(A) = 0\}$ be an admissible ideal in $\mathbb{N} \times \mathbb{N}$. Define a double sequence $f = \{f_{ij}\}$ in a fuzzy n–normed space $(F(X), N)$ by

$$f_{ij} = \begin{cases} (ij, 0, 0, \ldots, 0), & \text{if } i \text{ and } j \text{ are squares,} \\ (0, 0, \ldots, 0), & \text{otherwise.} \end{cases}$$

Let $f = (0, 0, \ldots, 0)$. Then, for every $\epsilon > 0$ and $\omega_k \in F(X)$ where $k = 2, 3, \ldots, n$, we have

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_k\|_\alpha > \epsilon\} \subseteq \{1, 4, 9, \ldots, (ij)^2, \ldots\}.$$

Then, we have

$$\delta(ij \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_k\|_\alpha > \epsilon\}) = 0,$$

for every $\epsilon > 0$ and for all $\omega_k \in F(X)$, $k = 2, 3, \ldots, n$. This implies that

$$I_\delta\text{–}\lim_{ij \to \infty} \|f_{ij} - f, \omega_k\|_\alpha = 0.$$

But the sequence $\{f_{ij}\}$ is not convergent in $(F(X), N)$.

Theorem 2.4. Let $(F(X), N)$ be fuzzy n–normed space. Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$. Let $\{f_{ij}\}$ and $\{h_{ij}\}$ be two double sequences in $F(X)$ such that, $I\text{–}\lim_{i,j \to \infty} \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha = 0$ and $I\text{–}\lim_{i,j \to \infty} \|h_{ij} - h, \omega_2, \ldots, \omega_n\|_\alpha = 0$, where $f, h \in F(X)$. Then, for all $\omega_2, \ldots, \omega_n \in F(X)$

(i) $I\text{–}\lim_{i,j \to \infty} \|f_{ij} + h_{ij} - (f + h), \omega_2, \ldots, \omega_n\|_\alpha = 0$.

(ii) $I\text{–}\lim_{i,j \to \infty} \|c(f_{ij} - f), \omega_2, \ldots, \omega_n\|_\alpha = 0$, for all scalar $c \in \mathbb{R}$.

(iii) $I\text{–}\lim_{i,j \to \infty} \|f_{ij} h_{ij} - fh, \omega_2, \ldots, \omega_n\|_\alpha = 0$.

Proof.

(i) Let $\epsilon > 0$ and for all $\omega_2, \ldots, \omega_n \in F(X)$ the sets

$$A_1 := \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha \geq \frac{\epsilon}{2}\right\} \in I,$$

and

$$A_2 := \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|h_{ij} - h, \omega_2, \ldots, \omega_n\|_\alpha \geq \frac{\epsilon}{2}\right\} \in I.$$

Let

$$A := \big\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|(f_{ij} + h_{ij}) - (f + h), \omega_2, \ldots, \omega_n\|_\alpha \geq \epsilon\big\}.$$

Then, the inclusion $A \subset A_1 \cup A_2$ holds and the statement follows.

(ii) It is trivial if $c = 0$. Let $c(\neq 0) \in \mathbb{R}$, since $I\text{–}\lim_{i,j \to \infty} \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha = 0$, then, for every $\epsilon > 0$, $\omega_2, \ldots, \omega_n \in F(X)$, we have

$$\left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha \geq \frac{\epsilon}{|c|}\right\} \in I.$$

Since $\|\cdot, \ldots, \cdot\|_\alpha$ is an α–n–normed, then, the inclusion

$$\big\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|cf_{ij} - cf, \omega_2, \ldots, \omega_n\|_\alpha \geq \epsilon\big\} \subseteq \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha \geq \frac{\epsilon}{|c|}\right\} \tag{2}$$

can be easily verified. The set on the right hand side of (2) belong to I. By definition of ideal the set on the left hand side of (2) belongs to I, too. This implies that $I\text{–}\lim_{i,j \to \infty} \|c(f_{ij} - f), \omega_2, \ldots, \omega_n\|_\alpha = 0$, for all scalar $c \in \mathbb{R}$ and for every $\omega_2, \ldots, \omega_n \in F(X)$.

(iii) Since $I-\lim_{i,j\to\infty} \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha = 0$ for every $\omega_2, \ldots, \omega_n \in F(X)$, we have

$$A = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha < 1\} \in \mathcal{F}(I),$$

for $i, j \in A$, we get

$$\|f_{ij}, \omega_2, \ldots, \omega_n\|_\alpha < \|f\|_\alpha + 1. \tag{3}$$

Now for all $\omega_2, \ldots, \omega_n \in F(X)$, we have

$$\begin{aligned}\|f_{ij}h_{ij} - fh, \omega_2, \ldots, \omega_n\|_\alpha &\le \|f_{ij}, \omega_2, \ldots, \omega_n\|_\alpha \|h_{ij} - h, \omega_2, \ldots, \omega_n\|_\alpha \\ &+ \|h\|_\alpha \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha.\end{aligned}$$

From inequality (3), we get

$$\|f_{ij}h_{ij} - fh, \omega_2, \ldots, \omega_n\|_\alpha \le (\|f\|_\alpha + 1)\|h_{ij} - h, \omega_2, \ldots, \omega_n\|_\alpha + \|h\|_\alpha \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha. \tag{4}$$

Now let $\epsilon > 0$ be given and for all $\omega_2, \ldots, \omega_n \in F(X)$. Choose $\delta > 0$ such that,

$$0 < \delta < \frac{\epsilon}{\|f\|_\alpha + \|h\|_\alpha + 1}. \tag{5}$$

Since, $I-\lim_{i,j\to\infty} \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha = 0$ and $I-\lim_{i,j\to\infty} \|h_{ij} - h, \omega_2, \ldots, \omega_n\|_\alpha = 0$, the sets

$$B = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha < \delta\} \in \mathcal{F}(I),$$

and

$$C = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \|h_{ij} - h, \omega_2, \ldots, \omega_n\|_\alpha < \delta\} \in \mathcal{F}(I).$$

Thus, we have $B \cap C \in \mathcal{F}(I)$. Obviously, $A \cap B \cap C \in \mathcal{F}(I)$ and for each $(i,j) \in \mathbb{N}\times\mathbb{N}$, from equations (4) and (5), it follows that

$$\begin{aligned}\|f_{ij}h_{ij} - fh, \omega_2, \ldots, \omega_n\|_\alpha &\le (\|f\|_\alpha + 1)\delta + (\|h\|_\alpha)\delta \\ &< \frac{(\|f\|_\alpha + 1)\epsilon}{\|f\|_\alpha + \|h\|_\alpha + 1} + \frac{(\|h\|_\alpha)\epsilon}{\|f\|_\alpha + \|h\|_\alpha + 1} \\ &< \frac{(\|f\|_\alpha + \|h\|_\alpha + 1)\epsilon}{\|f\|_\alpha + \|h\|_\alpha + 1} \\ &< \epsilon.\end{aligned}$$

This implies that

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : \|f_{ij}h_{ij} - fh, \omega_2, \ldots, \omega_n\|_\alpha \ge \epsilon\} \in I,$$

i.e, $I-\lim_{i,j\to\infty} \|f_{ij}h_{ij} - fh, \omega_2, \ldots, \omega_n\|_\alpha = 0$.

Theorem 2.5. Let I be an admissible ideal in $\mathbb{N}\times\mathbb{N}$ with the property (AP). Let $(F(X), N)$ be a fuzzy n–normed space. Then, $\{f_{ij}\}$ is an I–convergent double sequence in $(F(X), N)$ if and only if, there is a double sequence $\{h_{ij}\}$ convergent to f and such that $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij} \ne h_{ij}\} \in I$.

Proof. Suppose that $\{f_{ij}\}$ is I–convergent in $(F(X), N)$. For each $(s,t) \in \mathbb{N}$ and for every $\omega_2, \ldots, \omega_n \in F(X)$, let

$$K_{st} = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha < \frac{1}{st}\right\}.$$

Then, $K_{st} \in \mathcal{F}(I)$. Since, I is admissible ideal in $\mathbb{N}\times\mathbb{N}$ with the property (AP), by Lemma (1.2), there is $K \subset \mathbb{N}\times\mathbb{N}$ such that, $K \in \mathcal{F}(I)$ and the set $K \backslash K_{st}$ is finite for each $s, t \in \mathbb{N}$. Observe that $\{f_{ij}\}$ is convergent to f in K, that is for every $\epsilon > 0$, there exists $k = k(\epsilon) \in \mathbb{N}$ such that $i, j \ge k$ and $(i,j) \in K$ imply $\|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha < \epsilon$. Define a double sequence

$\{h_{ij}\}$ in $F(X)$ as

$$h_{ij} = \begin{cases} f_{ij}, & \text{if } (i,j) \in K \\ f, & \text{if} (i,j) \in K^c. \end{cases}$$

The double sequence $\{h_{ij}\}$ is convergent to f with respect to α–n–norm on $F(X)$. Thus, we have $\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij} \neq h_{ij}\} \in I$.

Conversely, suppose that $\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij} \neq h_{ij}\} \in I$ and $\{h_{ij}\}$ is convergent to f. Let $\epsilon > 0$ and $\omega_2, \ldots, \omega_n \in F(X)$ be given. Then, for each $k = k(\epsilon) \in \mathbb{N}$, we can write

$$\begin{aligned} \{i,j \leq k : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha \geq \epsilon\} &\subseteq \{i,j \leq k : f_{ij} \neq h_{ij}\} \\ &\cup \{i,j \leq k : \|h_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha \geq \epsilon\}. \end{aligned} \tag{6}$$

Since, the first set on the right hand side of (6) belongs to I and the second set is contained in a finite subset of $\mathbb{N}\times\mathbb{N}$, it belongs to I. This implies that $\{i,j \leq k : \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha \geq \epsilon\} \in I$.

Theorem 2.6. Let $\{f_{ij}\}$ be a double sequence in fuzzy n–normed space $(F(X), N)$. Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$ with the property (AP). Then, $\{f_{ij}\}$ is I–convergent to f if and only if, there exist $\{h_{ij}\}$ and $\{q_{ij}\}$ in $F(X)$ such that $f_{ij} = h_{ij} + q_{ij}$, $\{h_{ij}\}$ is convergent to f and $\sup(q_{ij}) = \{i,j \in \mathbb{N} : q_{ij} \neq \theta\} \in I$, where θ is the zero element of the fuzzy linear space $F(X)$.

Proof. Suppose $\{f_{ij}\}$ is I–convergent to f. Then, by Lemma (1), there is a set $K \in \mathcal{F}(I)$, $K = \{(i_n, j_m) : i_1 < i_2 < \ldots \text{ and } j_1 < j_2 < \ldots\} \subset \mathbb{N} \times \mathbb{N}$ such that $\{f_{i_n j_m}\}$ is convergent to f. Define the double sequence $\{h_{ij}\}$ in $F(X)$ as

$$h_{ij} = \begin{cases} f_{ij}, & \text{if } (i,j) \in K, \\ f, & (i,j) \in K^c. \end{cases} \tag{7}$$

It is clear that $\{h_{ij}\}$ is convergent to f. Further, we set $q_{ij} = f_{ij} - h_{ij}$ for each $(i,j) \in \mathbb{N} \times \mathbb{N}$. Since, $\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij} \neq h_{ij}\} \in K^c \in I$, we have $\{(i,j) \in \mathbb{N} \times \mathbb{N} : q_{ij} \neq \theta\} \in I$. It follows that $\sup(q_{ij}) \in I$ and by (7), we get $f_{ij} = h_{ij} + q_{ij}$.

Conversely, Suppose that, there exist two double sequences $\{h_{ij}\}$ and $\{q_{ij}\}$ in $F(X)$ such that $f_{ij} = h_{ij} + q_{ij}$, $\{h_{ij}\}$ convergent to f and $\sup(q_{ij}) \in I$. We prove that $\{f_{ij}\}$ is I–convergent to f. Let $K = \{(i_n, j_m) : i_1 < i_2 < \ldots \text{ and } j_1 < j_2 < \ldots\} \subset \mathbb{N} \times \mathbb{N}$ such that, $K = \{(i,j) \in \mathbb{N} \times \mathbb{N} : q_{ij} = \theta\}$. Since $\sup(q_{nk}) = \{(n,k) \in \mathbb{N} \times \mathbb{N} : q_{nk} \neq \theta\} \in I$, we have $K \in \mathcal{F}(I)$, therefore $f_{ij} = h_{ij}$ if $(i,j) \in K$. Thus, by lemma (1.3) we conclude that there exists a set $K = \{(i_n, j_m) : i_1 < i_2 < \ldots \text{ and } j_1 < j_2 < \ldots\} \subset \mathbb{N} \times \mathbb{N}$, $K \in \mathcal{F}(I)$, such that, $\{f_{i_n j_m}\}$ convergent to f, it follows that by lemma (1.3), we have $\{f_{ij}\}$ is I–convergent to f.

Theorem 2.7. Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$ and $(F(X), N)$ be a fuzzy n–normed space. A double sequence $\{f_{ij}\}$ in $F(X)$ is I–convergent with respect to α–n–norm, if and only if for all $\omega_2, \ldots, \omega_n \in (F(X), N)$ and every $\epsilon > 0$ there exist $s(\epsilon), t(\epsilon) \in \mathbb{N}$ such that,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|f_{ij} - f_{st}, \omega_2, \ldots, \omega_n\|_\alpha < \epsilon\} \in \mathcal{F}(I). \tag{8}$$

Proof. Suppose that the double sequence $\{f_{ij}\}$ is I–convergent to $f \in F(X)$. Then, for all $\omega_2, \ldots, \omega_n \in (F(X), N)$ and given $\epsilon > 0$, we have

$$A = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|f_{i,j} - f, \omega_2, \ldots, \omega_n\|_\alpha < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I)$$

Fix an integers $s = s(\epsilon)$ and $t = t(\epsilon)$ in A. Then, the following holds for all $(i,j) \in A$.

$$\|f_{ij} - f_{st}, \omega_2, \ldots, \omega_n\|_\alpha \leq \|f_{ij} - f, \omega_2, \ldots, \omega_n\|_\alpha + \|f - f_{st}, \omega_2, \ldots, \omega_n\|_\alpha < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, (8) holds.

Conversely, suppose that (8) holds for all $\omega_2, \dots, \omega_n \in (F(X), N)$ and every $\epsilon > 0$. Then, the set

$$B_\epsilon = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij} \in [f_{st} - \epsilon, f_{st} + \epsilon]\} \in \mathcal{F}(I), \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [f_{st} - \epsilon, f_{st} + \epsilon]$. Fixing $\epsilon > 0$, we have $B_\epsilon \in \mathcal{F}(I)$ and $B_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. Hence, $B_\epsilon \cap B_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \emptyset.$$

That is,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij} \in J\} \in \mathcal{F}(I).$$

and thus

$$\text{diam } J \leq \frac{1}{2} \text{ diam } J_\epsilon,$$

where the diam of J denotes the length of interval J. Proceeding in this way, by induction, we get a sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_m \supseteq \cdots$$

with the property

$$\text{diam } (I_m) \leq \frac{1}{2} \text{ diam } (I_{m-1}), \text{ for } m = 2, 3, \dots$$

and

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij} \in I_m\} \in \mathcal{F}(I).$$

Then, there exists $h \in \bigcap_{m \in \mathbb{N}} I_m$ and it is a routine work to verify that

$$I\text{–} \lim_{i,j \to \infty} \|f_{ij} - h, \omega_2, \dots, \omega_n\|_\alpha = 0.$$

§3 Conclusion

In this paper we defined the notions of I–convergence and I–Cauchy of double sequences in 2–fuzzy n–normed spaces with respect to α–n–norm, and some new concepts that basically relate to these two notions. Finally, some properties of these new notions were investigated.

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